ELECTRICITY AND MAGNETISM

Mathematical Preliminary

1- Coordinate systems

1-1 Cartesian system (Rectangular system)

One can locate the point $\boldsymbol{M}$*, in space, in the cartesian system by some number said the coordinates* $(\boldsymbol{x}, \boldsymbol{y}.\boldsymbol{z})$*, each expresses how far is the point from the origin of the cartesian basis* $(\vec{\imath}, \vec{\jmath}, \vec{k})$*, which can be orthonormal.*

- *One dimension*: $\overrightarrow{OM} = x\vec{\imath}$

  $\infty < x < \infty$: is the abcissa

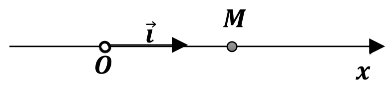


Fig.1-a

- *Two dimensions*: $\overrightarrow{OM} = x\vec{\imath} + y\vec{\jmath}$

  $\infty < x < \infty$: is the abcissa

  $\infty < y < \infty$: is the ordinate

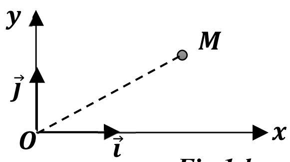


Fig.1-b

- *Three dimensions*: $\overrightarrow{OM} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$

  $\infty < x < \infty$: is the abcissa

  $\infty < y < \infty$: is the ordinate

  $\infty < z < \infty$: is the hight

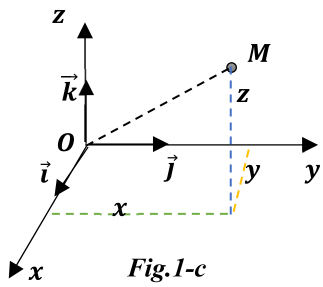


Fig.1-c

1-2 Polar system

The point is located in plane with the coordinates are the distance from the origin and the angle made between the datum (polar axis) and the direction $\overrightarrow{OM}$ *joining the pole* $\boldsymbol{O}$ *and the point* $\boldsymbol{M}$*. The basis is* $(\vec{u}_\rho, \vec{u}_\theta)$*.*

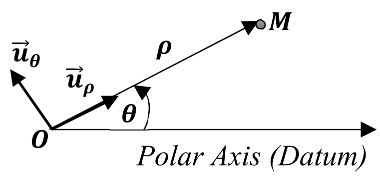


Fig.2

So, $\overrightarrow{OM} = \rho\vec{u}_\rho$

$\infty < \rho < \infty$

$0 < \theta < 2\pi$

1-3 Cylindrical system

The location of point is on the imaginary surface of a cylinder with axis $\overrightarrow{\boldsymbol{OZ}}$, and radius $\boldsymbol{\rho}$. The projection of the point on the base is defined by the radius and the angle $\boldsymbol{\theta}$ with respect to the datum, like the polar coordinates. The Hight of the point from the base of cylinder is the cote $\boldsymbol{Z}$

The position of point M is given by

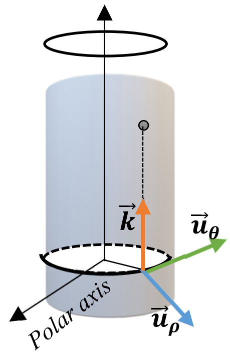


Fig.3

$$\overrightarrow{\boldsymbol{OM}} = \boldsymbol{\rho}\vec{\boldsymbol{u}}_{\boldsymbol{\rho}} + \boldsymbol{z}\vec{\boldsymbol{k}}$$

1-4 Spherical system

The location of point is on the imaginary surface of a sphere of radius $\boldsymbol{r}$, longitude $\boldsymbol{\varphi}$ and colatitude $\boldsymbol{\theta}$

- *The radius $0 \leq \boldsymbol{r} < \infty$*
- *The colatitude $0 \leq \boldsymbol{\theta} \leq \boldsymbol{\pi}$*
- *The longitude $0 \leq \boldsymbol{\varphi} \leq \boldsymbol{2\pi}$*

The position of point M is given by:

$$\overrightarrow{\boldsymbol{OM}} = \boldsymbol{r}\vec{\boldsymbol{u}}_{\boldsymbol{r}}$$

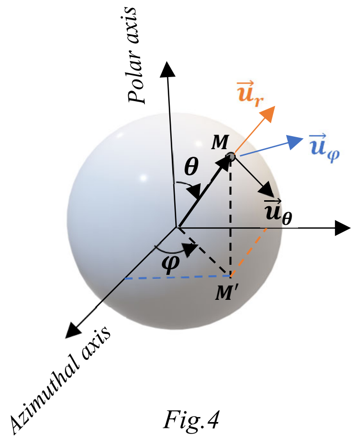


Fig.4

Vector Analysis

2- Vector Algebra

2-1-1 SCALAR PRODUCT

The scalar product between two vectors is given by

$$\vec{\boldsymbol{A}} \circ \vec{\boldsymbol{B}} = |\boldsymbol{A}||\boldsymbol{B}|\boldsymbol{cos}(\boldsymbol{A}, \boldsymbol{B}) = \boldsymbol{A}.\boldsymbol{B}.\boldsymbol{cos\theta}$$

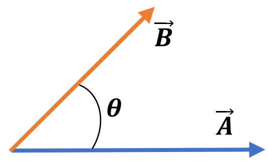


Fig.5

We use this definition which is equivalent to the sum of the product of the components of the vectors $\vec{\boldsymbol{A}}$ and $\vec{\boldsymbol{B}}$ taken two by two respectively

$$\vec{\boldsymbol{A}} \circ \vec{\boldsymbol{B}} = \boldsymbol{A_xB_x} + \boldsymbol{A_yB_y} + \boldsymbol{A_zB_z}$$

Remark:

To calculate (compute) the magnitude of any vector, we take only the square root of the scalar product of the with itself (whatever is the coordinate system)

$$|\boldsymbol{A}| = \sqrt{\vec{\boldsymbol{A}} \circ \vec{\boldsymbol{A}}} = \boldsymbol{A}$$

2-1-2 VECTORIAL PRODUCT

The scalar product between two vectors is given by

$$\vec{A} \wedge \vec{B} = |\vec{A}||\vec{B}| sin(\vec{A}, \vec{B})\, \vec{u} = A.B.sin\theta\, \vec{u}$$

With $\vec{u}$ *always orthogonal to the vector operands* $\vec{A}$ *and* $\vec{B}$ *(*$\vec{u} \perp \vec{A}$; $\vec{u} \perp \vec{B}$*)*

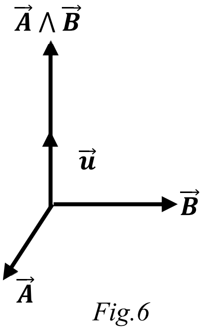


Fig.6

2-1-3 SCALAR AND VECTORIAL TRIPLE PRODUCT

- The scalar triple product between two vectors is given by

$$\vec{A} \circ (\vec{A} \wedge \vec{B})$$

- The vectorial triple product between two vectors is given by

$$\vec{A} \wedge (\vec{A} \wedge \vec{B})$$

3- Differential Calculus

3-1-1 DERIVATIVE AND DIFFERENTIAL

Differential and derivative are two fundamental concepts in calculus that are often used interchangeably. While they are related, they are not the same thing. Understanding the difference between differential and derivative is important in mastering calculus and its applications. Differential deals with infinitesimal change in some varying quantity. The derivative of a function represents an instantaneous rate of change in the value of a dependent variable with respect to the change in value of a dependent variable

