CHAPTER II: CONDUCTORS

I- Definition of electrostatic equilibrium

When there is no net motion of charge within a conductor.

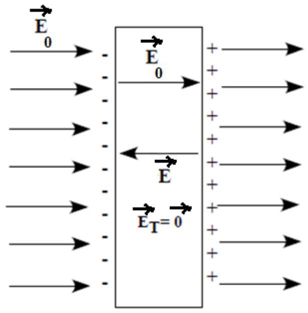
The conductors are materials where the electrons are free to move rather easily; however, when they are in electrostatic equilibrium, this means the charges are stationary in the object.

II- Properties of conductors in equilibrium

A conductor in electrostatic equilibrium has the following properties:

1. The electric field is zero everywhere inside the conductor.
2. Any net charge on an isolated conductor must reside entirely on its surface.
3. The E-field just outside a charged conductor is perpendicular to the conductor's surface and has a magnitude $\frac{\sigma}{\varepsilon_0}$, where σ is the surface charge density at that point.

1st Property



The 1st property can be understood by considering a conducting slab placed in an external electric field $\overrightarrow{E_0}$.

When the external electric field $\overrightarrow{E_0}$ is applied:

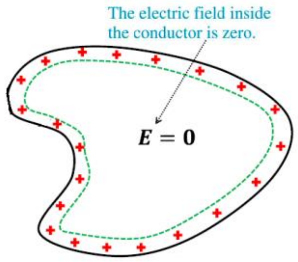
a) The negative charges (free electrons) will accelerate in the opposite direction of external electric field $\overrightarrow{E_0}$ and the positive charges move in the same direction of external electric field $\overrightarrow{E_0}$, causing a negative surface charge density on the left and a positive surface charge density on the right.

b) These induced charges will produce and induced electric field $\vec{E}$ that will oppose the external electric field $\overrightarrow{E_0}$ (from positive charges to negative charges).

c) Charge continues to accumulate until both electric fields cancel each other inside the conductor.

The time it takes a conductor to reach this equilibrium is 10^{-16} s.

2nd Property



Consider arbitrary shaped conductor. The conductor is positively charged.

Let's apply Gauss's law :

$$\emptyset = \oiint \vec{E}.d\vec{S} = \frac{Q_{enclosed}}{\varepsilon_0}$$

$$\vec{E} = \vec{0} \Rightarrow \emptyset = 0 \Rightarrow Q_{enclosed} = 0$$

So, all excess charges are located on the surface of the conductor.

Since E = 0 in the interior of the conductor, therefore dV = 0. i.e. **V = constant**

- **Poisson's and Laplace's Equation**

  Differential form of Gauss's law :

$$\text{div}\vec{E} = \frac{\rho}{\varepsilon_0} \Rightarrow \vec{\nabla}.\vec{E} = \frac{\rho}{\varepsilon_0} \quad \text{and } \vec{E} = -\vec{\nabla}V$$

$$\Rightarrow \vec{\nabla}.(-\vec{\nabla}V) = \frac{\rho}{\varepsilon_0} \Rightarrow -\vec{\nabla}.\vec{\nabla}V = \frac{\rho}{\varepsilon_0} \Rightarrow -\nabla^2 V = \frac{\rho}{\varepsilon_0}$$

$$\Rightarrow \nabla^2 V = \Delta V = -\frac{\rho}{\varepsilon_0}$$ **Poisson's and Laplace's equation**

In cartesian form : $\nabla^2 V = \Delta V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$

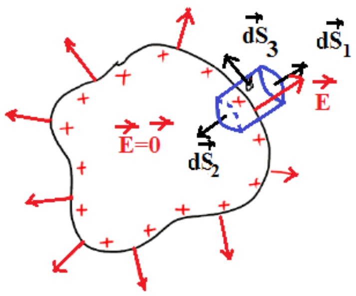
Since E= 0, the equation becomes Laplace's equation: $\nabla^2 V = 0$

