Properties of a fuzzy set: α-level sets

5. The *α-level sets* (or *α-cuts*):
   - The *α-level set* (where $\alpha \in [0,1]$) of the fuzzy set $\tilde{A}$ having the membership function $\mu_{\tilde{A}}(x)$ is the crisp set A_α for which $\mu_{\tilde{A}}(x) \geq \alpha$
   - We can define *strong α cut* as the crisp set A'_α for which $\mu_{\tilde{A}}(x) > \alpha$
   - In the example with the comfortable house, WHERE $\tilde{A} = \{(1,0.1),(2,0.5),(3,0.8),(4,1.0),(5,0.7),(6,0.2)\}$, the α-cuts of the fuzzy set $\tilde{A}$ are:
     - $A_{0.1} = \{1,2,3,4,5,6\} = supp\tilde{A}$ (the support of $\tilde{A}$)
     - $A_{0.2} = \{2,3,4,5,6\}$
     - $A_{0.5} = \{2,3,4,5\}$
     - $A_{0.7} = \{3,4,5\}$
     - $A_{0.8} = \{3,4\}$
     - $A_{1.0} = \{4\} = \text{core}\tilde{A}$

- It can be proved that for any fuzzy set $\tilde{A}$, it holds:

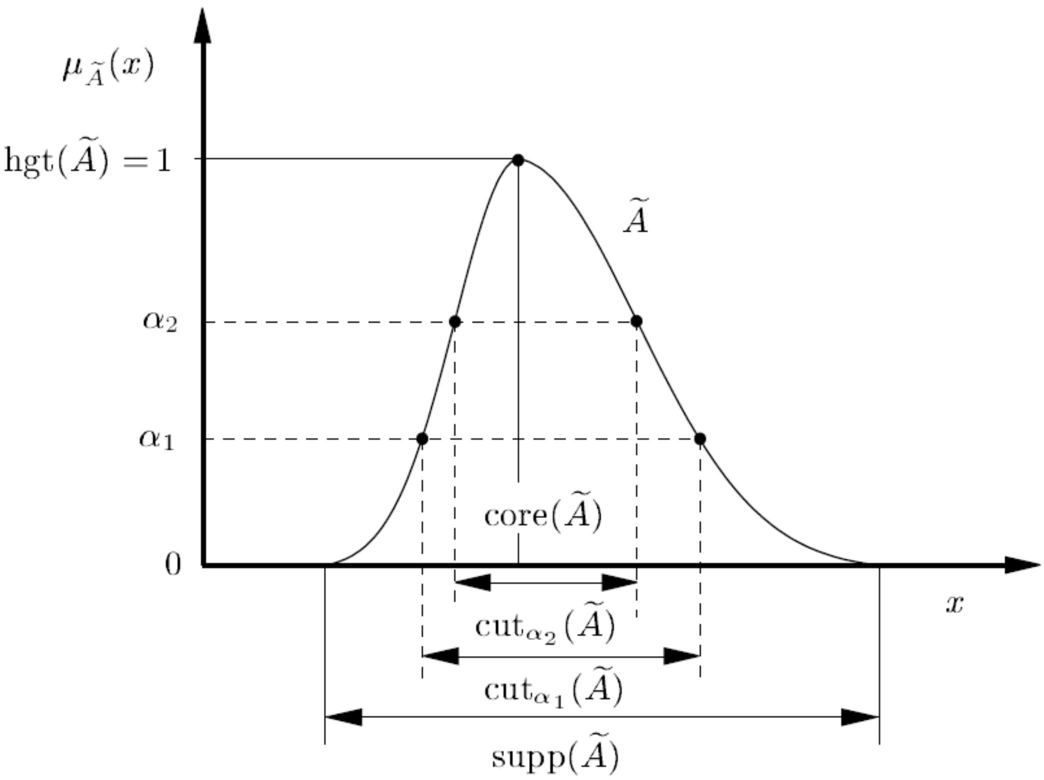
$$\tilde{A} = \bigcup_{\alpha} \alpha \cdot A_\alpha$$

- Which means that, any fuzzy set can be written as the union for all the values of α of the product between α and the α-cuts of the fuzzy set
- This property is very important and it connects the fuzzy and the crisp sets
- It is also very useful for proving different properties of fuzzy sets (some properties are easier to be proved for crisp sets)