Let the two variables x, y*, the infinitesimal change in these variables is* dx *and* dy*. But if the* x *variable is independent and* y *is a variable that depend on the variable. The rate of change of* y *with respect to the variable* x *is said the derivative of the function* $y(x)$*.*

3-1-2 DERIVATIVE AND DIFFERENTIAL

3-1-2-1 Function of one variable

Let x be an independent variable, and y *is the dependent variable, so each variation of the independent variable* Δx *produce the change* Δy*. So,* Δx *and* Δy *are the variations.*

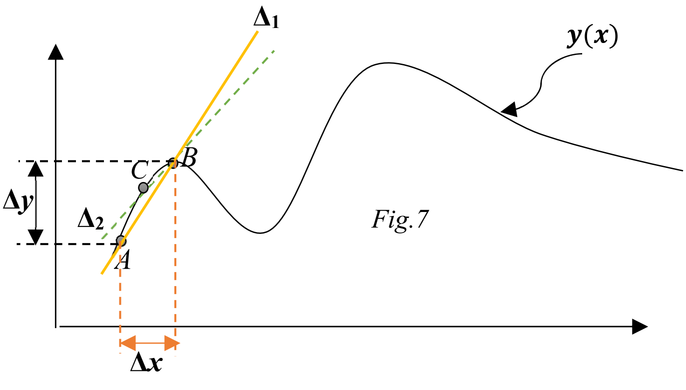


Fig.7

If each variation is very small or infinitesimal these variations tends to the *elementary variation* $\boldsymbol{dx}$ *and* $\boldsymbol{dy}$, *which we call differentials of* $\boldsymbol{x}$ *and* $\boldsymbol{y}$

$$\Delta x \rightarrow dx$$
$$\Delta y \rightarrow dy$$

The quotient between the two variations Δy *and* Δx *is the slope of the tangent of the line joining the points* ***A*** *and* ***B*** *in the figure (Line* Δ_1*).*

If we take the point ***A*** *such that its coordinates are* x_0 *and* y_0*, then after variations we get the point* ***B*** *with the coordinates* $x_0 + \Delta x$ *and* $y_0 + \Delta y$*. If these variations are infinitesimal (*$\Delta x \rightarrow 0; \Delta y \rightarrow 0$*), so, the point* ***A*** *tends to the point* ***B.*** *The line* Δ_1 *become the tangent of the curve at point* ***A*** $(x_0; y_0)$ *which is the line* Δ_2. This line represents the derivative of y with respect to x

$$\lim_{\Delta x \to 0} \left(\frac{y(x_0 + \Delta x) - y(x_0)}{\Delta x} \right) = \left. \frac{dy}{dx} \right|_{x_0} = y'(x_0)$$

Differential equations are equations that contain unknown functions and some of their derivatives.

What are Derivatives?

The concept of derivative of a function is one of the most powerful concepts in mathematics. The derivative of a function is usually a new function which is called as the derivative function or the rate function.

The derivative of a function represents an instantaneous rate of change in the value of a dependent variable with respect to the change in value of the independent variable. It's a fundamental tool of calculus which can also be interpreted as the slope of the tangent line. It measures how steep the graph of a function is at some given point on the graph. In simple terms, a derivative is the rate at which function changes at some particular point.

Differences: Differential and Derivative

While differential and derivative are related, they are not the same thing. The main difference between differential and derivative is that a differential is an infinitesimal change in a variable, while a derivative is a measure of how much the function changes with respect to its input.

Another difference is that the differential is a function of two variables, while the derivative is a function of one variable. The differential of a function is given by

***df(x) = f '(x) dx**, which is a function of both x and dx. The derivative, on the other hand, is given by f'(x) or **dy/dx**, which is a function of only x.*

The differential is often used in applications of calculus to approximate changes in a function, while the derivative is used to find the rate of change of a function at a given point. The differential is also used in optimization problems to find the maximum or minimum value of a function, while the derivative is used in a variety of applications, including physics, economics, and engineering.

The following table highlights the major differences between Differentials and Derivatives:

3-1-1 DIFFERENTIAL OPERATOR " $\vec{\nabla}$ " (OPERATOR NABLA OR DEL)

*The rule that assigns for each point of coordinates **x, y** and **z** a scalar is said to a scalar function (temperature, charge, masse, …)*

*The rule that assigns for each point of coordinates **x, y** and **z** a vector is said to a vector function (velocity, force, …)*

The DEL operator is an operator that acts on a scalar or vectorial function to give its differential

$$\vec{\nabla} = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$$

As mentioned above, the derivative of a function of one variable $f(x)$, is given, geometrically, by the slope of the tangent of the curve representing this function.

$$\frac{df(x)}{dx} = f'(x)$$

Let now the function depends on several variables. Let take two independent variables x and y, then the differential or the infinitesimal variation of that function is given by

$$df(x,y) = \frac{\partial f(x,y)}{\partial x}dx + \frac{\partial f(x,y)}{\partial y}dy = \left(\frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy\right)f(x,y)$$

$\frac{\partial f(x,y)}{\partial x}$ *and* $\frac{\partial f(x,y)}{\partial y}$ *are the partial derivative*

- $\frac{\partial f(x,y)}{\partial x}$ *is the partial derivative of the function $f(x,y)$ with respect to the variable x i.e. the derivative when varying x and y taken fixed (constant).*
- $\frac{\partial f(x,y)}{\partial x}$ *is the partial derivative of the function $f(x,y)$ with respect to the variable y i.e. the derivative when varying y and x taken fixed (constant).*

In the case of real space, the variables are x, y and z. The differential of the scalar function $f(x,y,z)$ is given as

$$\boxed{df(x,y,z) = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = \left(\frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy + \frac{\partial}{\partial z}dz\right)f(x,y,z)}$$

Remarque:

If we have only one of these variables x, y and z which varies the other are constant, then the differential of the function is:

$$df = \frac{\partial f}{\partial x}dx \quad \Rightarrow \quad \frac{df}{dx} = \frac{\partial f}{\partial x}$$

3-1-2 GRADIENT AND DIRECTIONAL DERIVATIVE

A- GRADIENT

As set above the differential of the function $f(x,y,z)$ is given by

$$df(x,y,z) = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

Which we can write in the form

$$df(x,y,z) = \left(\frac{\partial f}{\partial x}\vec{\imath} + \frac{\partial f}{\partial y}\vec{\jmath} + \frac{\partial f}{\partial z}\vec{k}\right) \circ \left(dx\,\vec{\imath} + dy\,\vec{\jmath} + dz\,\vec{k}\right)$$

$$\Rightarrow \quad \boxed{df(x,y,z) = \vec{\nabla}f \circ d\vec{r}}$$

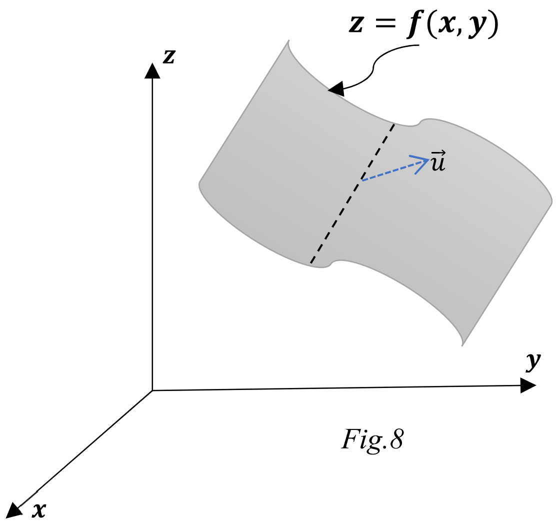


Fig.8

*This is the general form of the differential of the function **f (x, y, z)** whatever is the coordinate system.*

$\vec{\nabla}f$ **:** *is said the gradient of the function "**f**"*

$d\vec{r}$ **:** *is the infinitesimal displacement*

B- A DIRECTIONNAL DERIVATIVE

For the function of one variable de direction of the variation is well defined, but when we deal with the variation of the function of several variables, the question is, in what direction? So, we speak about a directional derivative.

