# Finite Expansion

## (Solved Exercises)

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**Exercise 03**

Construct the Finite Expansion at 0 of the following functions

1. $(1+\arctan x)(e^x+2\sin x)$ (ordre 3)
2. $(1+2\cos(2x))(x-\ln(1+x))$ (ordre 5)
3. $\dfrac{1+\arctan x}{\cos x}$ (ordre 4)
4. $\dfrac{\ln(1+x^3)}{x-\sin x}$ (ordre 3)
5. $(1+x)^{\frac{1}{x}}$ (ordre 2)
6. $\ln\dfrac{\sin x}{x}$ (ordre 4)

**Exercice 04**

1. Find the third order Maclaurin series for the function

$$f(x) = (1+x)^{\frac{1}{\sin x}}$$

2. Deduce $\lim_{x \to 0} f(x)$

**Exercise 05**

1. Construct the second order Taylor polynomial at 0 for the function

$$f(x) = \frac{e^{e^x} - e^{e^{-x}}}{\ln(1+x)}$$

2. Deduce $\lim_{x \to 0} f(x)$

**Exercise 06**

The same questions of axercise 4 for the function

$$f(x) = \frac{e^{\left(\frac{1}{\cos x} + \frac{x^2}{\sin x}\right)} - e}{\ln(1+x)}$$

**Exercise 07**

Using Taylor expansion, evaluate the limits

$$\lim_{x\to 0}\frac{1-\cos x}{x\ln(1+x)},\quad \lim_{x\to 0}\frac{x-\arcsin x}{\sin^3 x},\quad \lim_{x\to 0}\left(\frac{\sin x}{\sinh x}\right)^{\frac{1}{x^2}}.$$

# Practice sheets n 01

Using the Taylor expansion, study the position of the graphe of the function in relation of its tangent at $x_0 = 0$ in the following cases

1. $f(x) = \cos(2x) - 2\sin x$
2. $g(x) = \dfrac{x}{1+x^2} - xe^{-x}$
3. $h(x) = \ln\left(\dfrac{1+x}{1-x}\right)$

Two electrical charges of equal magnitude and opposite signs located near one another are called
an electrical dipole. The charges $Q$ and $-Q$ are a distance $d$ apart. The electric field, $E$, at the point $P$ is given by

$$E = \frac{Q}{R^2} - \frac{Q}{(R+d)^2}$$

Use series to investigate the behavior of the electric field far away from the dipole. Show that when $R$ is large in comparison to $d$, the electric field is approximately proportional to $\frac{1}{R^3}$

1)

$(1+\arctan x)(e^x+2\sin x)$ en 0 à l'ordre 3

$$(1+\arctan x) = 1 + x - \frac{x^3}{3} + \circ\left(x^3\right)$$

$$\begin{aligned} e^x + 2\sin x = & \left(1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \circ\left(x^3\right)\right) + 2\left(x - \frac{x^3}{3!} + \circ\left(x^3\right)\right) \\ & = 1 + 3x + \frac{x^2}{2} - \frac{x^3}{6} + \circ\left(x^3\right) \end{aligned}$$

$$\begin{aligned} (1+\arctan x)(e^x+2\sin x) & = \left(1 + x - \frac{x^3}{3}\right)\left(1 + 3x + \frac{x^2}{2} - \frac{x^3}{6}\right) + \circ\left(x\right. \\ & = 1 + 4x + \frac{7}{2}x^2 + \circ\left(x^3\right) \end{aligned}$$

# Exercise 3

2)

$$(1+2\cos(2x))(x-\ln(1+x) \text{ order } 5$$

$$(x-\ln(1+x)) = x - \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}\right) + \circ\left(x^5\right) =$$

$$\frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} - \frac{x^5}{5} + \circ\left(x^5\right)$$

$$\cos(2x) = 1 - \frac{(2x)^2}{2} + \frac{(2x)^4}{4!} + \circ\left(x^5\right) = 1 - 2x^2 + \frac{16x^4}{24} + \circ\left(x^5\right)$$

$$1+2\cos(2x) = 1 + 2\left(1 - 2x^2 + \frac{16x^4}{24} + \circ\left(x^5\right)\right) =$$

$$3 - 4x^2 + \frac{16x^4}{12} + \circ\left(x^5\right)$$

$$(1+2\cos(2x))(x-\ln(1+x)) = \left(3 - 4x^2 + \frac{16x^4}{12} + \circ\left(x^5\right)\right)\left(\frac{x^2}{2} - \frac{x^3}{3}\right.$$

$$= \frac{3}{2}x^2 - x^3 - \frac{5}{4}x^4 + \frac{11}{15}x^5 + \circ\left(x^5\right)$$