3rd Property

Consider a conductor with surface charge density σ. Construct a Gaussian surface in the shape of a small cylinder with the end faces parallel to the surface. Part of the cylinder is just outside the surface and the rest is inside.

$$\emptyset = \oiint \vec{E}.d\vec{S} = \frac{Q_{enclosed}}{\varepsilon_0}$$

$$\Rightarrow \emptyset = \oiint \vec{E}.\overrightarrow{dS_1} + \oiint \vec{E}.\overrightarrow{dS_2} + \oiint \vec{E}.\overrightarrow{dS_3}$$



$$\oiint \vec{E}.\overrightarrow{dS_1} = E.S_1 \quad (\vec{E}//\overrightarrow{dS_1} \textit{ and in the same direction})$$

$$\oiint \vec{E}.\overrightarrow{dS_2} = 0 \quad (\text{E = 0 inside the conductor})$$

$$\oiint \vec{E}.\overrightarrow{dS_3} = 0 \; (\vec{E} \perp \overrightarrow{dS_3})$$

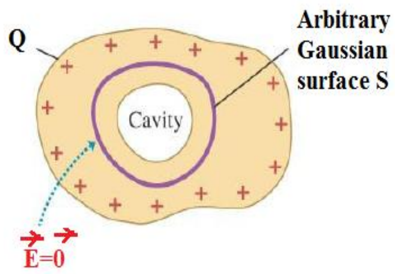
$$\Rightarrow \emptyset = E.S_1 = \frac{Q_{enclosed}}{\varepsilon_0} \Rightarrow \emptyset = E.S_1 = \frac{\sigma.S_1}{\varepsilon_0} \Rightarrow \boldsymbol{E} = \frac{\boldsymbol{\sigma}}{\boldsymbol{\varepsilon_0}}$$

This result applies to any conductor in electrostatic equilibrium. It also shows that you have large electric field where the surface charge density is large.

Because equipotential surfaces are always perpendicular to the electric field, the surface of a conductor in electrostatic equilibrium must be an equipotential surface.

III- Hollow conductor in electrostatic equilibrium

III.1- The cavity doesn't contain any charges:

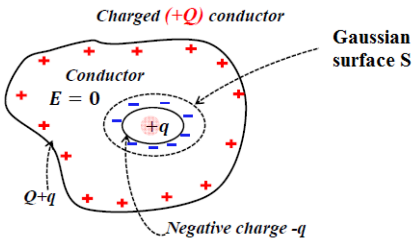


The behaviour of a hollow conductor without any charge on the cavity, is exactly the same as if it was a solid conductor.

Because the electric field inside the conductor is zero, the electric field at all points on the Gaussian surface must be zero. So, there aren't charges on inner surface of conductor being all the charge placed on the outer surface.

The electric potential on cavity equals the potential on conductor (constant).

III.2- An isolated charge q placed in the cavity:



Let's apply Gauss's law :

$$\emptyset = \oiint \vec{E}.d\vec{S} = \frac{Q_{enclosed}}{\varepsilon_0}$$

The electric flux is zero through the Gaussian surface, since E=0 inside the conductor.

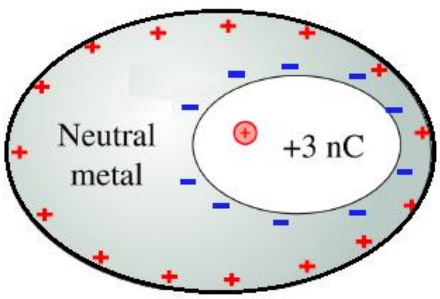
$$\vec{E} = \vec{0} \Rightarrow \emptyset = 0 \Rightarrow Q_{enclosed} = 0$$

So $Q_{enclosed}$ = 0 (the net charge inside the Gaussian surface). But, we know that there is +q inside, it means that there must be –q on the interior surface (+q charge induced –q on the surface).