In the figure on side, the derivative of the function can be in any direction, but if we want to see at what rate the change is done in the $\vec{u}$ direction for example,

$$f_u{}' = \vec{\nabla}f \circ \vec{u}$$

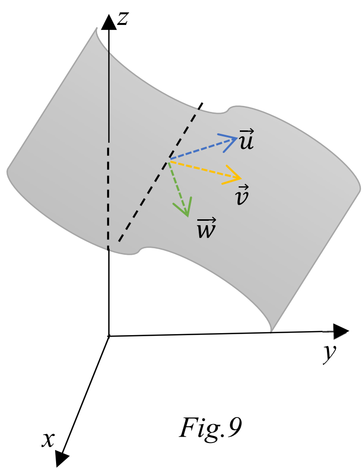


Fig.9

Finally, if we take the derivative in a certain direction such that the differential be maximized, so

$$f_n' = \vec{\nabla} f \circ \vec{n} = |\vec{\nabla} f|\,|\vec{n}|\,cos(\vec{\nabla} f;\vec{n})$$

The scalar product is a maximum, if the angle between the vectors is " π/2 "

$$|\vec{\nabla} f| = df$$

The differential is null when $df(x,y,z) = 0$

So, $\vec{\nabla} f \circ d\vec{r} = 0 \;\; \Rightarrow \;\; \vec{\nabla} f \perp d\vec{r}$

The gradient is equal to the maximum rate of change in the normal direction of the surface f(x ,y)

When the operator DEL multiplied directly with a scalar function, it produces its gradient which is a vectorial quantity

Example

What is the directional derivative of the function… in the direction u?

3-1-3 DIVERGENCE

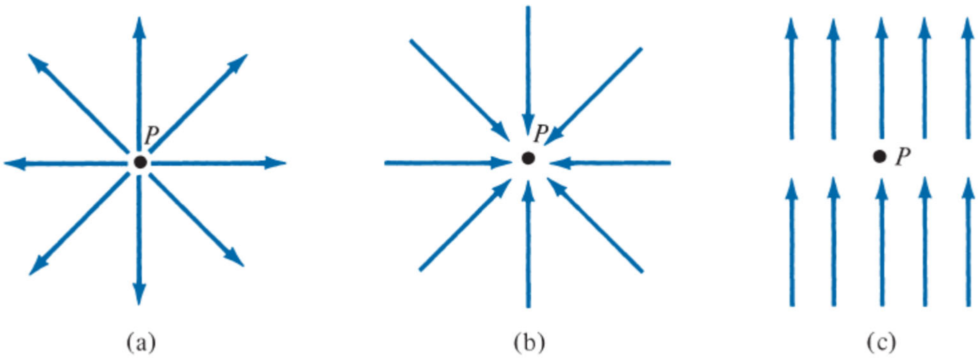


Fig.10

The divergence expresses how much the vector function $\vec{G}(x,y,z)$ *is spread from the point in question*

Let a vector function $\vec{G}(x,y,z) = G_x\,\vec{i} + G_y\,\vec{j} + G_z\,\vec{k}$. *The divergence of this function is found by applying the DEL operator scalarly upon it*

$$Div(\vec{G}(x,y,z) = \vec{\nabla} \circ \vec{G}(x,y,z) = \frac{\partial G_x}{\partial x} + \frac{\partial G_y}{\partial y} + \frac{\partial G_z}{\partial z}$$

When the operator DEL act, scalarly, upon a vector, it produces the divergence, which is a scalar quantity

- *The divergence can be positive. So, we have a source (fig. 10-a)*
- *The divergence can be negative. So, we have a sink (fig. 10-b)*
- *The divergence can be null. So, the vector is solenoidal (fig. 10-c)*

3-1-4 CURL

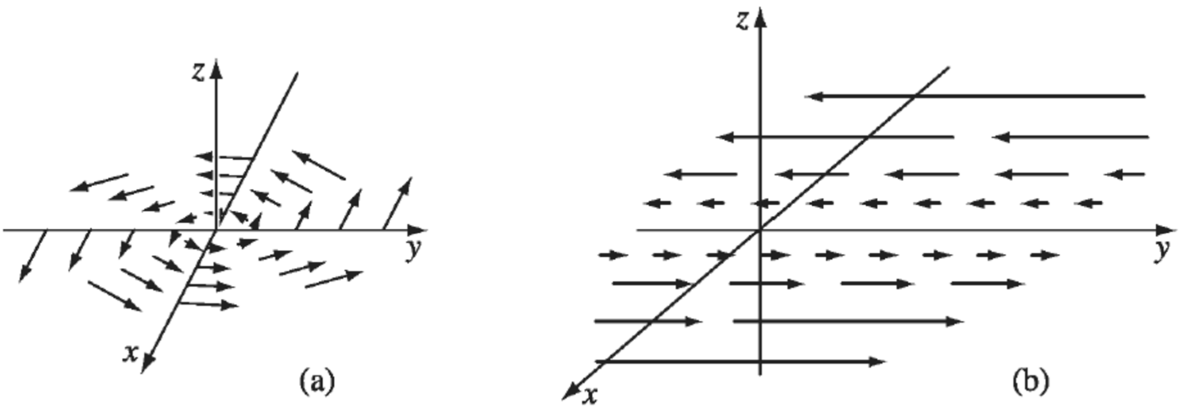


Fig.11

The curl expresses how much the vector function $\vec{\mathrm{G}}(x, y, z)$ swirls around the point in consideration

Let a vector function $\vec{\mathrm{G}}(x, y, z) = G_x\,\vec{\imath} + G_y\,\vec{\jmath} + G_z\,\vec{k}$. The curl of this function is found by applying the DEL operator vectorially upon it

$$\overrightarrow{Curl}(\vec{G}) = \vec{\nabla}\wedge\vec{\mathrm{G}}(x, y, z) = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ G_x & G_y & G_z \end{vmatrix} = \left(\frac{\partial G_z}{\partial y} - \frac{\partial G_y}{\partial z}\right)\vec{\imath} + \left(\frac{\partial G_x}{\partial z} - \frac{\partial}{\partial x}\right)\vec{\jmath} + \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial z}\right)\vec{k}$$

When the operator DEL act, vectorially, upon a vector, it produces the curl, which is a vectorial quantity.

3-1-5 THE SECOND DERIVATIVE

The gradient, the divergence and the curl are the first derivative only. When we apply the DEL operator twice, we obtain the second derivative, but tacking care when applying the second time. Because the gradient is obtained when we apply the DEL operator upon the scalar function, while the divergence and the curl are contained when applying the DEL operator upon the vector function. So, it is possible that certain actions are not permissible.

