Introduction to Probability

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Objectives

The objective of this document is to provide a comprehensive understanding of the basic principles and advanced applications of combinatorial counting techniques. Readers will be introduced to fundamental principles such as basic counting principles, permutations, and combinations. Subsequently, further examples and explanations will be provided to deepen these concepts. Finally, the multinomial coefficient will be introduced and discussed.

Introduction

Combinatorics is a branch of mathematics that deals with counting, arrangement, and combination of elements within mathematical structures. Combinatorial techniques are widely used in various fields, including mathematics, computer science, statistics, and engineering. This document aims to provide a solid foundation in combinatorics, starting with basic counting principles and progressing to more complex topics such as the multinomial coefficient.

Importance of Combinatorics

Combinatorics is not just about counting objects; it is about understanding the structure and properties of discrete systems. This understanding is crucial in fields such as cryptography, where secure communication relies on combinatorial algorithms, and in operations research, where optimization problems often require combinatorial solutions. In computer science, combinatorial algorithms are at the heart of search engines, data mining, and artificial intelligence.

Moreover, combinatorics plays a pivotal role in statistical modeling and probability theory. For instance, the study of random variables and their distributions often involves combinatorial methods to compute probabilities and expectations. Understanding these methods allows statisticians to model real-world phenomena more accurately.

Learning Outcomes

By the end of this document, readers should be able to:

- Understand and apply basic counting principles.
- Differentiate between permutations and combinations and solve problems involving both.
- Utilize generalized counting principles for complex scenarios.
- Compute and interpret multinomial coefficients.
- Apply combinatorial techniques to practical problems in various fields.

Combinatorics

1. Basic counting principle

The first fundamental rule of counting is to multiply.

In other words, if there are n possible results from one action and m possible outcomes from another, then there are $m \cdot n$ possible outcomes from carrying out both acts.

Basic counting principle *Definition*

Suppose that two experiments are to be performed. Then if experiment 1 can result in any one of m possible outcomes, and if for each outcome of experiment 1 there are n possible outcomes of experiment 2, then there are m*n possible outcomes of the two experiments together.

Example

Suppose we have 4 shirts of 4 different colors and 3 pants of different colors. How many different outfits are there? For each shirt there are 3 different colors of pants, so altogether there are $4 \times 3 = 12$ possibilities.

Example

How many different license plate numbers with 3 letters followed by 3 numbers are possible?

Solution: $(26)^3\ (10)^3$. Indeed, the English alphabet has 26 different letters, therefore there are 26 possibilities for the first place, 26 for the second, 26 for the third, 10 for the fourth, 10 for the fifth, and 10 for the sixth. We multiply.

2. Permutations

How many ways can one arrange letters a, b, c? We can list all possibilities, namely,

abc acb bac bca cab cba.

There are 3 possibilities for the first position. Once we have chosen the letter in the first position, there are 2 possibilities for the second position, and once we have chosen the first two letters, there is only 1 choice left for the third. So there are $3 \times 2 \times 1 = 6 = 3!$ arrangements. In general, if there are n distinct letters, there are n! dierent arrangements of these letters

Example

What is the number of possible batting orders (in baseball) with 9 players?

Solution: $9! = 362880$.

In permutations the order does matter as is illustrated by the next example.

Example

How many ways can one arrange the letters a, a, b, c? Let us label them first as A, a, b, c. There are 4! = 24 ways to arrange these letters.

But we have repeats: we could have Aa or aA which are the same. So we have a repeat for each possibility, and so the answer should be $\frac{4!}{2!} = 12$. If there were 3 as, 4 bs, and 2 cs, we would have

$\frac{9!}{3!4!2!} = 1260.$

Permutations *Definition*

The number of permutations of n objects is equal to $n! := 1 \cdots n$, with the usual convention$0! = 1$.

3. Combinations.

Now let us look at what are known as combinations.

Example

How many ways can we choose 3 letters out of 5? If the letters are a, b, c, d, e and order matters, then there would be 5 choices for the first position, 4 for the second, and 3 for the third, for a total of 5 × 4 × 3. Suppose now the letters selected were a, b, c. If order does not matter, in our counting we will have the letters a, b, c six times, because there are 3! ways of arranging three letters. The same is true for any choice of three letters. So we should have

$\frac{5\times4\times3}{3!}$. We can rewrite this as

$\frac{5\cdot4\cdot3}{3!} = \frac{5!}{3!2!} = 10$

This is often written as $\binom{5}{3}$, read 5 choose 3. Sometimes this is written C_5^3

or $5C_3$.