Let's count all charges inside the conductor to find the amount of charge on the exterior surface.

$$-q + Q_{outer\ surface} = +Q \Rightarrow Q_{outer\ surface} = +Q + q$$

Example



Charge +3 nC is in a hollow cavity inside a large chunk of metal that is electrically neutral. What is the total charge on the exterior surface of the metal?

The electric field is zero inside the conductor.

$$\vec{E} = \vec{0} \Rightarrow \emptyset = 0 \Rightarrow Q_{enclosed} = 0$$

$$\Rightarrow Q_{enclosed} = q + 3 = 0 \Rightarrow q = -3\ nC$$

Metal is electrically neutral$\Rightarrow -3 + Q_{outer\ surface} = 0$

$$\Rightarrow Q_{outer\ surface} = 3\ nC$$

IV- Two Spherical Conductors

1- Two spherical conductors are separated by a large distance.

They each carry the same positive charge Q. Conductor A has a larger radius than conductor B.

$$V_A = k\frac{Q}{R_A} \quad , \quad V_B = k\frac{Q}{R_B}$$

$$R_A > R_B \Rightarrow V_B > V_A$$

2- Two spherical conductors are separated by a large distance and connected by the wire.

They each carry the same positive charge Q. Conductor A has a larger radius than conductor B.

Before connection: $V_A = k\frac{Q}{R_A}$, $V_B = k\frac{Q}{R_B}$

after connection: $\grave{V}_A = k\frac{Q_A}{R_A}$, $\grave{V}_B = k\frac{Q_B}{R_B}$

$$\grave{V}_A = \grave{V}_B \quad \textbf{and} \quad 2Q = Q_A + Q_B$$

$$\grave{V}_A = \grave{V}_B \Rightarrow \frac{Q_A}{R_A} = \frac{Q_B}{R_B} \Rightarrow Q_A = \frac{R_A}{R_B} Q_B$$

$$2Q = Q_A + Q_B \Rightarrow 2Q = \frac{R_A}{R_B} Q_B + Q_B \Rightarrow Q_B = 2\frac{R_B}{R_A + R_B} Q$$

$$\Rightarrow Q_A = 2\frac{R_A}{R_A + R_B} Q$$

$$R_A > R_B \Rightarrow Q_A > Q_B$$

So, Q_A increases: since B initially has a higher potential, charges move from B to A.

$$\frac{E_A}{E_B} = \frac{Q_A}{R_A^2} \cdot \frac{R_B^2}{Q_B} = \frac{R_B^2}{R_A^2} \cdot \frac{Q_A}{Q_B} \Rightarrow \frac{E_A}{E_B} = \frac{R_B^2}{R_A^2} \frac{2\frac{R_A}{R_A+R_B} Q}{2\frac{R_B}{R_A+R_B} Q}$$

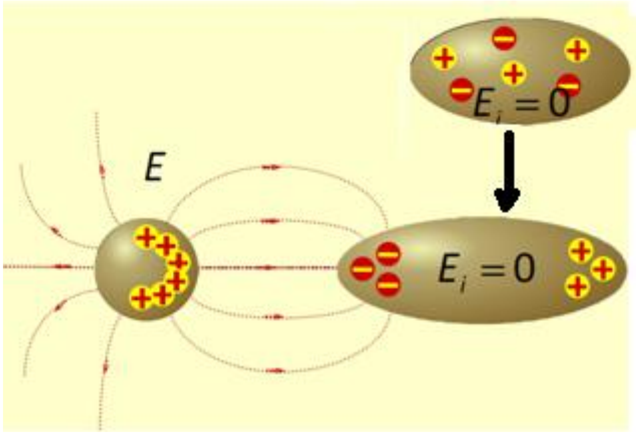
$$\Rightarrow \frac{E_A}{E_B} = \frac{R_B}{R_A}$$

$$R_A > R_B \Rightarrow E_B > E_A$$

The electric field is higher near the conductor which has lower radius.