$\vec{\nabla}\left(\vec{\nabla}\circ\vec{G}(x,y,z)\right)$ *Gradient of the divergence*

$\vec{\nabla}\circ\left(\vec{\nabla}\wedge\vec{G}(x,y,z)\right)$ *Divergence of the curl*

$\vec{\nabla}\circ\left(\vec{\nabla}f(x,y,z)\right)$ *Divergence of the gradient*

$\vec{\nabla}\wedge\left(\vec{\nabla}\wedge\vec{G}(x,y,z)\right)$ *Curl of the curl*

$\vec{\nabla}\wedge\left(\vec{\nabla}f(x,y,z)\right)$ *Curl of the gradient*

From this combination we found an important object that produced when applying the divergence of the gradient

$$\vec{\nabla}\circ\left(\vec{\nabla}f(x,y,z)\right)=\left(\frac{\partial}{\partial x}\vec{\imath}+\frac{\partial}{\partial y}\vec{\jmath}+\frac{\partial}{\partial z}\vec{k}\right)\circ\left(\frac{\partial f}{\partial x}\vec{\imath}+\frac{\partial f}{\partial y}\vec{\jmath}+\frac{\partial f}{\partial z}\vec{k}\right)=\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}$$

$$\boxed{\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}=\nabla^2 f=\Delta f}$$

$\nabla^2=\Delta$ *This operator is called <u>THE LAPLACIAN</u>*

4- INTEGRAL CALCULUS

4-1 ELEMENTARY DISPLACEMENT- DIFFERENTIAL DISPLACEMENT

We determine the displacement in certain direction for infinitesimal change or differential increment of the position

4-1-1 CARTESIAN COORDINATES

We have three directions in all space $\vec{\imath}$, $\vec{\jmath}$ *and* $\vec{k}$

- *<u>$\overrightarrow{ox}$ direction</u>*

The direction of the displacement is in the $\vec{\imath}$ *direction so the change from* x_1 *to* x_2 *is given by* Δx, *for infinitesimal displacement we have so* dx

- *<u>$\overrightarrow{oy}$ direction</u>*

The direction of the displacement is in the $\vec{\jmath}$ *direction, so the change from* y_1 *to* y_2 *is given by* Δy, *for infinitesimal displacement we have so* dy

- *<u>$\overrightarrow{oz}$ direction</u>*

The direction of the displacement is in the $\vec{k}$ *direction so the change from* z_1 *to* z_2 *is given by* Δz, *for infinitesimal displacement we have so* dz

4-1-2 POLAR COORDINATES

We have two directions in plane $\vec{u}_\rho$ *and* $\vec{u}_\theta$

- *In the radial direction*

The direction of the displacement is in the radiale direction so the change from ρ_1 *to* ρ_2 *is given by* $\Delta\boldsymbol{\rho}$*, for infinitesimal displacement we have so* $\boldsymbol{d\rho}$

- *In transversal direction*

The direction of the displacement is in the transversal direction, so the change from θ_1 *to* θ_2 *is given by* $\boldsymbol{\rho}\Delta\boldsymbol{\theta}$*, for infinitesimal displacement we have so* $\boldsymbol{\rho d\theta}$

4-1-3 CYLINDRICAL COORDINATES

We have in all space the $\boldsymbol{z}$ *coordinate with those of polar coordinates* (ρ, θ, z)*, three directions are* $\vec{\boldsymbol{u}}_{\boldsymbol{\rho}}$*,* $\vec{\boldsymbol{u}}_{\boldsymbol{\theta}}$ *and* $\vec{\boldsymbol{k}}$

- *In the radial direction*

The direction of the displacement is in the radiale direction so the change from ρ_1 *to* ρ_2 *is given by* $\Delta\boldsymbol{\rho}$*, for infinitesimal displacement we have so* $\boldsymbol{d\rho}$

- *In transversal direction*

The direction of the displacement is in the transversal direction, so the change from θ_1 *to* θ_2 *is given by* $\boldsymbol{\rho}\Delta\boldsymbol{\theta}$*, for infinitesimal displacement we have so* $\boldsymbol{\rho d\theta}$

- *In the axial direction*

The direction of the displacement is in the axial direction, so the change from z_1 *to* z_2 *is given by* $\Delta\boldsymbol{z}$*, for infinitesimal displacement we have so* $\boldsymbol{dz}$

4-1-4 SPHERICAL COORDINATES

We have in all space the radial coordinate $\boldsymbol{r}$*, the longitude and the colatitude* $\boldsymbol{\theta}$

- *In the radial direction*

The direction of the displacement is in the radiale direction so the change from $\boldsymbol{r_1}$ *to* $\boldsymbol{r_2}$ *is given by* $\Delta\boldsymbol{r}$*, for infinitesimal displacement we have so* $\boldsymbol{dr}$

- *In the longitude direction*

The direction of the displacement is in the longitude direction, so the change from φ_1 *to* φ_2 *is given by* $\boldsymbol{r\,sin\theta}\,\Delta\boldsymbol{\varphi}$*, for infinitesimal displacement we have so* $\boldsymbol{sin\theta d\varphi}$

- *In the latitude direction*

The direction of the displacement is in the latitude direction, so the change from θ_1 *to* θ_2 *is given by* $\Delta\boldsymbol{z}$*, for infinitesimal displacement we have so* $\boldsymbol{dz}$

4-2 ELEMENTARY SURFACE- DIFFERENTIAL SURFACE

The element of surface is a result of product of two variations, like a small square, in different direction.

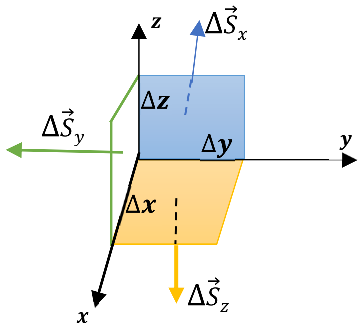
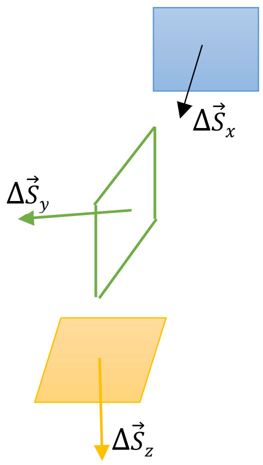


Fig.12

4-2-1 CARTESIAN COORDINATES

The infinitesimal displacements are in three directions $\boldsymbol{dx}$, $\boldsymbol{dy}$ *and* $\boldsymbol{dz}$

The surface is a vector whose magnitude is the area of that surface and the direction is orthogonal to it.



