THE INTEGRALS

0.1 Motivation

We will introduce the integral using an example. We consider the exponential function $f : x \mapsto e^x$. We want to calculate the area A below the curve of f and between the lines of equation $x = 0,\ x = 1,$ and x-axis.

We approximate this area by sums of areas of the rectangles located under the curve. More precisely, let $n \geq 1$ an integer; let's divide our interval $[0, 1]$ using subdivision

$$(0, \frac{1}{n}, \frac{2}{n}, ..., \frac{i}{n}, ..., \frac{n-1}{n}, 1).$$

each interval has $[\frac{i-1}{n}, \frac{i}{n}]$ as a basis and $f(\frac{i-1}{n}) = e^{\frac{i-1}{n}}, \quad i = 1, ..., n$ as height, then the area of each rectangle equal

$$(\frac{i}{n} - \frac{i-1}{n}).e^{\frac{i-1}{n}} = \frac{1}{n}.e^{\frac{i-1}{n}}$$

The sum of the areas is then calculated as the sum of a geometric sequence: We consider the "lower rectangles" A_{inf}

$$\sum_{n=1}^{n} \frac{e^{\frac{i-1}{n}}}{n} = \frac{1}{n}\sum_{n=1}^{n}(e^{\frac{1}{n}})^{i-1} = \frac{1}{n}.\frac{1-(e^{\frac{1}{n}})^n}{1-e^{\frac{1}{n}}} = \frac{\frac{1}{n}}{e^{\frac{1}{n}}-1}(e-1) \longrightarrow (e-1) \text{ lorsque } n \to +\infty$$

Now the «upper rectangles » A_{sup}, having the same base $[\frac{i-1}{n}, \frac{i}{n}]$ but the height is $f(\frac{i}{n}) = e^{\frac{i}{n}}$. a similar calculate prove that $\sum_{n=1}^{n} \frac{e^{\frac{i}{n}}}{n} \longrightarrow (e-1) \text{ lorsque } n \to +\infty$

The area A of our region is greater than the sum of lower rectangles area and it is less of the sum of upper rectangles area. If we consider too smaller subdivisions (or partitions) (that is to way if n tends to $+\infty$) then we obtain in the limit that the area A of our region is farmed by two areas tends to $e - 1$. Then $A = e - 1$.

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0.2 Riemann integrals

Let f be a function defined on a closed interval $[a, b]$. we reply the same construction as previously. What will replace the rectangles will be *step functions*. If the limit of the areas below equals the limit of the areas above we call this common limit **the integral** of f that we notes

$$\int_a^b f(x)dx$$

Remark 0.2.1 *It is not always true that these limits (that's way, the limit of area bellow and the limit of area above) are equal, the integral is only defined for integrable functions. Fortunately we will see that if the function f is continue then it is integrable.*

0.2.1 The integral of a step function

Let f a function defined on closed and bounded interval of $\mathbb{R}$, $(-\infty < a < b < +\infty)$. We call a partition of $[a, b]$ a strictly increasing finite sequence of numbers $x_0, x_1, ..., x_n$ such that

$$a = x_0 < x_1 < ..., x_n = b$$

Definition 0.2.2 *A function $f : [a, b] \to \mathbb{R}$ is a step function if there is a partition $x_0, x_1, ..., x_n$ and real numbers $c_1, c_2, ..., c_n$ such that for all $i \in \{1, 2, ..., n\}$ we are*

$$\forall x \in]x_{i-1}, x_i[, \quad f(x) = c_i$$

In other words f is a constant function on each of the subintervals of the partition.

Definition 0.2.3 *For a step function, its integral is the real $\int_a^b f(x)$ defined by*

$$\int_a^b f(x)dx = \sum_{i=1}^{n} c_i(x_i - x_{i-1})$$

<u>Remark:</u>
Note that each term $c_i(x_i - x_{i-1})$ is the area of the rectangle between the x-coordinates x_{i-1} and x_i and height c_i. We just have to take care that we count the area with a $(+)$ sign if $c_i > 0$ and a $(-)$ sign if $c_i < 0$.
The integral of a step function is indeed a real number which measures the algebraic area (i.e. with sign) between the curve of f and the x-axis.