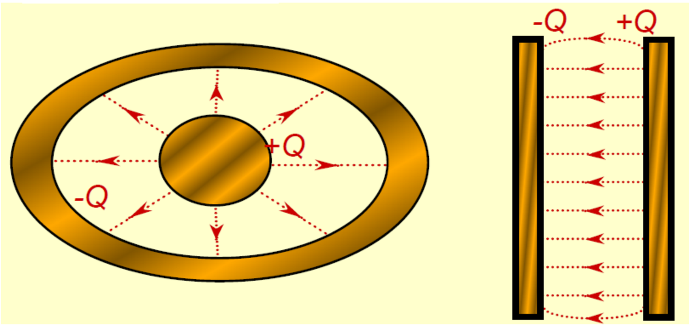
V- Electrostatic influence

When we put an electric charge near a conductor, electrostatic influence divides the charge inside the conduct.



- **Total electrostatic influence**

Total Electrostatic influence between two conductors occurs when all the field lines starting from a conductor and end in the other conductor. Surfaces with total influence have the same charge but different sign.



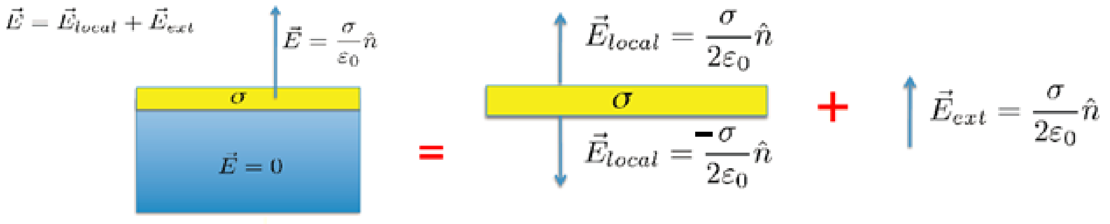
VI- Electrostatic pressure

The external field exerts a force at surface of a conductor. The force per unit area acting on the surface of the conductor (electrostatic pressure) always acts outward, and is given by:

$$P = \frac{F}{S} = \frac{Q.E_{ext}}{S} = \frac{\sigma.S.E_{ext}}{S}$$

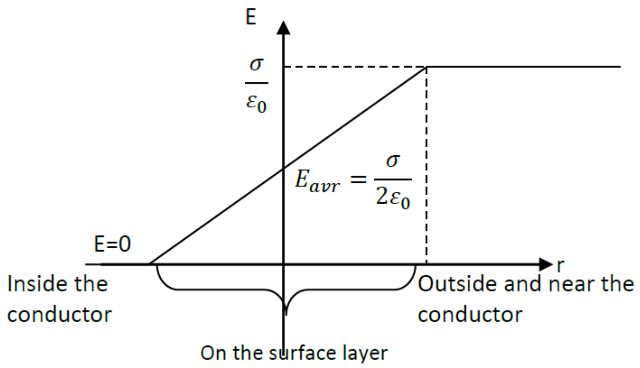
$$\Rightarrow \boldsymbol{P = \sigma.E_{ext}}$$

$$\vec{\boldsymbol{E}} = \overrightarrow{\boldsymbol{E_{local}}} + \overrightarrow{\boldsymbol{E_{ext}}}$$

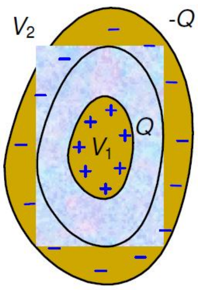


$$\boldsymbol{E_{ext}} = \frac{\boldsymbol{\sigma}}{\boldsymbol{2\varepsilon_0}} \Rightarrow \boldsymbol{P} = \boldsymbol{\sigma}.\frac{\boldsymbol{\sigma}}{\boldsymbol{2\varepsilon_0}}$$

$$\Rightarrow \boldsymbol{P} = \frac{\boldsymbol{\sigma^2}}{\boldsymbol{2\varepsilon_0}}$$



VII - Capacitance of a conductor and a capacitor



A capacitor is a system of two conductors separated by a distance, exerting total influence between them. The both conductors have equal and opposite charges.