# Exercise 3

3)

$$\frac{1+\arctan x}{\cos x} \quad (\text{order } 4)$$

$$1+\arctan x = 1 + x - \frac{x^3}{3} + \circ\left(x^4\right)$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + \circ\left(x^4\right)$$

$$\frac{1+\arctan x}{\cos x} = \frac{1+x-\frac{x^3}{3}+\circ\left(x^4\right)}{1-\frac{x^2}{2}+\frac{x^4}{4!}+\circ\left(x^4\right)}; \text{ Using } \left(\frac{1}{1-y}\right) = 1 + y + y^2 + \ldots$$

$$= \left(1 + x - \frac{x^3}{3} + \circ\left(x^4\right)\right) \frac{1}{1-\left(\frac{x^2}{2}-\frac{x^4}{4!}+\circ\left(x^4\right)\right)}$$

$$= \left(1 + x - \tfrac{x^3}{3} + \circ\left(x^4\right)\right)\left(1 + \left(\tfrac{x^2}{2} - \tfrac{x^4}{4!}\right) + \left(\tfrac{x^2}{2} - \tfrac{x^4}{4!}\right)^2 + \right.$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{5}{24}x^4 + \circ\left(x^4\right)$$

# Exercise 3

4)

$$\frac{\ln(1+x^3)}{x-\sin x} \quad \text{(order 3)}$$

$$x-\sin x = x-\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}\right)+\circ\left(x^6\right)=\frac{x^3}{6}-\frac{x^5}{120}+\circ\left(x^6\right)$$

The first non-nul term in the denominator is of the third degree . So let's apply the Taylor expansion of order 6

$$\ln(1+x^3)=x^3-\frac{\left(x^3\right)^2}{2}+\circ\left(x^6\right)=x^3-\frac{x^6}{2}+\circ\left(x^6\right)$$

$$\frac{\ln(1+x^3)}{x-\sin x} = \frac{x^3-\frac{x^6}{2}+\circ\left(x^6\right)}{\frac{x^3}{6}-\frac{x^5}{120}+\circ(x^6)}$$

$$=\frac{1-\frac{x^3}{2}+\circ\left(x^3\right)}{\frac{1}{6}-\frac{x^2}{120}+\circ(x^3)}=6.\frac{1-\frac{x^3}{2}+\circ\left(x^3\right)}{1-\frac{x^2}{20}+\circ(x^3)}$$

$$=6.\left(1-\frac{x^3}{2}+\circ\left(x^6\right)\right)\frac{1}{1-\frac{x^2}{20}+\circ(x^3)}$$

$$=6.\left(1-\frac{x^3}{2}+\circ\left(x^6\right)\right)\left(1+\frac{x^2}{20}+\circ\left(x^3\right)\right)$$

$$6+\frac{3}{10}x^2-3x^3+\circ\left(x^3\right)$$

# Exercise 3

5)

$$(1+x)^{\frac{1}{x}} \quad \text{(ordre 2)}$$

$(1+x)^{\frac{1}{x}} = e^{\ln(1+x)^{\frac{1}{x}}} = e^{\frac{\ln(1+x)}{x}}.$

$\frac{\ln(1+x)}{x} = \frac{x-\frac{x^2}{2}+\frac{x^3}{3}+\circ\left(x^3\right)}{x} = 1 - \frac{x}{2} + \frac{x^2}{3} + \circ\left(x^2\right)$

$e^{\frac{\ln(1+x)}{x}} = e^{\left(1-\frac{x}{2}+\frac{x^2}{3}+\circ\left(x^2\right)\right)} = e.e^{\left(-\frac{x}{2}+\frac{x^2}{3}+\circ\left(x^2\right)\right)}$

We have $e^y = 1 + y + \frac{y^2}{2} + \circ(y^2)$, with $y(0) = 0$, then

$$\begin{aligned} (1+x)^{\frac{1}{x}} = e^{\frac{\ln(1+x)}{x}} &= e.e^{\left(-\frac{x}{2}+\frac{x^2}{3}+\circ\left(x^2\right)\right)} \\ &= e\left(1+\left(-\frac{x}{2}+\frac{x^2}{3}+\circ\left(x^2\right)\right)+\frac{1}{2}\left(-\frac{x}{2}+\frac{x^2}{3}+\circ\left(x^2\right)\right)^2+\circ(x \\ &= e(1-\frac{x}{2}+\frac{11}{24}x^2)+\circ(x^2). \end{aligned}$$