0.2.2 Integrable function

Recall that a function $f : [a, b] \to \mathbb{R}$ is bounded if there exists $M \geq 0$ such that:

$$\forall x \in [a, b] : -M \leq f(x) \leq M.$$

we recall also that if we have two functions $f, g : [a, b] \to \mathbb{R}$ then we note

$$f \leq g \Leftrightarrow \forall x \in [a, b] : \quad f(x) \leq g(x)$$

We now assume that $f : [a, b] \to \mathbb{R}$ is any bounded function. We define two real numbers:

$$I^-(f) = \sum_{i=1}^{n} \inf_{[x_{i-1}, x_i]} f(x)(x_i - x_{i-1})$$

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$$I^+(f) = \sum_{i=1}^{n} \sup_{[x_{i-1},x_i]} f(x)(x_i - x_{i-1})$$

For $I^-(f)$ we take all the step functions (with all possible partitions) which remain less than f. We take the largest area among all these step functions, as we is not sure that this maximum exists we take the upper limit. For $I^+(f)$ is the same principle but the step functions are greater than f and we seek the smallest possible area.

Definition 0.2.4 *The bounded function $f : [a, b] \to \mathbb{R}$ is said to be integrable (in Riemann's sense) if $I^-(f) = I^+(f)$. We then call this number the Riemann integral of f over $[a, b]$ and we denote it*

$$\int_a^b f(x)dx$$

Example 0.2.5 *Let $f : [a, b] \to \mathbb{R}, \quad f(x) = x^2$. Show that it is integrable and calculate $\int_0^1 f(x)dx$. Let $n \geq 1$ and consider the follow regulate partition of $[0, 1]$ $(0, \frac{1}{n}, \frac{2}{n}, ..., \frac{i}{n}, ..., \frac{n-1}{n}, 1)$. in the interval $[\frac{i-1}{n}, \frac{i}{n}]$ we have*

$$\forall x \in [\frac{i-1}{n}, \frac{i}{n}] : \quad (\frac{i-1}{n})^2 \leq x^2 \leq (\frac{i}{n})^2.$$

$$I^+(f) = \sum_{i=1}^{n} \frac{i^2}{n^2}(\frac{i}{n} - \frac{i-1}{n}) = \sum_{i=1}^{n} \frac{i^2}{n^2}\frac{1}{n} = \frac{1}{n^3}\sum_{i=1}^{n} i^2 = \frac{(n+1)(2n+1)}{6n^2}$$

$$I^-(f) = \sum_{i=1}^{n} \frac{(i-1)^2}{n^2}(\frac{i}{n} - \frac{i-1}{n}) = \sum_{i=1}^{n} \frac{(i-1)^2}{n^2}\frac{1}{n} = \frac{1}{n^3}\sum_{i=1}^{n-1}(i-1)^2 = \frac{(n-1)(2n-1)}{6n^2}$$

If n tends to $+\infty$ then $I^+(f), I^-(f)$ tends to $\frac{1}{3}$. We deduce that $I^+(f) = I^-(f) = \frac{1}{3}$. So f is integrable and $\int_0^1 x^2dx = \frac{1}{3}$

Theorem 0.2.6 *A sufficient condition that a function to be Rieman-integrable is that it be continuous, or piecewise continuous.*

0.2.3 Properties of the integrals

- **Linearity**: The integral is a linear form on the vector space of piecewise continuous functions on a segment $[a, b]$. i.e.

$$\forall(\lambda, \mu) \in \mathbb{R}^2, \forall f, g \text{ intégrables} : \int_a^b (\lambda f + \mu g)(x)dx = \lambda \int_a^b f(x)dx + \mu \int_a^b g(x)dx.$$

- $$\int_a^b f(x)dx = -\int_b^a f(x)dx.$$

- **Relationship of Charles**:

$$\forall c \in [a, b] : \int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

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- **Positivity**:
$$(\forall x \in [a,b], f(x) \geq 0) \Rightarrow (\int_a^b f(x)dx) \geq 0.$$
Moreover if
$$(f\text{continuous on}[a,b], f \neq 0; \forall x \in [a,b], f(x) \geq 0) \Rightarrow (\int_a^b f(x)dx) > 0.$$

- **Monotonicity**: Let f, g two functions on $[a,b]$ is integrable. If
$$\forall x \in [a,b], f(x) \leq g(x) \text{ alors } \int_a^b f(x)dx \leq \int_a^b g(x)dx$$

- **Average formula**: Let f is integrable on $[a,b]$, m, M the bonds of f on $[a,b]$, $(m \leq M)$; there exist a real α on $[m,M]$ such that
$$\int_a^b f(x)dx = \alpha(b-a).$$
It results that if f is continuous on $[a,b]$, there exists a real c of $[a,b]$ such that
$$\int_a^b f(x)dx = (b-a)f(c).$$