$\Delta\vec{\boldsymbol{S}}_{\boldsymbol{x}} = \Delta\boldsymbol{y}.\Delta\boldsymbol{z}\,\vec{\boldsymbol{\imath}} \quad \Rightarrow$ *The infinitesimal surface is:* $\boldsymbol{d}\vec{\boldsymbol{S}}_{\boldsymbol{x}} = \boldsymbol{dy}.\boldsymbol{dz}\,\vec{\boldsymbol{\imath}}$

This surface is obtained by fixing the $'\,\boldsymbol{x}\,'$ *coordinate*

$\Delta\vec{\boldsymbol{S}}_{\boldsymbol{y}} = \Delta\boldsymbol{x}.\Delta\boldsymbol{z}\,\vec{\boldsymbol{\jmath}} \quad \Rightarrow$ *The infinitesimal surface is:* $\boldsymbol{d}\vec{\boldsymbol{S}}_{\boldsymbol{y}} = \boldsymbol{dx}.\boldsymbol{dz}\,\vec{\boldsymbol{\jmath}}$

This surface is obtained by fixing the $'\,\boldsymbol{y}\,'$ *coordinate*

$\Delta\vec{\boldsymbol{S}}_{\boldsymbol{z}} = \Delta\boldsymbol{x}.\Delta\boldsymbol{y}\,\vec{\boldsymbol{k}} \quad \Rightarrow$ *The infinitesimal surface is:* $\boldsymbol{d}\vec{\boldsymbol{S}}_{\boldsymbol{z}} = \boldsymbol{dx}.\boldsymbol{dy}\,\vec{\boldsymbol{\imath}}$

This surface is obtained by fixing the $'\,\boldsymbol{z}\,'$ *coordinate*

4-2-2 POLAR COORDINATES

The area is spanned by the variation in the in the radiale direction with amount $\Delta\boldsymbol{\rho}$*,and the amount* $\boldsymbol{\rho}\Delta\boldsymbol{\theta}$ *in transversal direction. We have used the* $'\,\boldsymbol{\rho}\Delta\boldsymbol{\theta}\,'$*in the later variation such that the two displacements will be homogenous.*

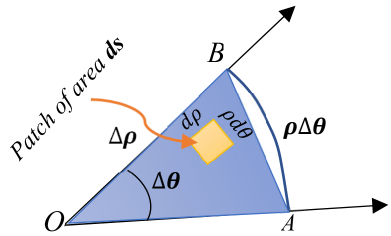


Fig.13

Let fixe the angle $\boldsymbol{\theta}$ *and varying the coordinate* $\boldsymbol{\rho}$*, we obtain the displacement* $\Delta\boldsymbol{\rho}$*. Now, let fix the distance* $\boldsymbol{\rho}$ *and varying the coordinate* $\boldsymbol{\theta}$*, we obtain the displacement* $\boldsymbol{\rho}\Delta\boldsymbol{\theta}$*. So, the area spanned for a small displacement is given by.*

$\Delta\vec{\boldsymbol{S}} = \boldsymbol{\rho}\,\Delta\boldsymbol{\rho}\,\Delta\boldsymbol{\theta}\,\vec{\boldsymbol{u}}$ *with* $\vec{\boldsymbol{u}} = \vec{\boldsymbol{u}}_{\boldsymbol{\rho}} \wedge \vec{\boldsymbol{u}}_{\boldsymbol{\theta}}$

In the limit case when we have an infinitesimal displacement, the area is given by the yellow patch in the figure and is: $\boldsymbol{d}\vec{\boldsymbol{S}} = \boldsymbol{\rho}\,\boldsymbol{d\rho}\,\boldsymbol{d\theta}\,\vec{\boldsymbol{u}}$

4-2-3 CYLINDRICAL COORDINATES

The variation that produces the elements of surface are: $\boldsymbol{\rho\Delta\theta}, \boldsymbol{\Delta\rho}$ *and* $\boldsymbol{\Delta z}$

- *When we fix the radial component (radial direction), the other components that span the area are* $'\boldsymbol{\rho\Delta\theta}'$ *and* $'\boldsymbol{\Delta z}'$. *So, the element of surface is:*

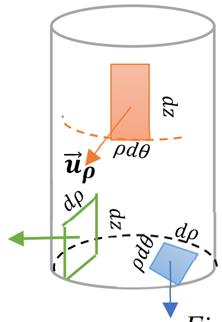


Fig.14

$\Delta\vec{S}_\rho = \boldsymbol{\rho\,\Delta\theta\Delta z\,\vec{u}_\rho} \;\Rightarrow$ *The infinitesimal surface is:* $\boldsymbol{d\vec{S}_\rho = \rho\, d\theta\, dz\,\vec{u}_\rho}$

- *When we fix the transversal component, the other components that span the area are* $'\boldsymbol{d\rho}'$ *'and* $'\boldsymbol{\Delta z}'$ *. So, the element of surface is:*

$\Delta\vec{S}_\theta = \Delta\rho\,\Delta z\;\; \boldsymbol{\vec{u}_\theta} \;\Rightarrow$ *The infinitesimal surface is:* $\boldsymbol{d\vec{S}_\theta = d\rho\, dz\,\vec{u}_\theta}$

- *When we fix the axial component (axial direction), the other components that span the area are* $'\boldsymbol{\rho\Delta\theta}'$ *and* $'\boldsymbol{\Delta\rho}'$. *So, the element of surface is:*

$\Delta\vec{S}_z = \boldsymbol{\rho\,\Delta\theta\, d\rho\;\vec{k}} \;\Rightarrow$ *The infinitesimal surface is:* $\boldsymbol{d\vec{S}_\rho = \rho\, d\theta\, d\rho\;\vec{k}}$

4-2-4 SPHERICAL COORDINATES

The variation that produces the elements of surface are: $\boldsymbol{r\Delta\theta}, \boldsymbol{\Delta r}$ *and* $\boldsymbol{r\,sin\theta\Delta\varphi}$

- *When we fix the radial component (radial direction), the other components that span the area are* $'\boldsymbol{r\Delta\theta}'$ *and* $'\boldsymbol{r\,sin\theta\Delta\varphi}'$. *So, the element of surface is:*

$\Delta\vec{S}_r = \boldsymbol{r\,sin\theta\Delta\varphi\, r\Delta\theta\,\vec{u}_r} \;\Rightarrow$ *The infinitesimal surface is:* $\boldsymbol{d\vec{S}_r = r^2\,sin\theta\, d\theta\, d\varphi\,\vec{u}_r}$

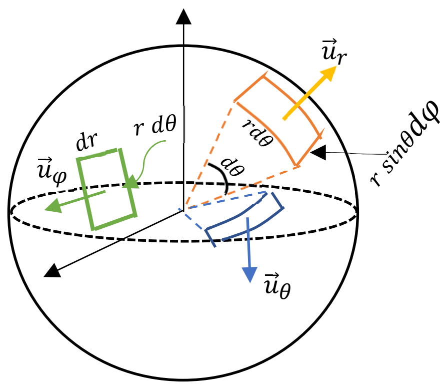


Fig.15

- *When we fix the colatitude component, the other components that span the area are* $'\boldsymbol{\Delta r}'$ *'and* $'\boldsymbol{r\,sin\theta\Delta\varphi}'$ *. So, the element of surface is:*

$\Delta\vec{S}_\theta = \boldsymbol{r\,sin\theta\,\Delta\varphi\,\Delta r\;\vec{u}_\theta} \;\Rightarrow$ *The infinitesimal surface is:* $\boldsymbol{d\vec{S}_\theta = r\,sin\theta dr\, d\varphi\,\vec{u}_\theta}$

- *When we fix the longitude component, the other components that span the area are* $'\boldsymbol{\Delta r}'$ *and* $'\boldsymbol{r\,\Delta\theta}'$. *So, the element of surface is:*

$\Delta\vec{S}_\varphi = \boldsymbol{r\,\Delta\theta\,\Delta r\;\vec{u}_\varphi} \;\Rightarrow$ *The infinitesimal surface is:* $\boldsymbol{d\vec{S}_\varphi = r\, dr d\theta\,\vec{u}_\varphi}$

4-3 ELEMENTARY VOLUME- DIFFERENTIAL VOLUME

The element of volume is a result of product of three variations, like a small cube, in different direction