- We will illustrate this property on the example with the comfortable house:
  - $\alpha \cdot A_\alpha$ is the fuzzy set in which each element will hace the membership function equal with α.
  - $0.1 \cdot A_{0.1} = \{(1, 0.1), (2, 0.1), (3, 0.1), (4, 0.1), (5, 0.1), (6, 0.1)\}$
  - $0.2 \cdot A_{0.2} = \{(2, 0.2), (3, 0.2), (4, 0.2), (5, 0.2), (6, 0.2)\}$
  - ...
  - $0.8 \cdot A_{0.8} = \{(3, 0.8), (4, 0.8)\}$
  - $1.0 \cdot A_{1.0} = \{(4, 1.0)\}$
  - The union of two or more fuzzy sets is defined as the maximum between their membership function, hence
  - $0.1 \cdot A_{0.1} \cup 0.2 \cdot A_{0.2} \cup \ldots \cup 0.8 \cdot A_{0.8} \cup 1.0 \cdot A_{1.0} =$ $= \{(1, 0.1), (2, max(0.1, 0.2)), (3, max(0.1, 0.2, \ldots, 0.8)),$ $(4, max(0.1, \ldots, 0.8, 1)), \ldots (6, max(0.1, 0.2)\} = \tilde{A}$

The set $A_{\alpha+}$ with

$$\mathrm{cut}_{\alpha+}(\widetilde{A}) = A_{\alpha+} = \left\{x \in X \mid \mu_{\widetilde{A}}(x) > \alpha\right\}$$



Properties (Basic properties of α-cuts)Let A, B are two a fuzzy subset on a universe X and $\alpha, \beta \in [0, 1]$

(1) if $\alpha \leq \beta$, then $A_\beta \subseteq A_\alpha$

(2) $(A \cap B)_\alpha = A_\alpha \cap B_\alpha$

(3) $(A \cup B)_\alpha = A_\alpha \cup B_\alpha$

Theorem (Decomposition theorem)Any fuzzy subset A of the reference set X is defined from its α-cuts by:

$$\forall x \in X \mu_A(x) = \sup_{0<\alpha\leq 1} \alpha \cdot \chi_{A_\alpha}(x)$$

χ_{A^α} is the characteristic function of A^α.

Proof. Let $x \in X$ and put $\mu(x) = \alpha, \alpha \in [0, 1]$ we have,

$$\begin{cases} \mu_\alpha(x) = 1 & \text{if} \quad \mu_\alpha(x) \geq \alpha \\ \mu_\alpha(x) = 0 & \text{if} \quad \mu_\alpha(x) < \alpha \end{cases}$$

So, $\alpha\mu_\alpha(x) = \alpha = \mu(x)$;

From where,

$$\sup_{\alpha\in[0,1]} (\alpha\mu_\alpha(x)) \geq \mu(x)$$

On the other hand we have:

$$\text{for all } \alpha \in [0,1], \begin{cases} \mu_\alpha(x) = 1 & \text{if} \quad \mu_\alpha(x) \geq \alpha \\ \mu_\alpha(x) = 0 & \text{if} \quad \mu_\alpha(x) < \alpha \end{cases}$$

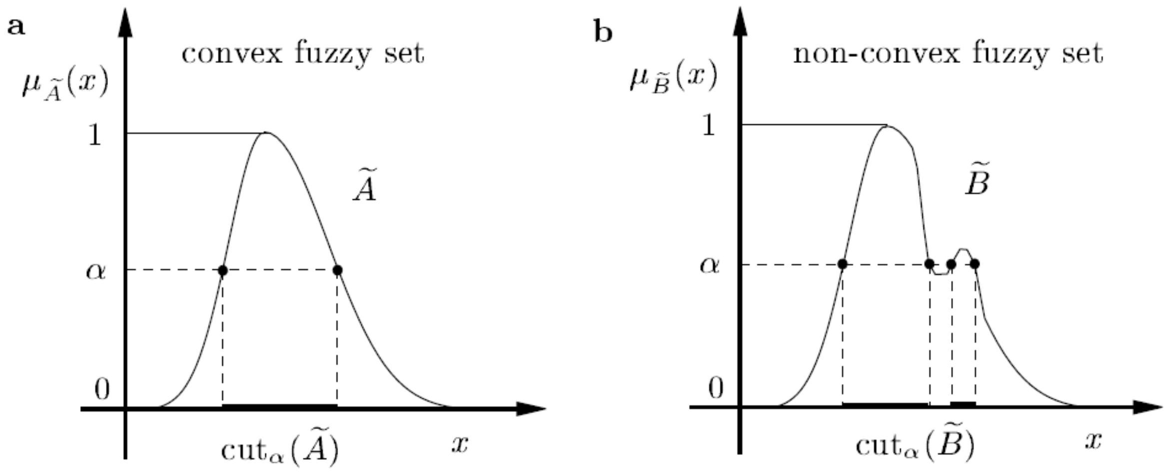
we have two cases: $\alpha\mu_\alpha(x) \leq \alpha \quad \forall\alpha \in [0,1]$

Hence,

$$\sup_{\alpha\in[0,1]} (\alpha\mu_\alpha(x)) \leq \mu(x)$$

According to $(*)$ and $(**)$ then $\forall x \in X \quad \mu(x) = \sup_{\alpha\in[0,1]} (\alpha\mu_\alpha(x))$

