DIFFERENTIAL EQUATIONS

0.1 First order differential equations

Definition 0.1.1 *An equation of the form*

$$y' = F(x, y, y') = 0;$$

where y is a function of x is the unknown, its called ***first order differential equation.***

0.1.1 Some types of 1st order differential equations

0.1.1.1 Separable differential equations:

They are of the form

$$f(y)y' = g(x).$$

The general solution is given by

$$\int f(y)dy = \int g(x)dx, \quad \text{car } y' = \frac{dy}{dx}.$$

Example 0.1.2 *We consider the differential equation*

$$y' - 4y = 0 \Leftrightarrow \int \frac{dy}{y} = \int 4dx \Leftrightarrow y = C.e^{4x}, \quad C \in \mathbb{R}$$

Exercise 1: Integrate the following two differential equations:

$$(y^2 + 1)y' = x + 1. \qquad (x - 1)y' + \sqrt{1 - y^2} = 0.$$

0.1.1.2 Homogeneous differential equations:

They are of the form

$$y' = F(\frac{y}{x}).$$

The resolution (or integration) of this type tends to the resolution of separable equation $\frac{y}{x} = t \Leftrightarrow y = tx$, then $y' = xt' + t$ ou bien $\frac{dy}{dx} = x\frac{dt}{dx} + t$.
The general solution is given by

$$x = C.e^{\int \frac{dt}{F(t) - t}}$$

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Example 0.1.3

$$xy' - 3y = x \Leftrightarrow y' = 3(\frac{y}{x}) + 1, \quad donc\, x = C.e^{\int \frac{dt}{2t+1}} = C\sqrt{2(\frac{y}{x}) + 1}$$

Exercise 2: Integrate the following homogeneous equation:

$$x^2 y dx - y^3 dy = x^3 dy$$

0.1.1.3 Linear differential equations:

They are in the form

$$y' + f(x)y + = g(x) ....(1)$$

where f, g are continuous functions on some interval I in $\mathbb{R}$, $g(x)$ is called the right hand side of the equation.

1. **Solve the equation without the second member:**

   $$y' + f(x)y = 0,$$

   it is a separable equation.

   $$y_H = Ce^{-\int f(x)dx}, \quad C \in \mathbb{R}$$

2. **Solve with second member:** The general solution of the complete equation (1)

   $$y_G = y_H + y_P,$$

   où y_G : general solution of (1), y_H : general solution without second member, y_P : particular solution of (1)
   Question: How to find y_P?

   - By observation with the naked eye. Else
   - By the variation of constant method

   We put $y_P(x) = C(x).y_1(x)$, where $y_1(x) = e^{-\int f(x)dx}$. So

   $$y_p'(x) = C'(x)e^{-\int f(x)dx} - C(x)f(x)e^{-\int f(x)dx}.$$

   Then we substitute in (1), we find

   $$C'(x) = g(x)e^{\int f(x)dx} \Rightarrow C(x) = \int g(x)e^{\int f(x)dx}, \quad \text{alors } y_P(x) = \int g(x)e^{\int f(x)dx}dx.e^{-\int f(x)dx}$$

Example 0.1.4 *We solve the equation*

$$xy' - 2y = x.$$

- *By observation, ($-x$ is particular solution) Resolution without second member:*

  $$xy' - 2y = 0 \Rightarrow y = C.x^2. \quad donc\ y_G = C.x^2 - x.$$

- *By the method of variation of constant:*

  $$y' = C'.x^2 + 2x.C \quad alors\ C' = \frac{1}{x^2} \Rightarrow C = -\frac{1}{x} + K.$$

  then

  The general solution of complete equation $y = Kx^2 - x.$

Exercise: Solve the linear equation

$$(1 - x^2)y' - xy = x.$$

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0.1.1.4 Differential equations of Bernoulli:

they are of the form

$$y' + f(x)y = y^\alpha g(x)$$

- $y = 0$, is particular solution.
- $\alpha = 0$, complete differential equation.
- $\alpha = 1$, linear equation without second member.
- $\alpha \neq 0,\ \alpha \neq 1,\ y \neq 0$, Bernoulli equation.

Solving this equation is done as follows: we multiply by $y^{-\alpha}$, we have

$$y^{-\alpha}y' + y^{1-\alpha}f(x) + = g(x).$$

we make $z = y^{1-\alpha} \Rightarrow z' = (1-\alpha)y^{-\alpha}y'$. So

$$\frac{z'}{1-\alpha} + f(x)z = g(x).$$

We are in the case of a linear equation.