# Exercise 3

6 $\quad \ln \dfrac{\sin x}{x} \quad$ (ordre 4)

$$\frac{\sin x}{x} = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} + \circ(x^5)}{x} = 1 - \frac{x^2}{6} + \frac{x^4}{120} + \circ(x^4)$$

me have $\ln(1+y) = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \circ\left(y^4\right)$, with $y(0) = 0$

by sustitution $y$ by $\left(-\frac{x^2}{6} + \frac{x^4}{120} + \circ(x^4)\right)$ we obtain

$$\begin{aligned} \ln \frac{\sin x}{x} = \ & \ln\left(1 + \left(-\frac{x^2}{6} + \frac{x^4}{120} + \circ(x^4)\right)\right) \\ & = \left(-\frac{x^2}{6} + \frac{x^4}{120}\right) - \frac{1}{2}\left(-\frac{x^2}{6} + \frac{x^4}{120}\right)^2 + \circ(x^4) \\ & = -\frac{x^2}{6} - \frac{x^4}{180} + \circ(x^4). \end{aligned}$$

# Exercice 4

$$f(x) = (1+x)^{\dfrac{1}{\sin x}}$$

$$f(x) = (1+x)^{\dfrac{1}{\sin x}} = e^{\dfrac{\ln(1+x)}{\sin x}}$$

$$\frac{\ln(1+x)}{\sin x} = \frac{x - \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \circ\left(x^4\right)}{x - \frac{x^3}{3} + \circ(x^4)} =$$

$$= \left(1 - \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \circ\left(x^3\right)\right)\left(1 + \frac{x^2}{3} + \circ\left(x^3\right)\right)$$

$$= 1 - \frac{x}{2} + \frac{2x^2}{3} + \frac{x^3}{6} + \circ\left(x^3\right)$$

# Exercise 4

$$\frac{\ln(1+x)}{\sin x} = \quad = 1 - \frac{x}{2} + \frac{2x^2}{3} + \frac{x^3}{6} + \circ\left(x^3\right) \quad \textit{then}$$

$$f(x) = (1+x)^{\dfrac{1}{\sin x}} = e^{\left(1-\frac{x}{2}+\frac{2x^2}{3}+\frac{x^3}{6}+\circ\left(x^3\right)\right)}$$

$$= ee^{\left(-\frac{x}{2}+\frac{2x^2}{3}+\frac{x^3}{6}+\circ\left(x^3\right)\right)}$$

$$= e\left[1 + \left(-\frac{x}{2} + \frac{2x^2}{3} + \frac{x^3}{6} + \circ\left(x^3\right)\right) + \frac{1}{2}\left(-\frac{x}{2} + \right.\right.$$

$$= e(1 - \tfrac{x}{2} + \tfrac{19}{24}x^2 + \tfrac{17}{48}x^3 + \circ\left(x^3\right)$$

$$f(x) = \frac{e^{e^x} - e^{e^{-x}}}{\ln(1+x)}.$$

$$e^{e^x} = e^{\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\circ\left(x^3\right)\right)} = e.e^{\left(x+\frac{x^2}{2}+\frac{x^3}{6}+\circ\left(x^3\right)\right)}$$

$$= e.\left[1+\left(x+\frac{x^2}{2}+\frac{x^3}{6}\right)+\frac{1}{2}\left(x+\frac{x^2}{2}\right.\right.$$

$$= e\left(1+x+x^2+\frac{5}{6}x^3\right)+\circ\left(x^3\right)$$

# Exercise 5

$$
\begin{aligned}
e^{e^x} &= e\left(1+x+x^2+\tfrac{5}{6}x^3\right)+\circ\left(x^3\right) \quad \textit{then}\\
e^{e^{-x}} &= e\left(1-x+x^2-\tfrac{5}{6}x^3\right)+\circ\left(x^3\right)
\end{aligned}
$$

$$
f(x)=\frac{e^{e^x}-e^{e^{-x}}}{\ln(1+x)}=\frac{e\left(2x+\frac{5}{3}x^3\right)+\circ\left(x^3\right)}{x-\frac{x^2}{2}+\frac{x^3}{3}+\circ(x^3)}=e\frac{\left(2+\frac{5}{3}x^2\right)+\circ\left(x^2\right)}{1-\frac{x}{2}+\frac{x^2}{3}+\circ(x^2)}
$$

$$
=e\left(2+x-\frac{x^2}{2}\right)+\circ\left(x^2\right)
$$

$$
\lim_{x\longrightarrow 0} f(x)=2e
$$

# Exercise 6 (homework)

$$f(x) = \frac{e^{\left(\frac{1}{\cos x} + \frac{x^2}{\sin x}\right)} - e}{\ln(1+x)}$$