The space between capacitors may be a vacuum (vacuum capacitor) or an insulating material (**dielectric**).

Capacitors are used to store electric charge and energy and they have a lot of applications on electrical circuits.

When a difference of potential $V = V_1 - V_2$ is applied to the plates of a capacitor, a movement of charges is produced from a conductor to another, until the difference of potential between both plates equals the applied difference of potential.

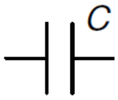
Therefore, the quantity of charge Q on the capacitor depends on the applied difference of potential V, on geometry and size of capacitor and on the material between plates.

$$\boldsymbol{Q = C\,V = C(V_1 - V_2)}$$

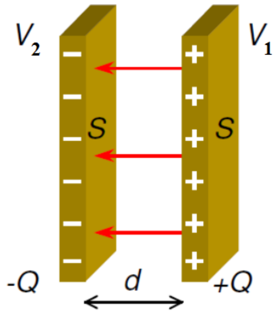
C: **capacitance**. It determines **t**he amount of storage in a capacitor.

The value of the capacitance only depends on geometry and size of capacitor and on insulating material between plates.

Unit of capacitance: **Farad** (F) = Coulomb/Volt.

Its graphic representation on the circuits is: 

VII.1- Parallel plate flat capacitor

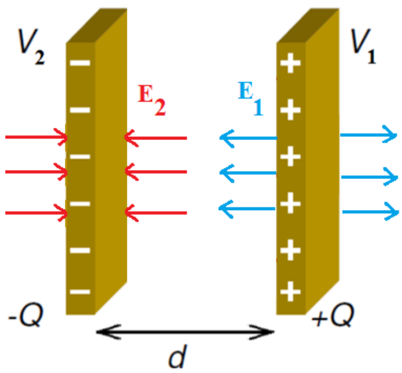


A parallel plate flat capacitor has their plates parallel of surface *S* at a distance *d*, very low compared with S. We'll consider the vacuum between plates.

When it is charged, both conductors are in electrostatic equilibrium, with the charge uniformly distributed across its surface.

The surface density of charge will be: $\boldsymbol{\sigma = \frac{Q}{S}}$

The electric field in the space between two conductors is:



$$\boldsymbol{E = E_1 + E_2 = \frac{\sigma}{2\varepsilon_0} + \frac{\sigma}{2\varepsilon_0} = \frac{\sigma}{\varepsilon_0} = \frac{Q}{S.\varepsilon_0}}$$

$$\boldsymbol{V_2 - V_1 = -\int_0^d E.dx = -E.d = -\frac{Q}{S.\varepsilon_0}.d}$$

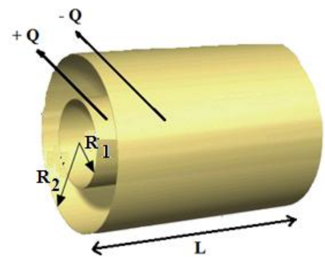
$$\boldsymbol{\Rightarrow V = V_1 - V_2 = \frac{Q}{S.\varepsilon_0}.d}$$

We have: $\boldsymbol{Q = C\,V = C(V_1 - V_2) \Rightarrow C = \frac{Q}{V}}$

$$\boldsymbol{\Rightarrow C = \frac{\varepsilon_0.S}{d}}$$

The capacitance depends exclusively on geometrical parameters.