- Let f is integrable on $[a,b]$ then
$$|\int_a^b f(x)dx| \leq \int_a^b |f(x)|dx$$

- **Cauchy Schwarz inequality**: si f and$g$ are continuous on $[a,b]$ we have the inequality
$$(\int_a^b f(x)dx)^2 \leq \int_a^b f^2(x)dx \int_a^b g^2(x)dx$$

0.3 Primitives

Definition 0.3.1 *A function F is a primitive of a function f on an interval if F is differentiable on the interval and satisfies the equation, $F'(x) = f(x)$, or, what is the same, $dF(x) = f(x)dx$*

Remark 0.3.2
- *The difference between two primitives of f is a constant. (false results if I is not interval)*

- *If x_0 is an element of I there exists a unique primitive of f zero at x_0 :*
$$F(x) = \int_{x_0}^x f(t)dt$$
For example:
$$\ln x = \int_1^x \frac{1}{t}dt$$

- *The camput of an integral then comes down to the comput of the primitives. At the end of this chapter we will find a table of common primitives to know by heart.*

- *If f is a continuous function on interval, we note $\int f(x)dx$ is a primitive of f on the interval.*

To find an a primitive of a function f one may be lucky enough to recognize that f is the derivative of a well-known function. This is unfortunately very rarely the case, and we do not know the primitives of most functions. However, we will see two techniques which allow us to calculate integrals and primitives: integration by parts and by change of variable.

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0.3.1 Integration by parts

Theorem 0.3.3 *Lets u and v two functions of the class $\mathcal{C}^1$ on an interval $[a, b]$.*

$$\int_a^b u(x)v'(x)dx = [u(x)v(x)]_a^b - \int_a^b u'(x)v(x)dx$$

0.3.2 Change of variable

Theorem 0.3.4 *Let u a function defined on I and $v : J \to I$ is bijection of class $\mathcal{C}^1$. For all $a, b \in J$*

$$\int_a^b u(v(t))v'(t)dx = \int_{v(a)}^{v(b)} u(x)dx$$

Indeed if we note $x = v(t)$ then b derivation we obtain $dx = v'(t)dt$. Therefore the substitution $\int_a^b u(v(t))v'(t)dx = \int_{v(a)}^{v(b)} u(x)dx$

0.3.3 Primitives of Rational Functions

Let $\frac{P(x)}{Q(x)}$ a rational function, where $P(x), Q(x) \neq 0$ are polynomials with real coefficients. we need decompose this fraction into simple elements of the first kind in the form

$$x \mapsto \frac{A}{(x-a)^n}$$

and of the second kind in the form

$$x \mapsto \frac{Ax + B}{[(x-a)^2 + b^2]^n}$$

where $A, B, a, b \in \mathbb{R}$ and $n \in \mathbb{N}$.
However, before this decomposition the following two rules must be verified

1. If $\mathring{d}(P(x)) \geq \mathring{d}(Q(x))$ we must do euclidean division

$$\frac{P(x)}{Q(x)} = N(x) + \frac{R(x)}{Q(x)} / \quad \mathring{d}(R(x)) < \mathring{d}(Q(x))$$

2. Then we decompose the fraction $\frac{R(x)}{Q(x)}$ that is to say we must put $Q(x)$ as a product of factors

   - Factor of $1 - st$ degree no repeat.
   - Factor of $1 - st$ degree repeat n times.
   - Factor of $2 - nd$ degree no repeat.
   - Factor of $2 - nd$ degree repeat n times.

For example:
We decompose the following fraction into simple fractions: $\frac{x+1}{(x-1)(x-2)^2(x^2+1)}$

- $x - 1$ factor of $1 - st$ degree no repeat.

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- $(x-2)^2$ factor of $1-st$ degree repeat 2 times.
- x^2+1 factor of $2-nd$ degree no repeat.

So $\dfrac{x+1}{(x-1)(x-2)^2(x^2+1)} = \dfrac{A}{x-1} + \dfrac{B}{x-2} + \dfrac{C}{(x-2)^2} + \dfrac{Dx+E}{x^2+1}$ After calculation we find $A=1,\ B=\frac{-22}{25},\ C=\frac{3}{5},\ D=\dfrac{3}{25},\ E=\dfrac{1}{25}$.

A rational fraction is written in a unique way as the sum of a finite number of simple fractions and a polynomial. So, calculating the primitive of a rational fraction amounts to calculating the primitive of simple fractions.