Combinations (binomial coefficients) *Definition*

The number of different groups of k objects chosen from a total of n objects is equal to

$\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Note that this is true when the order of selection is irrelevant, and if the order of selection is relevant, then there are

$n \cdot (n-1) \cdot \ldots \cdot (n-k+1) = \frac{n!}{(n-k)!}$

ways of choosing k objects out of

Example

Suppose there are 8 men and 8 women. How many ways can we choose a committee that has 2 men and 2 women?

Solution: we can choose 2 men $\binom{8}{2}$ ways and 2 woman in $\binom{8}{2}$ ways.

The number of possible committees is then the product

$\$\binom{8}{2} * \binom{8}{2} = 28 * 28 = 784.\$$

Example

For any $k \leq n$ we have

$\binom{n}{k} = \binom{n}{n-k}$

k . Indeed, the left-hand side gives the number of different groups of k objects chosen from a total of n objects which is the same to choose n − k objects not to be in the group of k objects which is the number on the right-h

Combinations (multinomial coecients) **Definition**

The number of ways to divide n objects into one group of n1 objects, one group of n2, . . ., and a kth group of n_k objects, where $n = n1 + \cdots + n_k$, is equal to

$\binom{n}{n_1 \cdots n_k} = \frac{n!}{n_1! n_2! \cdots n_k!}$

Example

Suppose we have 4 Americans and 6 Canadians.

- (a) How many ways can we arrange them in a line?
- (b) How many ways if all the Americans have to stand together?
- (c) How many ways if not all the Americans are together?
- (d) Suppose you want to choose a committee of 3, which will be all Americans or all Canadians.

  How many ways can this be done?
- (e) How many ways for a committee of 3 that is not all Americans or all Canadians

Solution:

- (a) This is just the number of arrangements of 10 elements, that is, 10!
- (b) Consider the Americans as one group (element) and each Canadian as a distinct group (6 elements); this gives 7 distinct groups (elements) to be arranged, which can be done in 7! ways. Once we have these seven groups arranged, we can arrange the Americans within their group in 4! ways, so we get 4!7! by the basic counting principle.
- (c) This is the answer to (a) minus the answer to (b): 10! − 4!7!
- (d) We can choose a committee of 3 Americans in $\binom{4}{3}$ ways and and a committee of 3 Canadians in $\binom{6}{3}$ ways, so the answer is $\binom{4}{3} + \binom{6}{3}$.
- (e) We can choose a committee of 3 out of 10 in $\binom{10}{3}$ ways, so the answer is $\binom{10}{3} - \binom{4}{3} - \binom{6}{3}$.

  finally,Finally, we consider three interrelated.

Further examples and explanations

II

1. Introduction

This section will provide additional examples and explanations to solidify the understanding of the concepts introduced in the previous section.

2. Generalized counting principle

Here we expand on the basic counting principle formulated in Section1 ,One can visualize this principle by using the box method below. Suppose we have two experiments to be performed, namely, one experiment can result in n outcomes, and the second experiment can result in m outcomes. Each box represents the number of possible outcomes in that experiment.

$$\boxed{\text{Experiment 1}} \qquad \boxed{\text{Experiment 2}} = \boxed{\text{Experiment 1 and 2 together}}$$

$$\boxed{\mathbf{m}} \qquad \boxed{\mathbf{n}} = \boxed{\mathbf{mn}}$$

Example

There are 20 teachers and 100 students in a school. How many ways can we pick a teacher and student of the year?

Solution: using the box method we get 20 × 100 = 2000

Generalized counting principle ***Definition***

Suppose that k experiments are to be performed, with the number of possible outcomes being ni for the **i**th experiment. Then there are

$n1 \cdot \cdots \cdot nk$

possible outcomes of all k experiments together.

Example

How many functions defined on k points are possible if each function can take values as either 0 or 1.

Solution: the counting principle or the box method on the 1, . . . , k points gives us 2^k possible functions. This is the generalized counting principle with n1 = n2 = ... = nk = 2.

3. Permutations

Now we give more examples on permutations, and we start with a more general results on the number of possible permutations.

Permutations revisited *Definition*

The number of different permutations of n objects of which n1 are alike, n2 are alike, ..., n2 are alike is equal to

$\frac{n!}{n1!\cdots nr!}$

Example

How many ways can one arrange the letters a, a, b, b, c, c?

Solution: let us rst re-label the letters by A, a, B, b, C, c. Then there are 6! = 720 ways to arrange these letters.

But we have repeats (for example, Aa or aA) which produce the same arrangement for the original letters.