Example 0.1.5

$$xy' - y = y^2 \ln x.$$

E.D de Bernoulli , $\alpha = 2$. On multiplier par y^{-2}

$$xy^{-2}y' - y^{-1} = \ln x. \quad \textit{on pose } z = y^{-1} \Rightarrow z' = -y^2y',$$

on trouve

$$z' + \frac{1}{x}z = \frac{\ln x}{x}. \quad \textit{E.D linéaire complète}$$

0.2 Second Order Differential Equations

Definition 0.2.1 *A second order differential equation is an relation in the form*

$$F(x, y, y', y'') = 0,$$

between the variable x, the function $y(x)$ and these two first derivatives.

Example 0.2.2

- $y'' + \omega^2 y = 0$, *admits for solutions on $\mathbb{R}$ the functions*

$$\varphi_1(x) = \sin\omega x,\ \varphi_2(x) = \cos\omega x$$

- $y'' = 0$, *admits for solutions all polynomial of the form $ax+b$, with a et b two arbitrary constants.*

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0.2.1 Equations not containing (or without) y

Let

$$F(x, y', y'') = 0.$$

We make $y' = z$, then the equation become

$$F(x, z, z') = 0.$$

Example 0.2.3

$$y'' + y'^2 = 0, \quad \textit{on pose } y' = z, \ \textit{alors } z' + z^2 = 0 \Rightarrow -\frac{dz}{z^2} = dx \Rightarrow z = \frac{1}{x+c} \Rightarrow dy = \frac{dx}{x+C}$$

$$y = \ln(x + C) + K, \quad C, \ K \textit{ étant des constantes}$$

Exercise: Resolve the equation $xy'' + 2y' = 0$.

0.2.2 Second order linear differential equations

Definition 0.2.4 *A second order linear differential equation is an equation of the form*

$$a(x)y'' + b(x)y' + c(x)y = f(x),$$

où $a(x), \ b(x), \ c(x)$ *et* $f(x)$ *are functions.*

We associate with this equation the equation without a second member called **homogeneous equation**, i.e.

$$a(x)y'' + b(x)y' + c(x)y = 0,$$

with right hand side called **complete equation.**

Theorem 0.2.5 *The general solution is obtained by adding to a particular solution of the complete equation the general solution of the homogeneous equation (i.e. without a second member)*

$$y_G = y_P + y_H.$$

0.2.2.1 Solving the equation without a second member

If we know $y_1, \ y_2$ a particular solutions, then the general solution is written as the form $y = \lambda_1 y_1 + \lambda_2 y_2$ and the two solutions must be linearly independent, which resulting in

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0. \quad \text{Is called the Wronskian}$$

0.2.3 Linear differential equations with constant coefficients

They are of the form

$$ay'' + by' + cy = f(x),$$

where $a, \ b, \ c,$ are real constants and $f(x)$ is a function. To solve this type of equations

1. We search the general solution of the homogeneous equation, i.e $f(x) = 0$. in the form $y = e^{rx}$ and we replace in $ay'' + by' + cy = 0$, we obtain

$$e^{rx}(ar^2 + br + c) = 0 \Rightarrow ar^2 + br + c = 0, \quad \text{called the } \textbf{characteristic equation}$$

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- If $\Delta > 0$, then $r_1,\ r_2$ two distinct real roots. The general solution of the homogeneous equation is in the form

$$y = C_1 e^{r_1 x} + C_2 e^{r_2 x}, \quad C_1,\ C_2 \text{are real constants.}$$

- If $\Delta = 0$, the characteristic equation admits a double root $r_1 = r_2 = r$. The general solution is of the form

$$y = (C_1 x + C_2)e^{rx}.$$

- If $\Delta < 0$, so $r_1,\ r_2$ two complex roots $r_1 = \alpha + i\beta, \quad r_2 = \alpha - i\beta$. the general solution is of the form

$$y = e^{\alpha x}(A\cos(\beta x) + B\sin(\beta x))$$

2. we search the general solution of complete equation i.e. $f(x) \neq 0$.