1. 1-st species

$$\int \frac{1}{(x-a)^n}dx = \begin{cases} \ln|x-a| & \text{si } n=1 \\ \dfrac{-1}{(n-1)(x-a)^n} & \text{si } n>1 \end{cases}$$

2. 2-nd species

$$\int \frac{Ax+B}{((x-a)^2+b^2)^n}dx$$

Example 0.3.5 *Application example*

$$\int \frac{x^5-x^2+2x+1}{x^3-1}dx = \frac{x^3}{3} + \ln|x-1| - \frac{1}{2}\ln(x^2+x+1) + \sqrt{3}\arctan(\frac{2}{\sqrt{3}}(x+\frac{1}{2})) + c$$

Exercise: Give the primitive functions of

$$\int \frac{x^3+x^2+2}{(x^2+2)^2}dx; \quad \int \frac{1}{(x^2+2x+2)^2}dx$$

0.3.4 Primitives of the Form

$$\int f(\sin x, \cos x)dx.$$

Several methods exists for computing the integral $\int f(\sin x, \cos x)dx$. One of which is completely, although not always the most efficient. We make the change of variable $t = \tan\dfrac{x}{2}$. then we have:

$$\sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}, \quad \tan x = \frac{2t}{1-t^2}, \quad dx = \frac{2}{1+t^2}dt.$$

Therefore the integral becomes:

$$\int f(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2})\frac{2}{1+t^2}dt$$

Example 0.3.6

$$\int \frac{\sin x}{1-\sin x}dx = \int \frac{4t}{(1+t^2)(t-1)^2}dt = -x - \frac{2}{\tan\frac{x}{2}-1} + c.$$

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0.3.5 Primitives of the Form

$$\int f(e^x)dx, \textbf{ or } \int f(chx, shx)dx.$$

We make $t = e^x$

Example 0.3.7

$$\int \frac{e^x}{1+e^{3x}} = \int \frac{t}{1+t^3} = \frac{t}{(1+t)(1+t+t^2)}dt = -\frac{\sqrt{3}}{3}\arctan(\frac{2e^x-1}{\sqrt{3}}) + c.$$

Example 0.3.8

$$I = \int \frac{shx}{chx+shx}dx. \quad \textit{We have} \quad chx = \frac{e^x+e^{-x}}{2}, \; shx = \frac{e^x-e^{-x}}{2}$$

$$I = \frac{1}{2}x + \frac{1}{4e^{2x}} + c$$

0.4 Primitives of irrational functions

0.4.1 Primitives of the Form

$$I = \int f(x, \sqrt{ax^2+bx+c}).$$

In this case we distinguish three cases:

1. **1-st case** $\Delta > 0$. So let α, β the roots of $ax^2+bx+c=0$. we put $\sqrt{ax^2+bx+c} = (x-\alpha)t \Rightarrow x = \dfrac{a\beta-\alpha t^2}{a-t^2}$. Then

$$I = \int f(\frac{a\beta-\alpha t^2}{a-t^2}, t(\frac{a\beta-\alpha t^2}{a-t^2}-\alpha))dt$$

   which is an primitive of a rational function in terms of t.

   Example 0.4.1 *Computing the integral* $J = \int \dfrac{dx}{\sqrt{x^2+3x-4}}$

   $x^2+3x-4 = (x+4)(x-1) = (x+4)^2t^2$

$$J = \int \frac{\frac{10t}{(1-t^2)^2}}{\frac{5t}{1-t^2}} = \ln|\frac{1+t}{1-t}| + c$$

2. **2-nd case** $\Delta = 0$, $a \geq 0$ obvious. If $a < 0$ impossible to resolve.

3. **3-rd case** $\Delta < 0$, $a < 0$ impossible. But if $a > 0$ we make the change variable $\sqrt{ax^2+bx+c} = \sqrt{a}x + t \Rightarrow x = \dfrac{t^2-c}{b-2\sqrt{a}t}$

Example 0.4.2

$$I = \int \frac{\sqrt{x^2+x+1}}{x} dx.$$

We have $\Delta = -3 < 0,\ a = 1 > 0,$ *we put* $\sqrt{x^2+x+1} = x + t \Rightarrow x = \dfrac{1-t^2}{2t-1}$ *et* $dx = \dfrac{-t^2+2t-2}{(2t-1)^2}dt$ *then the integral* I *becomes*

$$I = -2\int \frac{(t^2+t-1)^2}{(2t-1)^2(t^2-1)}dt$$

The student must complete the solution

Exercise: Find the primitive function of J et K such that

$$J = \int \sqrt{x^2+1}; \quad K = \int \sqrt{x^2-4}.$$

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