Fuzzy Relations

5.1 CRISP AND FUZZY RELATIONS

A *crisp relation* represents the presence or absence of association, interaction or interconnectedness between the elements of two or more sets. This concept can be generalized to allow for various degrees or strengths of association or interaction between elements. Degrees of association can be represented by membership grades in a *fuzzy relation* in the same way as degrees of set membership are represented in the fuzzy set. In fact, just as the crisp set can be viewed as a restricted case of the more general fuzzy set concept, the crisp relation can be considered to be a restricted case of the fuzzy relation.

Throughout this chapter, the concepts and properties of crisp relations are briefly discussed as a refresher and in order to demonstrate their generalized application to fuzzy relations.

A *relation* among crisp sets $X_1, X_2, \ldots, X_n$ is a subset of the Cartesian product $\times_{i \in \mathbb{N}_n} X_i$. It is denoted either by $R(X_1, X_2, \ldots, X_n)$ or by the abbreviated form $R(X_i | i \in \mathbb{N}_n)$. Thus,

$$R(X_1, X_2, \ldots, X_n) \subseteq X_1 \times X_2 \times \ldots \times X_n,$$

so that for relations among sets $X_1, X_2, \ldots, X_n$, the Cartesian product $X_1 \times X_2 \times \ldots \times X_n$ represents the universal set. Because a relation is itself a set, the basic set concepts such as containment or subset, union, intersection, and complement can be applied without modification to relations.

Each crisp relation R can be defined by a characteristic function which assigns a value of 1 to every tuple of the universal set belonging to the relation and a 0 to every tuple not belonging to it. Denoting a relation and its characteristic function by the same symbol R, we have

$$R(x_1, x_2, \ldots, x_n) = \begin{cases} 1 & \text{iff } \langle x_1, x_2, \ldots, x_n \rangle \in R, \\ 0 & \text{otherwise} \end{cases}$$

The membership of a tuple in a relation signifies that the elements of the tuple are related to or associated with one another. For instance, let R represent the relation of marriage between

Example 5.1

Let R be a relation among the three sets X = {English, French}, Y = {dollar, pound, franc, mark} and Z = {US, France, Canada, Britain, Germany}, which associates a country with a currency and language as follows:

$$R(X, Y, Z) = \{\langle\text{English, dollar, US}\rangle, \langle\text{French, Franc, France}\rangle, \langle\text{English, dollar, Canada}\rangle, \langle\text{French, dollar, Canada}\rangle, \langle\text{English, pound, Britain}\rangle\}.$$

This relation can also be represented with the following three-dimensional membership array:

English

	US	*Fra*	*Can*	*Brit*	*Ger*
dollar	1	0	1	0	0
pound	0	0	0	1	0
franc	0	0	0	0	0
mark	0	0	0	0	0

French

	US	*Fra*	*Can*	*Brit*	*Ger*
dollar	0	0	1	0	0
pound	0	0	0	0	0
franc	0	1	0	0	0
mark	0	0	0	0	0

the set of all men and the set of all women. Of all the possible pairings of men and women, then, only those pairs who are married to each other will be assigned a value of 1 indicating that they belong to this relation. A relation between two sets is called *binary*; if three, four, or five sets are involved, the relation is called *ternary*, *quaternary*, or *quinary*, respectively. In general, a relation defined on n sets is called *n-ary* or *n-dimensional*.

A relation can be written as a set of ordered tuples. Another convenient way of representing a relation $R(X_1, X_2, \ldots, X_n)$ involves an n-dimensional membership array: $\mathbf{R} = [r_{i_1, i_2, \ldots, i_n}]$. Each element of the first dimension i_1 of this array corresponds to exactly one member of X_1 and each element of dimension i_2 to exactly one member of X_2, and so on. If the n-tuple $\langle x_1, x_2, \ldots, x_n \rangle \in X_1 \times X_2 \times \ldots \times X_n$ corresponds to the element $r_{i_1, i_2, \ldots, i_n}$ of $\mathbf{R}$, then

$$r_{i_1, i_2, \ldots, i_n} = \begin{cases} 1 & \text{if and only if } \langle x_1, x_2, \ldots, x_n \rangle \in R, \\ 0 & \text{otherwise.} \end{cases}$$

BINARY FUZZY RELATIONS

Binary relations have a special significance among n-dimensional relations since they are, in some sense, generalized mathematical functions. Contrary to functions from X to Y, binary relations $R(X, Y)$ may assign to each element of X two or more elements of Y. Some basic operations on functions, such as the inverse and composition, are applicable to binary relations as well.