# Exercise 07

Evaluate the limits

1. $$\lim_{x\to 0}\frac{1-\cos x}{x\ln(x+1)}=\lim_{x\to 0}\frac{1-(1-\frac{x^2}{2}+\circ(x^2))}{x^2+\circ(x^2)}=\frac{1}{2}$$

2. $$\lim_{x\to 0}\frac{x-\arcsin x}{\sin^3 x}=\lim_{x\to 0}\frac{x-\left(x+\frac{1}{2}.\frac{x^3}{3}\right)+\circ\left(x^3\right)}{x^3+\circ\left(x^3\right)}=\frac{-1}{6},$$

3. $$\lim_{x\to 0}\left(\frac{\sin x}{\sinh x}\right)^{\frac{1}{x^2}}=\lim_{x\to 0}e^{\ln\left(\frac{\sin x}{\sinh x}\right)^{\frac{1}{x^2}}}=\lim_{x\to 0}e^{\frac{1}{x^2}\ln\left(\frac{\sin x}{\sinh x}\right)}$$

$$\left(\frac{\sin x}{\sinh x}\right)=\frac{x-\frac{x^3}{6}+\circ\left(x^3\right)}{x+\frac{x^3}{6}+\circ\left(x^3\right)}=\frac{1-\frac{x^2}{6}+\circ\left(x^2\right)}{1+\frac{x^2}{6}+\circ\left(x^2\right)}=\left(1-\frac{x^2}{6}+\circ\left(x^2\right)\right).\frac{1}{1+\frac{x^2}{6}+\cdots}$$

$$=\left(1-\frac{x^2}{6}+\circ\left(x^2\right)\right)\left(\frac{1}{1-\left(-\frac{x^2}{6}\right)+\circ\left(x^2\right)}\right)$$

$$=\left(1-\frac{x^2}{6}+\circ\left(x^2\right)\right)\left(1+\left(-\frac{x^2}{6}\right)+\circ\left(x^2\right)\right)$$

# Exercise 07

Then

$\ln\left(\frac{\sin x}{\sinh x}\right) = \ln\left(1 - \frac{x^2}{3} + \circ\left(x^2\right)\right) = -\frac{x^2}{3} + \circ\left(x^2\right)$

$\frac{1}{x^2}\ln\left(\frac{\sin x}{\sinh x}\right) = \frac{1}{x^2}\left(-\frac{x^2}{3} + \circ\left(x^2\right)\right) = \dfrac{-1}{3}$

Hence

$$\lim_{x\longrightarrow 0}\left(\frac{\sin x}{\sinh x}\right)^{\dfrac{1}{x^2}} = \lim_{x\longrightarrow 0} e^{\frac{1}{x^2}\ln\left(\frac{\sin x}{\sinh x}\right)} = e^{\frac{-1}{3}}$$

# Exercise 08

Using the Taylor expansion, study the position of the graphe of the function in relation of its tangent at $x_0 = 0$ in the following cases

1. $f(x) = \cos(2x) - 2\sin x = 1 - \dfrac{4x^2}{2} - 2\left(x - \frac{x^3}{6}\right) + \circ(x^3) = 1 - 2x - 2x^2 + \frac{1}{3}x^3 + \circ(x^3)$

$y = 1 - 2x$ is the tangent equation of the graphe at 0,
$f(x) - y = -2x^2 + \circ(x^2) < 0$, then the graphe is under the tangent

.

2 $g(x) = \dfrac{x}{1+x^2} - xe^{-x} = x\left(1-x^2\right) - x\left(1 - x + \frac{x^2}{2}\right) + \circ(x^2) = x^2 + + \circ\left(x^2\right)$

$y = 0$ is the tangent equation of the graphe at 0,
$f(x) - y = x^2 + \circ(x^2) > 0$, then the graphe is above the tangent

1. $h(x) = \ln\left(\dfrac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x) = \left(x - \frac{x^2}{2} + \frac{x^3}{3}\right) - \left(-x - \frac{x^2}{2} - \frac{x^3}{3}\right) + \circ(x^3)$

$= 2x + \frac{2}{3}x^3 + \circ(x^3)$
$y = 2x$ is the tangent equation of the graphe at 0,
$f(x) - y = \frac{2}{3}x^3 + \circ(x^3)$, then the graphe is above the tangent for $x > 0$ and under the tangent for $x < 0$
So 0 is an inflection point

# Thank you

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Thank you for your attention