So dividing by the number of repeats for A, a, B, b and C, c, so the answer is $\frac{6!}{(2!)^3} = 90.$

Example

How many dierent letter arrangements can be formed from the word PEPPER? Solution: There are three copies of P and two copies of E, and one of R. So the answer is\frac{ 6! }{3!2!1!} = 60.

4. Combinations

Below are more examples on combintations

Example

Suppose there are 9 men and 8 women. How many ways can we choose a committee that has 2 men and 3 women?

Solution: We can choose 2 men in $\binom{9}{2}$ ways and 3 women in $\binom{8}{3}$ ways.

The number of committees is then the product

$\binom{9}{2} * \binom{8}{3}$

Example

Suppose somebody has n friends, of whom k are to be invited to a meeting.

1. How many choices do exist for such a meeting if two of the friends will not attend together?
2. How many choices do exist if 2 of the friends will only attend together?

Solution:

1. We can divide all possible groups into two (disjoint) parts: one is for groups of friends none of which are these two, and another which includes exactly one of these two friends. There are $\binom{n-2}{k}$ groups in the rst part, and $\binom{n-2}{k-1}$ in the second. For the latter we also need to account for a choice of one out of these two incompatible friends. So altogether we have $\binom{n-2}{k} + \binom{2}{1} \cdot \binom{n-2}{k-1}$
2. Again, we split all possible groups into two parts: one for groups which have none of the two inseparable friends, and the other for groups which include both of these two friends. Then $\binom{n-2}{k} + 1 \cdot 1 \cdot \binom{n-2}{k-2}$

Theorem: **The binomial theorem** — *Fundamental*

$(x+y)n = \sum_{k=0}^{n} = \binom{n}{k} x^k y^{n-k}$

Example

We can use combinatorics to show that $\binom{10}{4} = \binom{9}{3} + \binom{9}{4}$ without evaluating these expressions explicitly.

Solution: the left-hand side represents the number of committees consisting of 4 people out of the group of 10 people.

$$\binom{0}{0}$$
$$\binom{1}{0} \quad \binom{1}{1}$$
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$
$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$
$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$
$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$
$$\binom{6}{0} \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad \binom{6}{6}$$
$$\binom{7}{0} \quad \binom{7}{1} \quad \binom{7}{2} \quad \binom{7}{3} \quad \binom{7}{4} \quad \binom{7}{5} \quad \binom{7}{6} \quad \binom{7}{7}$$

Pascal's triangle

Now we would like to represent the right-hand side. Let's say Tom Brady is one these ten people, and he might be in one of these committees and he is special, so we want to know when he will be there or not. When he is in the committee of 4, then there are $1 \cdot 9\ 3$ number of ways of having a committee with Tom Brady as a member, while 9 4 is the number of committees that do not have Tom Brady as a member. Adding it up gives us the number of commit

5. Multinomial Coefficient

Example

Suppose we are to assign 10 police officers: 6 patrols, 2 in station, 2 in schools. Then there are $\frac{10!}{6!2!2!}$ different assignments.

Example

We have 10 flags: 5 of them are blue, 3 are red, and 2 are yellow. These flags are indistinguishable, except for their color. How many different ways can we order them on a flag pole ?

Solution: $\frac{10!}{5!3!2!}$

Exercises and Solutions

1. Execises

The following exercises are designed to reinforce and expand your understanding of combinatorial principles and techniques. Combinatorics, with its roots in counting and arrangement, provides essential tools for solving problems in various fields such as mathematics, computer science, statistics, and engineering. By working through these exercises, you will develop a deeper appreciation for the methods used to count, arrange, and combine elements within different contexts.

Exercice1:

Suppose a license plate must consist of 7 numbers or letters. How many license plates are there if

- (A) there can only be letters?
- (B) the rst three places are numbers and the last four are letters?
- (C) the rst three places are numbers and the last four are letters, but there can not be any repetitions in the same license plate?

Exercice2:

A school of 50 students has awards for the top math, English, history and science student in the school

(A) How many ways can these awards be given if each student can only win one award?

(B) How many ways can these awards be given if students can win multiple awards?

Exercice 3:

A password can be made up of any 4 digit combination.

(A) How many different passwords are possible?

(B) How many are possible if all the digits are odd?

(C) How many can be made in which all digits are different or all digits are the same?

Exercice 4:

There is a school class of 25 people made up of 11 guys and 14 girls.

(A) How many ways are there to make a committee of 5 people?

(B) How many ways are there to pick a committee of all girls?

(C) How many ways are there to pick a committee of 3 girls and 2 guys?