Given a fuzzy relation $R(X, Y)$, its *domain* is a fuzzy set on X, dom R, whose membership function is defined by

$$\text{dom}\, R(x) = \max_{y \in Y} R(x, y)$$

for each $x \in X$. That is, each element of set X belongs to the domain of R to the degree equal to the strength of its strongest relation to any member of set Y. The *range* of $R(X, Y)$ is a fuzzy relation on Y, ran R, whose membership function is defined by

$$\text{ran}\, R(y) = \max_{x \in X} R(x, y)$$

for each $y \in Y$. That is, the strength of the strongest relation that each element of Y has to an element of X is equal to the degree of that element's membership in the range of R. In addition, the *height* of a fuzzy relation $R(X, Y)$ is a number, $h(R)$, defined by

$$h(R) = \max_{y \in Y} \max_{x \in X} R(x, y).$$

That is, $h(R)$ is the largest membership grade attained by any pair $\langle x, y \rangle$ in R.

A convenient representation of binary relation $R(X, Y)$ are *membership matrices* $\mathbf{R} = [r_{xy}]$, where $r_{xy} = R(x, y)$. Another useful representation of binary relations is a *sagittal diagram*. Each of the sets X, Y is represented by a set of nodes in the diagram; nodes corresponding to one set are clearly distinguished from nodes representing the other set. Elements of $X \times Y$ with nonzero membership grades in $R(X, Y)$ are represented in the diagram by lines connecting the respective nodes. These lines are labelled with the values of the membership grades.

The *inverse* of a fuzzy relation $R(X, Y)$, which is denoted by $R^{-1}(Y, X)$, is a relation on $Y \times X$ defined by

$$R^{-1}(y, x) = R(x, y)$$

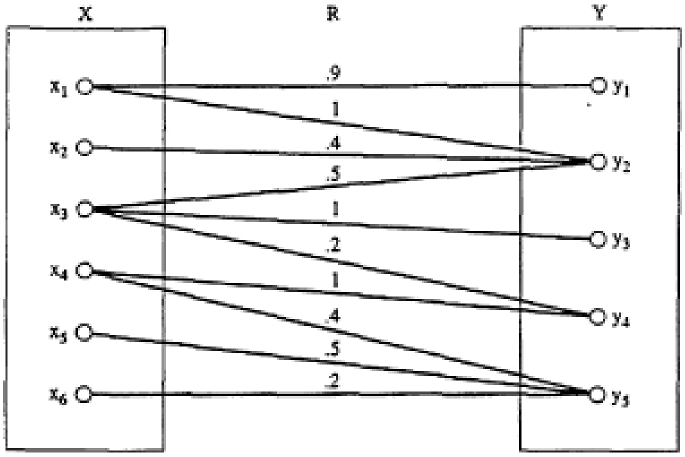
for all $x \in X$ and all $y \in Y$. A membership matrix $\mathbf{R}^{-1} = [r_{yx}^{-1}]$ representing $R^{-1}(Y, X)$ is the transpose of the matrix $\mathbf{R}$ for $R(X, Y)$, which means that the rows of $\mathbf{R}^{-1}$ equal the columns of $\mathbf{R}$ and the columns of $\mathbf{R}^{-1}$ equal the rows of $\mathbf{R}$. Clearly,

$$(\mathbf{R}^{-1})^{-1} = \mathbf{R}$$

for any binary fuzzy relation.

Consider now two binary fuzzy relations $P(X, Y)$ and $Q(Y, Z)$ with a common set Y. The *standard composition* of these relations, which is denoted by $P(X, Y) \circ Q(Y, Z)$, produces a binary relation $R(X, Z)$ on $X \times Z$ defined by

$$R(x, z) = [P \circ Q](x, z) = \max_{y \in Y} \min[P(x, y), Q(y, z)]$$



(a)

$$R = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{array} \begin{array}{c} \begin{array}{ccccc} y_1 & y_2 & y_3 & y_4 & y_5 \end{array} \\ \begin{bmatrix} .9 & 1 & 0 & 0 & 0 \\ 0 & .4 & 0 & 0 & 0 \\ 0 & .5 & 1 & .2 & 0 \\ 0 & 0 & 0 & 1 & .4 \\ 0 & 0 & 0 & 0 & .5 \\ 0 & 0 & 0 & 0 & .2 \end{bmatrix} \end{array}$$

composition of two binary fuzzy relations represented by their membership functions:

$$\begin{bmatrix} .3 & .5 & .8 \\ 0 & .7 & 1 \\ .4 & .6 & .5 \end{bmatrix} \circ \begin{bmatrix} .9 & .5 & .7 & .7 \\ .3 & .2 & 0 & .9 \\ 1 & 0 & .5 & .5 \end{bmatrix} = \begin{bmatrix} .8 & .3 & .5 & .5 \\ 1 & .2 & .5 & .7 \\ .5 & .4 & .5 & .6 \end{bmatrix}.$$