   - Case where $f(x)$. is a polynomial, then we search a particular solution as the polynomial form
   - Case where $f(x) = e^{\alpha x}P_n(x)$, we search a particular solution, we make $y = e^{\alpha x}P(x)$, (P any polynomial)
   - Case where $f(x) = A\cos(\alpha x) + B\sin(\alpha x), \quad \alpha,\ A,\ B \in \mathbb{R}$.
     • If $i\alpha$ is not solution of characteristic equation, we search y_P in the form $y_P = A'\cos(\alpha x) + B'\sin(\alpha x)$
     • If $i\alpha$ is a root of a characteristic equation (necessarily simple), we search y_P in the form $y_P = x(A'\cos(\alpha x) + B'\sin(\alpha x))$
   - **In the general case** , To solve the complete equation, we apply the variation of constants method. As follow: Let $y = C_1 y_1 + C_2 y_2$ general solution of the equation without a second member. Search for a particular solution by the variation of constants method. We make $y = C_1(x)y_1(x) + C_2(x)y_2(x)$ with $C_1'(x)y_1(x) + C_2'(x)y_2(x) = 0$. After calculation, we will have two equations with two unknowns

$$\begin{cases} C_1'(x)y_1'(x) + C_2'(x)y_2'(x) = f(x) \\ C_1'(x)y_1(x) + C_2'(x)y_2(x) = 0 \end{cases}$$

     This system has one and only one solution because $y_1,\ y_2$ are linearly independent. therefore $C_1(x),\ C_2(x)$ exists.

Example 0.2.6 *Solve the second order differential equation $y'' + y = x\sin x$.*

1. Solve the equation without second member
   Characteristic : $r^2 + 1 = 0 \Rightarrow r = \pm i$, so

$$y_H = C_1\cos x + C_2\sin x.$$

2. with second member:
   we use the variation of constants method.

$$\begin{cases} C_1'(x)y_1(x) + C_2'(x)y_2(x) = 0 \\ C_1'(x)y_1'(x) + C_2'(x)y_2'(x) = x\sin x \end{cases}$$

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$$\begin{cases} C_1' \cos x + C_2' \sin x = 0 \\ -C_1' \sin x + C_2' \cos x = x \sin x \end{cases}$$

$$\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1.$$

Then $C_1' = -x \sin^2 x, \quad C_2' = x \cos x \sin x$
After integration we have:
$C_1 = -\frac{x^2}{4} + \frac{x}{4} \sin 2x + \frac{1}{8} \cos 2x, \quad C_2 = \frac{x}{2} \cos 2x - \frac{1}{4} x + \frac{1}{8} \sin 2x$

We replace $C_1,\ C_2$ in y_H we obtain the general solution y_G.

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Series N°3 (*) for student

Exercise 01: Give the type of each **first-order** differential equations (the Solution is not required)

1. $xy' = (x-1)y, \qquad (1+y^2)dy = xdx, \qquad 2xyy' - y^2 + x = 0.$
2. $y' = \dfrac{x-y}{x+y}, \qquad y' - \dfrac{y}{1-x^2} = 1 + x.$

Exercise 02: Solve the following *separable* differential equations:

- $(1+e^x)yy' = e^x,$ ○ $\tan(x)\sin^2(y)dx + \cos^2(x)\cot(y)dy.$
- $\dfrac{e^y}{e^y+1}y' = \dfrac{1}{x},$ ○ $3e^x\tan(y)dx + \dfrac{1-e^x}{\cos^2 y}dy.$ (*)
- $y'\tan(x) = y$ (*), ○ $(x^2+1)y' = y^2 + 4.$ (*)

Exercise 03: Solve the following *homogeneous* differential equations:

- $y' = \dfrac{y}{x} - 1,$ ○ $y' = -\dfrac{x+y}{x}$ (*) ○ $(x-y)ydx - x^2dy = 0.$
- $(x^2+y^2)dx - 2xydy = 0,$ (*) ○ $y' = \dfrac{x-y}{x+y}$

Exercise 04: Find the general solution of following *linear* differential equations:

- $y' - \dfrac{y}{x} = x,$ (*) ○ $(1-x^2)y' + xy = 2x.$ (*)
- $y' - y\cos(x) = \sin(2x).$

Exercise 05: Find the particulars solutions satisfying the given initial conditions:

1. $xy' + y - e^x = 0; \quad y(a) = b;$ (*)
2. $y' - \dfrac{y}{1-x^2} - 1 - x = 0, \quad y(0) = 0.$
3. $y' - y\tan(x) = \dfrac{1}{\cos(x)}, \quad y(0) = 0.$ (*)

Exercise 06: Find the general solution of *Bernoulli* differential equations:

- $y' + \dfrac{y}{x} = -xy^2,$ ○ $y'\sin x\cos x - 3y = -3y^{\frac{2}{3}}\sin^3 x,$ (*) ○ $y' - y = 2\sqrt{y}e^{-x}.$

Exercise 07: Solve the following **second-order** differential equations:

1. $y'' + 3y' + 2y = 2x^2 + 8x + 7.$
2. $y'' - y = e^{2x} - e^x$

3. $y'' - 6y' + 9y = 2e^{3x} - e^{x}$.