6. Convexity of a fuzzy set
   - A fuzzy set $\tilde{A} \subset X$ is convex if and only if $\forall x_1, x_2 \in X$ and $\forall\lambda \in [0,1]$ the following relation takes place: $\mu_{\tilde{A}}(\lambda \cdot x_1 + (1-\lambda)\cdot x_2) \geq min(\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2))$
   - The expression $\lambda \cdot x_1 + (1-\lambda)\cdot x_2$ describes the segment situated between the points having the abscissa x_1 and x_2
   - The expression $\mu_{\tilde{A}}(\lambda \cdot x_1 + (1-\lambda)\cdot x_2)$ describes the image of this segment through the function $\mu_{\tilde{A}}(x)$
   - **Equivalently, a fuzzy set $\tilde{A}$ is convex iff all its α-level sets are convex**
   - Which means that, if a fuzzy set is not convex, there exist α-level sets of this fuzzy set that are not convex, i.e., there exist segments $x_1^\alpha x_2^\alpha$ which are "interrupted" (are not continues)



t-norms, t-conorms

A t-norm T is a function $z = T(a, b)$, $0 \leq a, b, z \leq 1$, having the following four properties:

1. $T(a, 1) = a$;
2. $T(a, b) = T(b, a)$;
3. if $b_1 \leq b_2$, then $T(a, b_1) \leq T(a, b_2)$;
4. $T(a, T(b, c)) = T(T(a, b), c)$.

Property 1 is a boundary condition implying $T(1, 1) = 1$, $T(0, 1) = 0$. Property 2 then says that $T(1, 0) = 0$ too. From Property 3 we see that $0 \leq 1$ implies $T(0, 0) \leq T(0, 1) = 0$ and $T(0, 0) = 0$. T has those three properties required of an intersection $i(a, b)$ given in the previous Section 3.2. In addition T is symmetric (property 2) and non-decreasing in both arguments (property 2 and 3). Also it is associative (Property 4) which we will need later in this section.

for some t-norm T.

The basic t-norms are

$$T_m(a, b) = \min(a, b)$$

$$T_b(a, b) = \max(0, a + b - 1)$$

$$T_p(a, b) = ab$$

$$T^*(a, b) = \begin{cases} a, & \text{if} \quad b = 1 \\ b, & \text{if} \quad a = 1 \\ 0, & \text{otherwise.} \end{cases}$$

T_m is called standard intersection, T_b is bounded sum, T_p is algebraic product and T^* is drastic intersection.

It is not too difficult to see (see the problems at the end of this section) that

$$T^*(a,b) \leq T_b(a,b) \leq T_p(a,b) \leq T_m(a,b),$$

for all a, b in $[0,1]$.

In fact, if T is any t-norm, then

$$T^*(a,b) \leq T(a,b) \leq T_m(a,b),$$

Triangular conorms

Definition 5 A triangular conorm (t-conorm for short) is a binary operation S on the unit interval $[0,1]$, i.e., it is a function $S : [0,1]^2 \to [0,1]$ such that for all $x, y, z \in [0,1]$: the following four axioms are satisfied:

(S1) $S(x,y) = S(y,x)$ (commutativity)

(S2) $S(x, S(y,z)) = S(S(x,y),z)$ (associativity)

(S3) $S(x,y) \leq S(x,z)$ whenever $y \leq z$ (monotonicity)

(S4) $S(x,0) = x$ (boundary condition)

Example

The following are the four basic t-norms S_M, S_P, S_L, and S_D given by, respectively:

$S_M(x,y) = \max(x,y)$	$(maximum)$
$S_P(x,y) = x + y - x \cdot y$	$(probabilisticsum)$
$S_L(x,y) = \min(x+y,1)$	$(Lukasiewiczt - conorm, boundedsum)$
$S_D(x,y) = \begin{cases} 1 \quad if(x,y) \in]0,1]^2 \\ \max(x,y) otherwise \end{cases}$	$(drasticsum)$

$T(x,y) = \frac{x+y}{(1+xy)}$	Einstein
$T(x,y) = \frac{x+y-2xy}{(1-xy)}$	Hamacher
$T(x,y) = \frac{x+y+xy-\min(x,y,1-\alpha)}{\max(1-\alpha,1-y,\alpha)}$	Dubois and Parade (1986) $\alpha \in [0,1]$

Proposition 4 Any t-conorm S satisfies $S(1, x) = S(x, 1) = 1$, for all $x \in [0, 1]$.

Proof We know that $S(x, 1) \in [0, 1]$, so $S(x, 1) \leq 1$, and we use the axiom (S3)(monotonicity) we obtient $S(x, 1) \geq S(0, 1) = 1$.