4-3-1 CARTESIAN COORDINATES

The three displacement that we have seen above for each direction are $\Delta \boldsymbol{x}$, $\Delta \boldsymbol{y}$ *and* $\Delta \boldsymbol{z}$*.So, the element of volume is given by:*

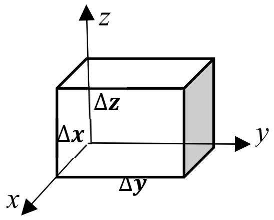


Fig.16

$$\Delta \boldsymbol{V} = \Delta \boldsymbol{x}.\Delta \boldsymbol{y}.\Delta \boldsymbol{z}$$

$\Rightarrow$ *The infinitesimal volume is:* $\boldsymbol{dV} = \boldsymbol{dx}.\boldsymbol{dy}.\boldsymbol{dz}$

4-3-2 CYLINDRICAL COORDINATES

The three displacement that we have seen above for each direction are $\Delta \boldsymbol{r}$, $\boldsymbol{r}\ \Delta \boldsymbol{\theta}$ *and* $\Delta \boldsymbol{z}$*.So, the element of volume is given by*

$$\Delta \boldsymbol{V} = \boldsymbol{r}\ \Delta \boldsymbol{r}\ \Delta \boldsymbol{\theta}\ \Delta \boldsymbol{z}$$

$\Rightarrow$ *The infinitesimal volume is:* $\boldsymbol{dV} = \boldsymbol{r}\ \boldsymbol{dr}\ \boldsymbol{d\theta}\ \boldsymbol{dz}$

Fig.10

4-3-4 SPHERICAL COORDINATES

The three displacement that we have seen above for each direction are $\Delta \boldsymbol{r}$, $\boldsymbol{r}\ \Delta \boldsymbol{\theta}$ *and* $\Delta \boldsymbol{z}$*.So, the element of volume is given by*

$$\Delta \boldsymbol{V} = \boldsymbol{r}\ \boldsymbol{sin\theta}\ \Delta \boldsymbol{r}\ \Delta \boldsymbol{\theta}\ \Delta \boldsymbol{\varphi}$$

$\Rightarrow$ *The infinitesimal volume is:* $\boldsymbol{dV} = \boldsymbol{r}\ \boldsymbol{sin\theta}\ \boldsymbol{dr}\ \boldsymbol{d\theta}\ \Delta \boldsymbol{\varphi}$

4-4 INTEGRALS

4-4-1 SIMPLE INTEGRAL

If we deal with a function of single variable, the integral of that function or the primitive is given by:

$$\int \boldsymbol{f}(\boldsymbol{x})\ \boldsymbol{dx} = \boldsymbol{F}(\boldsymbol{x}) + \boldsymbol{C}$$

With $\boldsymbol{C}$ *is a constant*

and $\frac{\boldsymbol{dF(x)}}{\boldsymbol{dx}} = \boldsymbol{f}(\boldsymbol{x})$

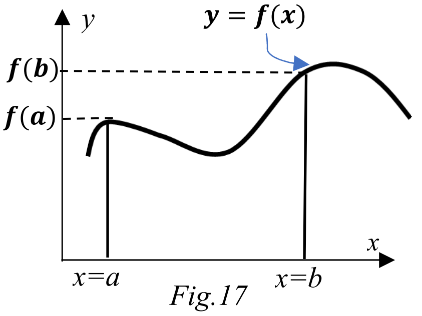


Fig.17

Because the functions $\boldsymbol{F}(\boldsymbol{x})$ *and* $\boldsymbol{F}(\boldsymbol{x}) + \boldsymbol{C}$ *are the integrals for the same function (integrand)* $\boldsymbol{f}(\boldsymbol{x})$*. Thus, for different value of* $\boldsymbol{C}$*, we obtain different integral of* $\boldsymbol{f}(\boldsymbol{x})$*. This implies that this integral is indefinite*

If the <u>boundary</u> is defined, we have the definite integral

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

The geometrical interpretation of this definite integral, it gives the area delimited by the curve and limit of integration. Lower limit $x = a$ *and upper limit* $x = b$

4-4-2 DOUBLE INTEGRAL

If we deal with a function of two variables, we can use the integral for both differential variables dx *and* dy

$$\iint f(x,y)\,dxdy$$

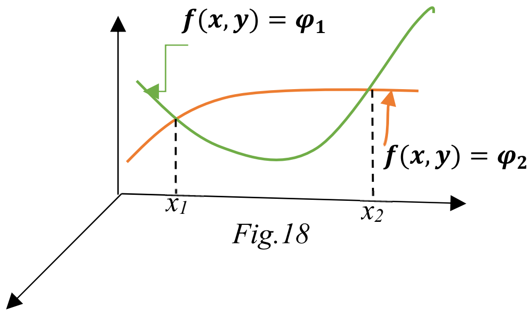


Fig.18

Two cases to report:

When the two variables x *and* y *are not dependent on each other, the double integral can be reduced like to a product of two simple integrals each concern the corresponding variable.*

$$I = \int dy \int f(x,y)\,dx$$

When the two variables x *and* y *are dependent on each other, the double integral can be done for one variable first* $y(x)$ *and then for the second variable.*

If $\varphi_1 \le y \le \varphi_2$, $x_1 \le x \le x_2$ *and then*

$$I = \int_{x_1}^{x_2} \left(\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y) dy \right) dx$$

In certain case it is convenient to reverse the order of integration $\psi_1 \le x \le \psi_2$ *and* $y_1 \le y \le y_2$

$$I = \int_{y_1}^{y_2} \left(\int_{\psi_1(y)}^{\psi_2(y)} f(x,y) dx \right) dy$$

Example

Let the function $f(x,y) = ye^{xy}$ *where* $1 \le x \le 2$ *and* $0 \le y \le 2$

1st case:

$$I_1 = \int_0^2 \left(\int_1^2 ye^{xy} dx \right) dy = \int_0^2 y \frac{e^{xy}}{y} \Big|_1^2 = \int_0^2 (e^{2y} - e^y)\,dy = \frac{1 + e^4 - 2e^2}{2}$$

$$I_2 = \int_1^2 \left(\int_0^2 y e^{xy} dy \right) dx = \int_1^2 \left(y \frac{e^{xy}}{x} - \int_0^2 \frac{e^{xy}}{x} dy \right) dx = \int_1^2 \left(\frac{e^{2x}}{x} - \left[\frac{e^{2x}}{x^2} - \frac{1}{x^2} \right] \right) dx$$

$$I_2 = \int_1^2 \left(\frac{e^{2x}}{x} - \frac{e^{2x}}{x^2} + \frac{1}{x^2} \right) dx$$

Notice that we can calculate the area of a surface using the double integral, also the volume.