Exercise*5* .Use the binomial theorem to show that

$\sum_k = 0^n \binom{n}{k} = 2^n,$

$\sum_k = 0^n (-1)^k \binom{n}{k} = 0$

Exercise6:

Prove the multinomial theorem

$(x_1 + \ldots + x_k)^n = \sum_{\substack{n_1+\ldots+n_k=n \\ n_1,\ldots,n_k\geq 0}} \binom{n}{n_1,\ldots,n_k} x_1^{n_1} \cdot \ldots \cdot x_k^{n_k}$

2. Solutions

Solution1

- A- in each of the seven places we can put any of the 26 letters giving 26^7 possible letter combinations.
- B-in each of the first three places we can place any of the 10 digits, and in each of the last four places we can put any of the 26 letters giving a total of $10^3 \cdot 26^4$.
- C- if we can not repeat a letter or a number on a license plate, then the number of license plates becomes (10 · 9 · 8) · (·26 · 25 · 24 · 23).

Solution2

- A- 50 · 49 · 48 · 47
- B- 50^4

Solution2

- A- 50 · 49 · 48 · 47
- B- 50^4

Solution3

- A- 10^4
- B- 5^4
- C- 10 · 9 · 8 · 7 + 10

Solution4

- A-$\binom{25}{5}$
- B-$\binom{14}{5}$
- C-$\binom{14}{3} \cdot \binom{11}{2}$

Solution1*

use the binomial formula $(x + y^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$ with $x = y = 1$ to see $2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{n-k} = \sum_{k=0}^{n} \binom{n}{k}$.

with x = −1, y = 1 $0 = (-1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{n-k} = \sum_{k=0}^{n} \binom{n}{k}$

Solution2*

we can prove the statement using mathematical induction on k. For k = 1

we have$(x_1)^n = \sum_{n1=n} \binom{n}{n_1} = x_1 = x_1^n$

which is true; for k = 2

we have

$\sum_{\substack{(n_1,n_2) \\ n_1+n_2=n}} \binom{n}{n_1,n_2} x_1^{n_1} \cdot \ldots \cdot x_k^{n_k}$

which is the binomial formula itself.

Now suppose the multinomial formula holds for k = K (induction hypothesis), that is,

$$(x_1 + \ldots + x_K)^n = \sum_{\substack{(n_1,\ldots,n_K) \\ n_1+\ldots+n_K=n}} \binom{n}{n_1,\ldots,n_K} x_1^{n_1} \cdot \ldots \cdot x_K^{n_K}$$

And we need to show

Denote

$$y_1 = x_1, .., y_{K-1} := x_{K-1}, y_K := x_K + x_{K+1}$$

then by the induction hypothesis

$$\begin{aligned}(x_1 + \ldots + x_{K+1})^n = (y_1 + \ldots + y_K)^n \\ &= \sum_{\substack{(n_1,\ldots,n_K) \\ n_1+\ldots+n_K=n}} \binom{n}{n_1,\ldots,n_K} y_1^{n_1} \cdot \ldots \cdot y_{K-1}^{n_{K-1}} \cdot y_K^{n_K} \\ &= \sum_{\substack{(n_1,\ldots,n_K) \\ n_1+\ldots+n_K=n}} \binom{n}{n_1,\ldots,n_K} x_1^{n_1} \cdot \ldots \cdot x_{K-1}^{n_{K-1}} \cdot (x_K + x_{K+1})^{n_K}\end{aligned}$$

By the binomial formula

$$(x_K + x_{K+1})^{n_K} = \sum_{n_K}^{m=1} \binom{n_K}{m} x_m^K \cdot x_n$$

$$(x_1 + \ldots + x_{K+1})^n = \sum_{\substack{(n_1,\ldots,n_K) \\ n_1+\ldots+n_K=n}} \binom{n}{n_1,\ldots,n_K} x_1^{n_1} \cdots x_{K-1}^{n_{K-1}} \left(\sum_{n_K}^{m=1} \binom{n_K}{m} x_m^K \cdot x_{n_K-m}^{K+1} \right)$$

It is easy to see (using the definition of multinomial coefficients)

$$\binom{n}{n_1,\ldots,n_K} \binom{n_K}{m} = \binom{n}{n_1,\ldots,n_K,m}, \quad n_1 + \ldots + n_K + m = n.$$

Indeed,

$$\binom{n}{n_1,\ldots,n_K} \binom{n_K}{m} = \frac{n!}{n_1!n_2!\cdots n_{K-1}!\cdot n_K!} \frac{n_K!}{m!(n_K-m)!} = \frac{n!}{n_1!n_2!\cdots n_{K-1}!\cdot m!(n_K-m)!} = \binom{n}{n_1,\ldots,n_K,m}.$$