II.2- Cylindrical capacitor

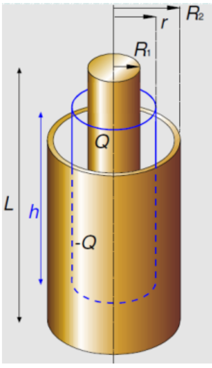


A cylindrical capacitor is made up by two cylindrical conductors, coaxial of radii R_1 and R_2 and length *L* much greater than the space between conductors ($L >> R_2$-R_1).

$$\boldsymbol{\sigma_1 = \frac{Q}{S_1} = \frac{Q}{2\pi R_1 L}}$$

$$\boldsymbol{\sigma_2 = \frac{-Q}{S_1} = \frac{-Q}{2\pi R_2 L}}$$

The electric field in the space between two conductors can be computed by applying Gauss's law to an enclosed cylindrical surface of radius *r* ($R_1 < r < R_2$) and height *h* < L:



$$\emptyset = \oiint \vec{E}.d\vec{S} = \frac{Q_{enclosed}}{\varepsilon_0}$$

$$\Rightarrow E.S_G = \frac{Q}{\varepsilon_0} \Rightarrow E.\boldsymbol{2\pi r h} = \frac{\sigma_1 S_1}{\varepsilon_0} \Rightarrow E.\boldsymbol{2\pi r h} = \frac{\frac{Q}{2\pi R_1 L}2\pi R_1 h}{\varepsilon_0}$$

$$\Rightarrow \boldsymbol{E} = \frac{\boldsymbol{Q}}{\boldsymbol{2\pi\varepsilon_0.L.r}}$$

$$\boldsymbol{V_2 - V_1} = -\int_{R_1}^{R_2} \boldsymbol{E.dr}$$

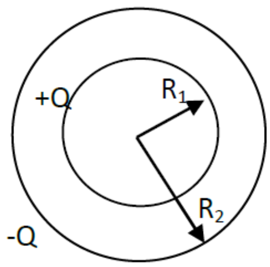
$$\Rightarrow \boldsymbol{V_1 - V_2} = \int_{R_1}^{R_2} \frac{Q}{2\pi\varepsilon_0.L.r}.dr = \frac{Q}{2\pi\varepsilon_0.L} ln\frac{R_2}{R_1}$$

$$\Rightarrow \boldsymbol{C} = \frac{\boldsymbol{Q}}{\boldsymbol{V_1 - V_2}} = \frac{\boldsymbol{Q}}{\frac{Q}{2\pi\varepsilon_0.L} ln\frac{R_2}{R_1}}$$

$$\Rightarrow \mathbf{C} = \frac{2\pi\varepsilon_0.L}{ln\frac{R_2}{R_1}}$$

VII.3- Spherical capacitor

Consider two concentric conducting spheres under total influence, one with charge +Q and the other with charge -Q.



The electric field in the space between two conductors can be computed by applying Gauss's law to an enclosed spherical surface of radius r ($R1<r<R2$):

$$\emptyset = \oiint \vec{E}.d\vec{S} = \frac{Q_{enclosed}}{\varepsilon_0}$$

$$\Rightarrow E.S_G = \frac{Q}{\varepsilon_0} \Rightarrow E.\boldsymbol{4\pi r^2} = \frac{\boldsymbol{Q}}{\varepsilon_0}$$

$$\Rightarrow E = \frac{\boldsymbol{Q}}{\boldsymbol{4\pi\varepsilon_0 r^2}}$$

$$V_1 - V_2 = \int_{R_1}^{R_2} E.dr = \int_{R_1}^{R_2} \frac{Q}{4\pi\varepsilon_0 r^2}.dr$$

$$\Rightarrow V_1 - V_2 = \frac{Q}{4\pi\varepsilon_0}\left(-\frac{1}{R_2} + \frac{1}{R_1}\right)$$

$$\Rightarrow V_1 - V_2 = \frac{Q}{4\pi\varepsilon_0}\left(\frac{-R_1 + R_2}{R_1 R_2}\right)$$

$$\Rightarrow V_1 - V_2 = \frac{Q}{4\pi\varepsilon_0}\left(\frac{R_2 - R_1}{R_1 R_2}\right)$$

$$\Rightarrow C = \frac{Q}{V_1 - V_2} = \frac{Q}{\frac{Q}{4\pi\varepsilon_0}\left(\frac{R_2 - R_1}{R_1 R_2}\right)}$$

$$\Rightarrow C = \frac{4\pi\varepsilon_0 R_1 R_2}{R_2 - R_1}$$

Remark

If we fill the region between the plates of a capacitor with an insulating material the capacitance will be increased by some numerical factor κ**:**

$$\mathbf{C = \kappa . C_{air}}$$

The number κ (which is unitless) is called the dielectric constant of the insulating material.