4-4-3 TRIPLE INTEGRAL

If we deal with a function of three variables, we can use the integral for both differential variables $\boldsymbol{dx}, \boldsymbol{dy}$ *and* $\boldsymbol{dz}$

$$I = \iiint f(x, y, z)\, dxdydz = \int_{x_1}^{x_2} \left(\int_{\varphi_1(x)}^{\varphi_2(x)} \left(\int_{\psi_1(x,y)}^{\psi_2(x,y)} f(x, y, z) dz \right) dy \right) dx$$

Two cases to report:

When the two variables $\boldsymbol{x}$*,* $\boldsymbol{y}$ *and* $\boldsymbol{z}$ *are not dependent on each other, the triple integral can be reduced like to a product of three simple integrals each concern the corresponding variable without regarding which is first.*

$$I = \int dx . \int dy . \int f(x, y, z)\, dz$$

When the three variables $\boldsymbol{x}$*,* $\boldsymbol{y}$ *and* $\boldsymbol{z}$ *are dependent on each other, the triple integral can be done for one variable first* $\boldsymbol{z(x, y)}$*, then for the second* $\boldsymbol{y(x)}$ *variable and then the third* $\boldsymbol{x}$*.*

Notice: we can compute the volume by using triple integral

4-4-4 PATH INTEGRAL OR LINE INTEGRAL: GRADIENT THEOREM

Line integral, which mean an integral along a curve in space, it deals with a vectorial quantity, and the expression given as follows:

$$\int_A^B \vec{V} \circ d\vec{l}$$

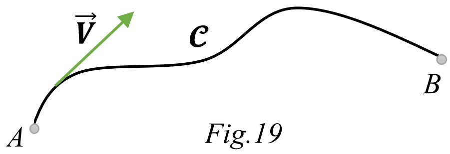


Fig.19

With $\vec{V}(x, y, z)$ *is a function vector and* $d\vec{l}$ *an infinitesimal displacement*

The transport of a vector $\vec{V}$ *along the path AB is given by the line integral.*

The vector $\vec{V}$ *has the components, in cartesian system,* $V_x(x, y, z)$*,* $V_y(x, y, z)$ *and* $V_z(x, y, z)$*. So,* $\vec{V}(x, y, z) = V_x\vec{i} + V_y\vec{j} + V_z\vec{k}$ *and* $d\vec{l} = dx\vec{i} + dy\vec{j} + dz\vec{k}$

$$\int_A^B \vec{V} \circ d\vec{l} = \int_A^B (V_x\, dx + V_y\, dy + V_z\, dz)$$

When the trip is a closed loop the two ends $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *coincide and the integral is to be carried a long that path is said the circulation of the vector* $\vec{\boldsymbol{V}}$*.*

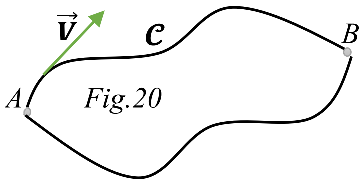


Fig.20

$$Circulation(\vec{\boldsymbol{V}}) = \oint \vec{\boldsymbol{V}} \circ \vec{\boldsymbol{dl}}$$

The line integral, in general, it depends on the path taken from $\boldsymbol{A}$ *to* $\boldsymbol{B}$*. But there is a special case of vector function for which the integral is independent of the path and is determined only by the end points.*

The vector function that has this property is said to be conservative.

If we take a scalar function $\boldsymbol{W(x,y,z)}$ *its variation or differential is given by:*

$$\boldsymbol{dW = \frac{\partial W}{\partial x}dx + \frac{\partial W}{\partial y}dy + \frac{\partial W}{\partial z}dz = \vec{\nabla}W \circ d\vec{l}}$$

So

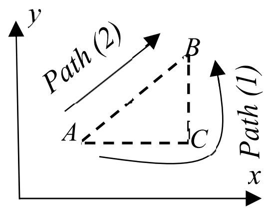
$$\int_{\boldsymbol{A}}^{\boldsymbol{B}} \boldsymbol{dW = W(B) - W(A)} = \int_{\boldsymbol{A}}^{\boldsymbol{B}} \boldsymbol{\vec{\nabla}W \circ d\vec{l}}$$

Which is the fundamental theorem of gradient

Example

Calculate the line integral of the function $\vec{G}(x,y) = y^2\vec{\imath} + 2x(y+1)\vec{\jmath}$*, from point* $\mathrm{A}(1,1,0)$ *to point* $\mathrm{B}(2,2,0)$ *along path (1) and path (2).*

What is the line integral of $\vec{G}(x,y)$ *for a loop that goes from* A *to* B *along path (1) and return to* A *along path (2)?*



<u>Solution</u>

$$d\vec{l} = dx\,\vec{\imath} + dy\,\vec{\jmath} + dz\,\vec{k}$$

<u>PATH (1)</u>

Path (1) consist of 2 pieces AC and CB

$$\int_{\text{Path (1)}} \vec{G} \circ d\vec{l} = \int_{\mathrm{A}}^{\mathrm{C}} \vec{G} \circ d\vec{l} + \int_{\mathrm{C}}^{\mathrm{b}} \vec{G} \circ d\vec{l}$$

Along AC:

$dy = dz = 0$ *and* $y = 1$ *so* $\vec{G} \circ d\vec{\mathrm{l}} = G_x dx = \mathrm{y}^2 dx = dx$

$\Rightarrow \int_{A}^{\mathrm{C}} \vec{G} \circ d\vec{\mathrm{l}} = \int_1^2 \mathrm{dx} = 1$

Along CB:

$\mathrm{dx} = dz = 0$ *and* $\mathrm{x} = 2$ *so* $\vec{G} \circ d\vec{\mathrm{l}} = G_x dx = 2x(y+1)dy = 4(y+1)dy$

$\Rightarrow \int_{\mathrm{C}}^{\mathrm{B}} \vec{G} \circ d\vec{\mathrm{l}} = \int_1^2 4(y+1)dy = 10$

So, along the path (1), line integral of $\vec{G}(x,y)$ *is:* $\int_{\text{Path (1)}} \vec{G} \circ d\vec{l} = 1 + 10 = 11$

PATH (2)

The line joining A to B is given by the equation $y = x \Rightarrow\ dy = dx$ *,* $dz=0$

$$\int_{\text{Path (2)}} \vec{G} \circ d\vec{l} = \int_{\text{Path (2)}} y^2\vec{\imath} + 2x(y+1)\vec{\jmath} \circ (dx\vec{\imath} + dy\vec{\jmath}) = \int_1^2 (3x^2 + 2x)dy = 10$$

Along path (1) the line integral is "11" and along the path (2) is " 10 ". So,

$$\int_{\text{Path (2)}} \vec{G} \circ d\vec{l} \neq \int_{\text{Path (1)}} \vec{G} \circ d\vec{l}$$

This allows us to conclude that the vector function $\vec{G}(x,y)$ *is not conservative.*

Line integral of $\vec{G}(x,y)$ *for a loop ABCA*

$$\oint \vec{G} \circ d\vec{l} = \int_{\text{Path (1)}} \vec{G} \circ d\vec{l} - \int_{\text{Path (2)}} \vec{G} \circ d\vec{l} = 11 - 10 = 1 \neq 0$$

The sign "$-$*" in the second term of righthand side, is due to parkouring the path in the reverse way.*

We conclude that; for a conservative vector function the line integral along a path which form a loop is null. Or the line integral is independent of the path between two points A and B

$$\oint \vec{G} \circ d\vec{\mathrm{l}} = 0 \quad \Leftrightarrow \quad \int_{\text{Path (1)}} \vec{G} \circ d\vec{\mathrm{l}} = \int_{\text{Path (2)}} \vec{G} \circ d\vec{\mathrm{l}}$$

4-4-5 SURFACE INTEGRAL: DIVERGENCE THEOREM GAUSS THEOREM (GREEN THEOREM)

Let a path (L) which is a loop that delimit the surface "$\vec{\mathrm{S}}$*", and let the vector function* $\vec{A} = A_x\vec{\imath} + A_y\vec{\jmath} + A_z\vec{k}$*.*