4. $y'' + y = x \sin x$. $(*)$

5. $y'' + 2y' + y = \cos^2 x$. $(*)$

6. $y'' - 2y' + 2y = \sin(x).e^{x}$ $(*)$

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0.3 Short solution

Exercise 01: The type of first order differential equations:

1. $xy' = (x-1)y \Leftrightarrow \frac{dy}{dx} = \frac{x-1}{x}dx$ Separable equation
2. $(1+y^2)dy = xdx$, obvious
3. $2xyy' - y^2 + x = 0 \Leftrightarrow y - \frac{1}{2x}y = -\frac{1}{2}y^{-1}$
4. $y' = \frac{x-y}{x+y} = \frac{1-\frac{y}{x}}{1+\frac{y}{x}}$ homogeneous equation
5. $y' - \frac{y}{1-x^2} = 1+x \Leftrightarrow y' - \frac{1}{1-x^2}y = 1+x$ linear equation

Exercise 02:

$$(2): \quad \tan(x)\sin^2(y)dx + \cos^2(x)\cot(y)dy = 0.$$

We divided by $\sin^2(y)\cos^2(x)$ such that: $y \neq (2k+1)\frac{\pi}{2}$ and $x \neq k\pi$ So

$$(2) \Leftrightarrow \int \frac{\sin x}{\cos^3 x}dx + \int \frac{\cos y}{\sin^3 y}dy = 0,$$

after integration we obtain

$$(2) \Leftrightarrow \frac{1}{\cos^2 x} + \frac{1}{\sin^2 y} = C. \quad C \in \mathbb{R}.$$

$$(4): \quad 3e^x\tan(y)dx + \frac{1-e^x}{\cos^2 y}dy$$

$$(4) \Leftrightarrow \int \frac{3e^x}{1-e^x}dx = -\int \frac{dy}{\sin x \cos y}$$

separable equation. The student must complete the integration.

Exercise 03:

1. $y' = \frac{y}{x} - 1$, we make the change of variable $\frac{y}{x} = t \Leftrightarrow y = tx \Leftrightarrow y' = t'x + t \Leftrightarrow \frac{dy}{dx} = \frac{dt}{dx}x + t$ so $y' = \frac{y}{x} - 1 \Leftrightarrow t'x = -1 \Leftrightarrow x = Ce^{-\frac{y}{x}}$.
2. $y' = -\frac{x+y}{x}$ the same work for this equation
3. $(x-y)ydx - x^2dy = 0 \Leftrightarrow \frac{dy}{dx} = \frac{y}{x} - (\frac{y}{x})^2$ the student must complete
4. $(x^2+y^2)dx - 2xydy = 0 \Leftrightarrow \frac{dy}{dx} = \frac{1}{2}(\frac{1}{\frac{y}{x}} + \frac{y}{x}) = F(\frac{y}{x})$ homogeneous equation make the change of variable as previous $y = tx$ next we resolve a separable equation.

5. $y' = \frac{x-y}{x+y} = \frac{1-\frac{y}{x}}{1+\frac{y}{x}} \Leftrightarrow \int \frac{dx}{x} = \int \frac{1+t}{1-2t-t^2} dt$ we have

$$\frac{1+t}{1-2t-t^2} = \frac{A}{t-1-\sqrt{2}} + \frac{B}{t-1+\sqrt{2}}$$

complete ...

Exercise 04: We chose

$$y' - y\cos(x) = \sin(2x).$$

- Solve without second member
  $\frac{dy}{dx} = y\cos(x) \Leftrightarrow y = Ce^{\sin x}$
- with second member: by the variation constant method
  $C' = \sin 2xe^{-\sin x} \Leftrightarrow C = e^{-\sin x}(-2\sin x - 1) + K$ then $y = Ke^{\sin x} - 2\sin x - 1$