Thus,

$$(x_1 + \ldots + x_{K+1})^n = \sum_{\substack{(n_1,\ldots,n_K) \\ n_1+\ldots+n_K=n}} \binom{n}{n_1,\ldots,n_K} \sum_{n_K}^{m=1} \binom{n_K}{m} x_{n_1}^1 \cdots x_{n_{K-1}}^{K-1} x_m^K x_{n_K-m}^{K+1}$$

Note that

$$(x_1 + \ldots + x_{K+1})^n = \sum_{(m_1,\ldots,m_K,m_{K+1}) m_1+\ldots+m_{K+1}=n} \binom{n}{m_1,\ldots,m_K,m_{K+1}} \times x_{m_1}^1 \cdots x_{m_{K-1}}^{K-1} x_{m_K}^K x_{m_{K+1}}^{K+1},$$

which is what we wanted.

Exam

1. Introduction

Combinatorics Final Exam Instructions: Answer all questions. Show all your work for full credit. You may use a calculator, but no other aids are allowed. The total time for the exam is 1 hour and 30 minutes.

2. Exam

2.1. Exercise 1: Permutations and Combinations

1. **(3 points)** A company has 8 distinct projects and wants to assign 4 of them to a team of 4 employees such that each employee gets exactly one project.

   a. **(1.5 points)** In how many different ways can the projects be assigned to the employees if the order in which the projects are assigned matters?

   b. **(1.5 points)** In how many different ways can the projects be assigned if the order in which the projects are assigned does not matter?

2.2. Exercise 2: Multinomial Coefficient

1. **(3 points)** A classroom has 12 students. The teacher wants to form three groups for a group project: one group with 5 students, one group with 4 students, and one group with 3 students.

   a. **(1.5 points)** In how many different ways can the teacher assign the students to these three groups?

   b. **(1.5 points)** If the teacher wants to designate one of the groups as the "presentation group" and the other two as "discussion groups," how many different ways can the assignments be made?

2.3. Exercise 3: Advanced Counting Principles

1. **(4 points)** A box contains 10 red balls, 7 blue balls, and 5 green balls. In how many ways can 8 balls be chosen such that there are at least 2 balls of each color?

   a. **(2 points)** Calculate the number of ways to choose 8 balls with at least 2 red, 2 blue, and 2 green balls.

   b. **(2 points)** If exactly 2 red balls must be chosen, how many different ways can 8 balls be selected from the box?

Solutions Outline

1. Exercise 1: Permutations and Combinations

1. a. **Solution:**
   - The number of ways to assign 4 projects to 4 employees (where order matters) is P(8,4)=8!(8−4)!=8×7×6×5=1680P(8,4)=(8−4)!8!=8×7×6×5=1680.

   b. **Solution:**
   - The number of ways to choose 4 projects from 8 (where order does not matter) is (84)×4!=8!4!(8−4)!×4!=8!4!4!×24=70×24=1680(48)×4!=4!(8−4)!8!×4!=4!4!8!×24=70×24=1680.

2. Exercise 2: Multinomial Coefficient

1. a. **Solution:**
   - The number of ways to divide 12 students into groups of 5, 4, and 3 is 12!5!4!3!=27,7205!4!3!12!=27,720.

   b. **Solution:**
   - If one group is designated as the "presentation group," then the number of ways is 3×12!5!4!3!=3×27,720=83,1603×5!4!3!12!=3×27,720=83,160.

3. Exercise 3: Advanced Counting Principles

1. a. **Solution:**
   - Use combinations to ensure at least 2 balls of each color and sum over valid combinations for the remaining 2 balls.

   b. **Solution:**
   - Fix 2 red balls and calculate combinations for choosing the remaining 6 balls from the remaining blue and green balls ensuring conditions.

Conclusion

Combinatorics is a fundamental area of mathematics that provides essential tools for counting, arranging, and combining elements. Through the exercises provided in this document, you have reinforced your understanding of basic counting principles, permutations, and combinations, and explored more advanced topics such as generalized counting principles and the multinomial coefficient.

By mastering these concepts, you are now better equipped to tackle a wide range of combinatorial problems in both theoretical and applied contexts. The skills you have developed through these exercises will be invaluable in fields such as computer science, statistics, operations research, and beyond.

Combinatorics not only enhances your problem-solving abilities but also opens up new avenues for research and application in various disciplines. Continue to practice and explore the vast world of combinatorial mathematics, and apply these techniques to solve complex problems and innovate in your chosen field.

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