For example,

$$\begin{aligned} .8(= r_{11}) &= \max[\min(.3, .9), \min(.5, .3), \min(.8, 1)] \\ &= \max[\min(p_{11}, q_{11}), \min(p_{12}, q_{21}), \min(p_{13}, q_{31})], \\ .4(= r_{32}) &= \max[\min(.4, .5), \min(.6, .2), \min(.5, 0)] \\ &= \max[\min(p_{31}, q_{12}), \min(p_{32}, q_{22}), \min(p_{33}, q_{32})]. \end{aligned}$$

BINARY RELATIONS ON A SINGLE SET

In addition to defining a binary relation that exists between two different sets, it is also possible to define a crisp or fuzzy binary relation among the elements of a single set X. A binary relation of this type can be denoted by $R(X, X)$ or $R(X^2)$ and is a subset of $X \times X = X^2$. These relations are often referred to as *directed graphs* or *digraphs*.

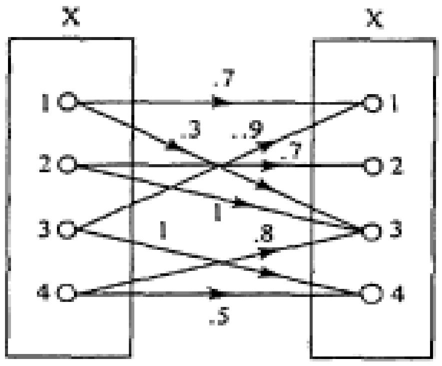
Binary relations $R(X, X)$ can be expressed by the same forms as general binary relations (matrices, sagittal diagrams, tables). In addition, however, they can be conveniently expressed in terms of simple diagrams with the following properties: (1) each element of the set X is represented by a single node in the diagram; (2) directed connections between nodes indicate pairs of elements of X for which the grade of membership in R is nonzero; and (3) each connection in the diagram is labeled by the actual membership grade of the corresponding pair in R.

A crisp relation $R(X, X)$ is *reflexive* iff $\langle x, x\rangle \in R$ for each $x \in X$, that is, if every element of X is related to itself. Otherwise, $R(X, X)$ is called *irreflexive*. If $\langle x, x\rangle \notin R$ for every $x \in X$, the relation is called *antireflexive*.

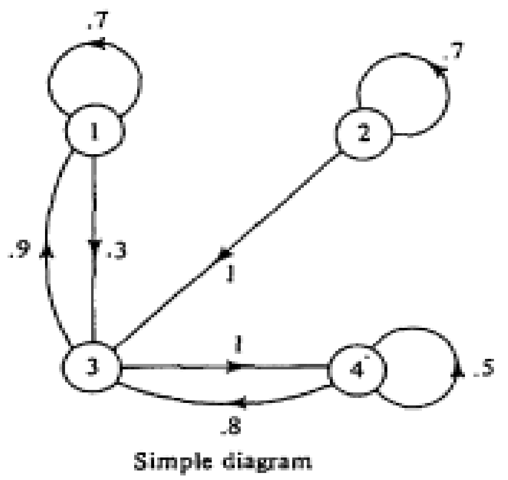
A crisp relation $R(X, X)$ is *symmetric* iff for every $\langle x, y\rangle \in R$, it is also the case that $\langle y, x\rangle \in R$, where $x, y, \in X$. Thus, whenever an element x is related to an element y through a symmetric relation, y is also related to x. If this is not the case for some x, y, then the relation is called *asymmetric*. If both $\langle x, y\rangle \in R$ and $\langle y, x\rangle \in R$ implies $x = y$, then the relation is called *antisymmetric*. If either $\langle x, y\rangle \in R$ or $\langle y, x\rangle \in R$, whenever $x \neq y$, then the relation is called *strictly antisymmetric*.