Exercise 05:

$$y' - \frac{y}{1-x^2} - 1 - x = 0, \quad y(0) = 0.$$

For $x \in]-1, 1[$

$$\frac{dy}{dx} = \frac{dx}{1-x^2} \Leftrightarrow y = Ce^{\arg thx}$$

by the variation constant method and after calculation we find

$$C = \frac{1}{2}\arcsin x + \frac{1}{4}\sin(2\arcsin x) + K$$

then the general solution given by

$$y = e^{\arg thx}(\frac{1}{2}\arcsin x + \frac{1}{4}\sin(2\arcsin x) + K)$$

So the solution satisfy the initial condition $y(0) = 0$ is

$$y = e^{\arg thx}(\frac{1}{2}\arcsin x + \frac{1}{4}\sin(2\arcsin x))$$

Exercise 06:

$$y' + \frac{y}{x} = -xy^2$$

Bernoulli equation we multiply by y^2 we obtain

$$y^{-2}y' + \frac{1}{x}y^{-1} = -x$$

Make the change of variable $y^{-1} = z \Leftrightarrow -y'y^{-2} = z'$ the Bernoulli equation become linear equation in the form

$$z' + \frac{1}{x}z = -x$$

We use the variation of constant method, the general solution of linear equation given by

$$z = \frac{K}{x} + \frac{x^2}{3}$$

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, and the general solution of Bernoulli equation is

$$y = \frac{1}{z} = \frac{1}{\dfrac{K}{x} + \dfrac{x^2}{3}}$$

Exercise 07:

$$y'' + 3y' + 2y = 2x^2 + 8x + 7..........(1)$$

- first the solution without second member The characteristic equation is

$$r^2 + 3r + 2 = 0 \Rightarrow r_1 = -1, \quad r_2 = -2$$

 then

$$y_H = C_1 e^{-x} + C_2 e^{-2x}, \quad C_1, C_2, \quad \text{are constants}$$

- second with second member we know that $y_G = y_H + y_P$
 The second member is polynomial then the particular solution

$$y_P = ax^2 + bx + c, \quad y_P' = 2ax + b, \quad y_P'' = 2a$$

 We replace in (1) and by identification $a = 1, \quad b = \frac{5}{2}, \quad c = \frac{-5}{4}$ hence

$$y_P = x^2 + \frac{5}{2}x + \frac{-5}{4}$$

 and

$$y_G = C_1 e^{-x} + C_2 e^{-2x} + x^2 + \frac{5}{2}x + \frac{-5}{4}$$

$$y'' - y = e^{2x} - e^{x}.......; (2)$$

Characteristic equation is

$$r^2 - 1 = 0, \quad r = \pm 1$$

Solution without second member is

$$C_1 e^{-x} + C_2 e^{x}$$

then we find y_P, use **the variation constant method** as follow: we put $y_1 = e^{-x}, \quad y_1 = e^{x}$ and solve the system equations

$$\begin{cases} C_1' y_1 + C_2' y_2 = 0 \\ C_1' y_1' + C_2' y_2' = f(x) \end{cases}$$

$$\begin{cases} C_1' e^{-x} + C_2' e^{x} = 0......(1) \\ -C_1' e^{-x} + C_2' e^{x} = f(x)...(2) \end{cases}$$

$$(1) + (2) : C_2' = \frac{1}{2}(e^{x} - 1) \Rightarrow C_2 = \frac{1}{2}e^{x} - \frac{x}{2} + K_2$$

$$(2) - (1) : C_1' = -\frac{1}{2}(e^{3x} - e^{2x}) \Rightarrow C_1 = -\frac{1}{6}e^{3x} + \frac{1}{4}e^{2x} + K_1$$

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then

$$y_G = K_1 e^{-x} + K_2 e^{x} + \frac{1}{3}e^{2x} + \frac{1}{4}(1-2x)e^{x}$$

$$y'' - 6y' + 9y = 2e^{3x} - e^{x}......(3)$$

Characteristic equation is:

$$r^2 - 6r + 9 = 0 \Leftrightarrow (x-3)^2 = 0, \quad r = 3\text{double root}$$

so

$$y_H = (C_1 x + C_2)e^{3x}$$

For find the particular solution we can use

- First method: because the second member is the form $P(x)e^{\alpha x}$. We search $y_P = y_{P_1} + y_{P_2}$ such that $y_{P_1} = Cx^2 e^{\alpha x}, \quad \alpha = 3, \quad y_{P_2} = Ce^{x}$

  $$y'_{P_1} = C(2x + 3e^{2x})e^{3x}, \quad y''_{P_1} = C(9x^2 + 12x + 2)e^{3x}$$

  After calculation we can find $C = 1$ then $y_{P_1} = x^2 e^{3x}$ in the same way, we find $y_{P_2} = -\frac{1}{4}e^{x}$ then

  $$y_G = (C_1 x + C_2)e^{3x} + x^2 e^{3x} - \frac{1}{4}e^{x}$$

- we can use: the variation of constant method as previous

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