VIII- Stored energy on a capacitor

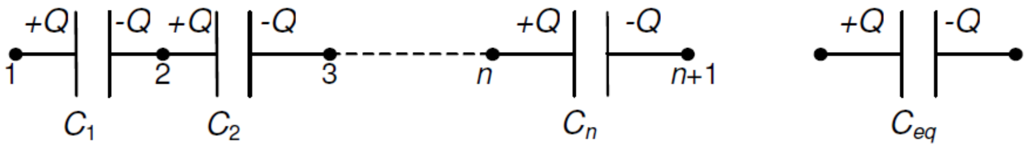
To charge a capacitor (initially discharged), a difference of potential *V* can be applied to the plates of capacitor, producing a flow of electrons, from the plate with positive potential to the plate with negative potential. The charging process progress from charge 0 to charge $Q = CV$.

So, needed energy to charge the capacitor from discharged to charge Q and difference of potential *V* is:

$$E = \frac{1}{2C} Q^2 = \frac{1}{2} CV^2$$

IX- Association of capacitors

IX.1- Association in series



$$\frac{1}{C_{eq}} = \sum_{i=1}^{n} \frac{1}{C_i}$$

For each capacitor: $V_i - V_{i+1} = \frac{Q}{C_i}$, i= 1,2,….,n

The difference of potential V between terminals of the association:

$$V_1 - V_{n+1} = \sum_{i=1}^{n}(V_i - V_{i+1}) \quad , i= 1,2,….,n$$

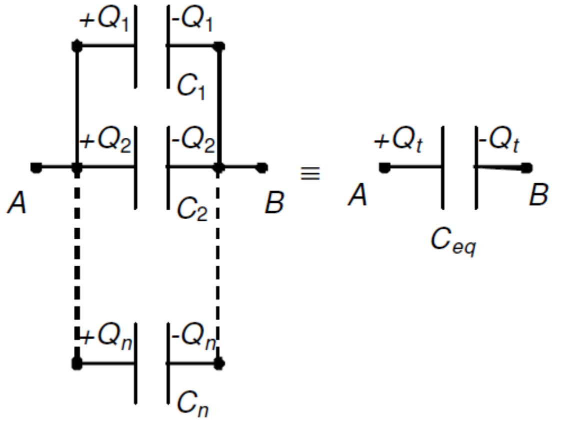
Therefore:

$$V = V_1 - V_{n+1} = \sum_{i=1}^{n}(V_i - V_{i+1}) = \sum_{i=1}^{n}\left(\frac{Q}{C_i}\right) = \frac{Q}{C_{eq}}$$

So, when *n* capacitors are associated in series, the equivalent capacitance is lower than the capacitance of each one of the associated capacitors.

IX.2- Association in parallel

A set of capacitors are associated in parallel when all the capacitors are connected to the same difference of potential V_A-V_B.



$$C_{eq} = \sum_{i=1}^{n} \boldsymbol{C_i}$$

The charge is distributed on plates of all capacitors:

$$Q_{Tot} = \sum_{i=1}^{n} \boldsymbol{Q_i}$$

For each capacitor: $\boldsymbol{Q_i} = (V_A - V_B)C_i$

So: $Q_{Tot} = \sum_{i=1}^{n} \boldsymbol{Q_i} = (V_A - V_B)\sum_{i=1}^{n} \boldsymbol{C_i}$

$$Q_{Tot} = (V_A - V_B)C_{eq}$$

So, when *n* capacitors are associated in parallel, the equivalent capacitance is greater than the capacitance of each one of the associated capacitors.