X	1	2	3	4
1	.7	0	.3	0
2	0	.7	1	0
3	.9	0	0	1
4	0	0	.8	.5

Membership matrix



Sagittal diagram



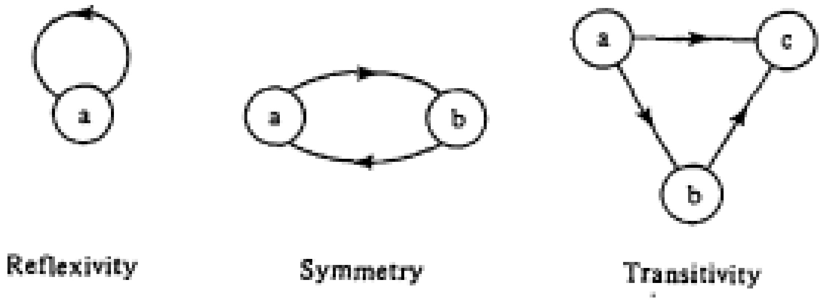
Simple diagram

x	y	R(x, y)
1	1	.7
1	3	.3
2	2	.7
2	3	1
3	1	.9
3	4	1
4	3	.8
4	4	.5

Table

A crisp relation $R(X, X)$ is called *transitive* iff $\langle x, z\rangle \in R$ whenever both $\langle x, y\rangle \in R$ and $\langle y, z\rangle \in R$ for at least one $y \in X$. In other words, the relation of x to y and of y to z implies the relation of x to z in a transitive relation. A relation that does not satisfy this property is called *nontransitive*. If $\langle x, z\rangle \notin R$ whenever both $\langle x, y\rangle \in R$ and $\langle y, z\rangle \in R$, then the relation is called *antitransitive*.

The properties of reflexivity, symmetry, and transitivity are illustrated for crisp relations $R(X, X)$ in Fig. 5.5. We can readily see that these properties are preserved under inversion of the relation.



FUZZY TOLERANCE AND EQUIVALENCE RELATIONS

A fuzzy relation, $\underset{\sim}{R}$, on a single universe X is also a relation from X to X. It is a fuzzy equivalence relation if all three of the following properties for matrix relations define it:

Reflexivity $\quad \mu_{\underset{\sim}{R}}(x_i, x_i) = 1 \qquad (3.21a)$

Symmetry $\quad \mu_{\underset{\sim}{R}}(x_i, x_j) = \mu_{\underset{\sim}{R}}(x_j, x_i) \qquad (3.21b)$

Transitivity $\quad \mu_{\underset{\sim}{R}}(x_i, x_j) = \lambda_1$ and $\mu_{\underset{\sim}{R}}(x_j, x_k) = \lambda_2 \longrightarrow \mu_{\underset{\sim}{R}}(x_i, x_k) = \lambda \qquad (3.21c)$

where $\lambda \geq \min[\lambda_1, \lambda_2]$.

Example 3.11. Suppose, in a biotechnology experiment, five potentially new strains of bacteria have been detected in the area around an anaerobic corrosion pit on a new aluminum–lithium alloy used in the fuel tanks of a new experimental aircraft. In order to propose methods to eliminate the biocorrosion caused by these bacteria, the five strains must first be categorized. One way to categorize them is to compare them to one another. In a pairwise comparison, the following ''similarity'' relation, $\underset{\sim}{R}_1$, is developed. For example, the first strain (column 1) has a strength of similarity to the second strain of 0.8, to the third strain a strength of 0 (i.e., no relation), to the fourth strain a strength of 0.1, and so on. Because the relation is for pairwise similarity it will be reflexive and symmetric. Hence,

$$\underset{\sim}{R}_1 = \begin{bmatrix} 1 & 0.8 & 0 & 0.1 & 0.2 \\ 0.8 & 1 & 0.4 & 0 & 0.9 \\ 0 & 0.4 & 1 & 0 & 0 \\ 0.1 & 0 & 0 & 1 & 0.5 \\ 0.2 & 0.9 & 0 & 0.5 & 1 \end{bmatrix}$$

is reflexive and symmetric. However, it is not transitive, e.g.,

$$\mu_{\underset{\sim}{R}}(x_1, x_2) = 0.8, \quad \mu_{\underset{\sim}{R}}(x_2, x_5) = 0.9 \geq 0.8$$

but

$$\mu_{\underset{\sim}{R}}(x_1, x_5) = 0.2 \leq \min(0.8, 0.9)$$

One composition results in the following relation:

$$\underset{\sim}{R}_1^2 = \underset{\sim}{R}_1 \circ \underset{\sim}{R}_1 = \begin{bmatrix} 1 & 0.8 & 0.4 & 0.2 & 0.8 \\ 0.8 & 1 & 0.4 & 0.5 & 0.9 \\ 0.4 & 0.4 & 1 & 0 & 0.4 \\ 0.2 & 0.5 & 0 & 1 & 0.5 \\ 0.8 & 0.9 & 0.4 & 0.5 & 1 \end{bmatrix}$$

where transitivity still does not result; for example,

$$\mu_{\underset{\sim}{R}^2}(x_1, x_2) = 0.8 \geq 0.5 \qquad \text{and} \qquad \mu_{\underset{\sim}{R}^2}(x_2, x_4) = 0.5$$

but

$$\mu_{\underset{\sim}{R}^2}(x_1, x_4) = 0.2 \leq \min(0.8, 0.5)$$