The surface integral of the function $\vec{A}(x,y,z)$ *over the surface* $\vec{\mathrm{S}}$ *delimited by the contour "*C*" is defined as follows:*

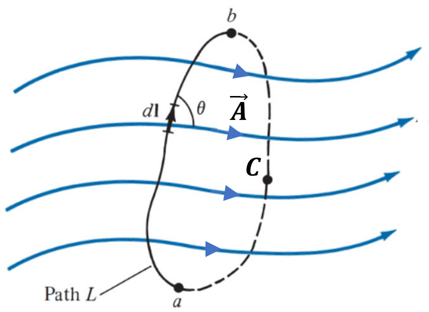


Fig.21-a

$$\iint \vec{\boldsymbol{A}} \circ \boldsymbol{d\vec{s}}$$

When then the surface is closed the integral is written as:

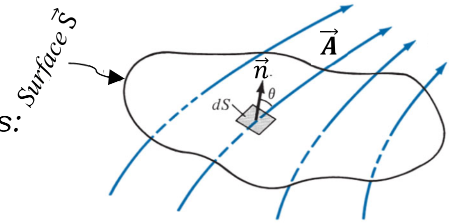


Fig.21- b

$$\oiint \vec{\boldsymbol{A}} \circ \boldsymbol{d\vec{s}}$$

$\vec{\boldsymbol{n}}$*: is the normal to the closed surface, which is always outward, but for the open surface, the direction is defined by right-hand rule when we follow the contour of that surface.*

The opened surface is bounded by the closed path (loop), whereas the closed surface defines a volume.

The flow of a field $\vec{\boldsymbol{A}}$ through the surface '$\boldsymbol{S}$' is called flux. The divergence theorem (Green's theorem or Gauss's Theorem) says that the flux of a vector field through a closed surface is equal to the volume integral of the divergence of that field.

$$\oiint \vec{\boldsymbol{A}} \circ \boldsymbol{d\vec{s}} = \iiint \vec{\boldsymbol{\nabla}} \circ \vec{\boldsymbol{A}}\ \boldsymbol{dv}$$

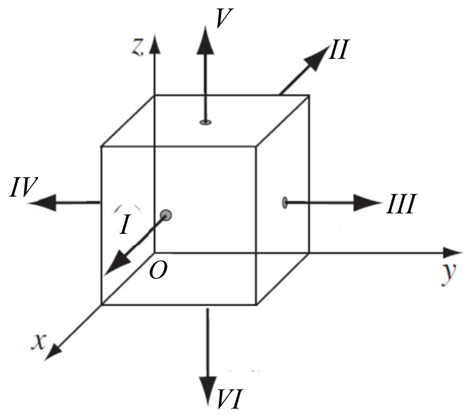
Example

Check the divergence theorem using the function $\vec{\boldsymbol{A}} = \boldsymbol{y^2}\ \vec{\boldsymbol{\imath}} + (\boldsymbol{2xy} + \boldsymbol{z^2})\ \vec{\boldsymbol{\jmath}} + \boldsymbol{2yz}\ \vec{\boldsymbol{k}}$ *and a unit cube at the origin.*

<u>*Solution*</u>

The six face that make a closed surface are:

Faces I, II, III, IV, V, VI



$$\vec{\boldsymbol{\nabla}} \circ \vec{\boldsymbol{A}} = \left(\frac{\partial}{\partial x}\vec{\imath} + \frac{\partial}{\partial x}\vec{\jmath} + \frac{\partial}{\partial x}\vec{k}\right) \circ \left(y^2\ \vec{\imath} + (2xy + z^2)\ \vec{\jmath} + 2yz\ \vec{k}\right)$$

$$\vec{\boldsymbol{\nabla}} \circ \vec{\boldsymbol{V}} = \boldsymbol{2(x + y)}$$

$$\Rightarrow \quad \iiint \vec{\boldsymbol{\nabla}} \circ \vec{\boldsymbol{A}}\ \boldsymbol{dv} = \int_0^1 \int_0^1 \int_0^1 \boldsymbol{2(x + y)\ dx\ dy\ dz}$$

$$\boldsymbol{dv} = \boldsymbol{dx\ dy\ dz} \qquad 0 \leq \boldsymbol{x} \leq 1;\ \ 0 \leq \boldsymbol{y} \leq 1;\ \ 0 \leq \boldsymbol{z} \leq 1$$

$$\iiint \vec{\boldsymbol{\nabla}} \circ \vec{\boldsymbol{A}}\ \boldsymbol{dv} = \int_0^1 \left(\int_0^1 \left(\int_0^1 \boldsymbol{2(x + y)\ dx}\right) \boldsymbol{dy}\right) \boldsymbol{dz} = 2$$

The right-hand side of the theorem

$$\oiint \vec{\boldsymbol{A}} \circ \boldsymbol{d\vec{s}} = \underbrace{\int_0^1 \int_0^1 \vec{\boldsymbol{A}} \circ \boldsymbol{d\vec{s}}}_{\text{Surface I}} + \underbrace{\int_0^1 \int_0^1 \vec{\boldsymbol{A}} \circ \boldsymbol{d\vec{s}}}_{\text{Surface II}} + \underbrace{\int_0^1 \int_0^1 \vec{\boldsymbol{A}} \circ \boldsymbol{d\vec{s}}}_{\text{Surface III}} + \underbrace{\int_0^1 \int_0^1 \vec{\boldsymbol{A}} \circ \boldsymbol{d\vec{s}}}_{\text{Surface IV}} + \underbrace{\int_0^1 \int_0^1 \vec{\boldsymbol{A}} \circ \boldsymbol{d\vec{s}}}_{\text{Surface V}} + \underbrace{\int_0^1 \int_0^1 \vec{\boldsymbol{A}} \circ \boldsymbol{d\vec{s}}}_{\text{Surface VI}}$$

$d\vec{s}_I = dy\ dz\ \vec{\imath},\ d\vec{s}_{II} = -dy\ dz\ \vec{\imath}\ ;$

$d\vec{s}_{III} = dx\ dz\ \vec{\jmath};\ \ d\vec{s}_{IV} = -dx\ dz\ \vec{\jmath};$

$d\vec{s}_{V} = dx\ dy\ \vec{k}\ ;\ d\vec{s}_{VI} = -dx\ dy\ \vec{k}$

$$\oiint \vec{A} \circ d\vec{s} = \int_0^1 \int_0^1 y^2 dydz - \int_0^1 \int_0^1 y^2 dydz + \int_0^1 \int_0^1 (2x + z^2)\ dxdz - \int_0^1 \int_0^1 z^2 dxdz + \int_0^1 \int_0^1 2y\ dxdy + \int_0^1 \int_0^1 0dxdy$$

Finally, the total flux is

$$\oiint \vec{\boldsymbol{A}} \circ \boldsymbol{d\vec{s}} = 2$$

Fig.10

4-4-6 CURL THEOREM: STOKES THEOREM

This theorem states that the integral of a curl of a vector function $\vec{\boldsymbol{A}}$ *equals the integral of that function over the boundary of that surface, i.e., the line integral a long the closed path that delimits the surface*

$$\oint \vec{\boldsymbol{A}} \circ \boldsymbol{d}\vec{\boldsymbol{l}} = \iint \left(\vec{\boldsymbol{\nabla}} \wedge \vec{\boldsymbol{A}}\right) \circ \boldsymbol{d}\vec{\boldsymbol{s}}$$