Numerical Analysis 1 (Chapter 2)

Dr :Kamel BENYETTOU

National Higher School of Mathematics,

Scientic and Technology Hub of Sidi Abdellah,

P.O. Box 75, Algier 16093, Algeria.

Email : kamel.benyettou@nhsm.edu.dz[1]

04/02/2024

National Higher school of mathematics
K. BENYETTOU

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Chapter2 : Numerical resolution ofan algebraic equation

1. Objectives

objectives

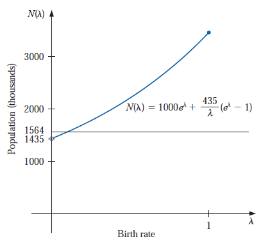
In summary, the main objective of numerically solving an algebraic equation is to obtain accurate, efficient and reliable solutions that meet the requirements of the problem or application at hand.

2. Problem statement

The growth of a population can often be modeled over short periods of time by assuming that the population grows continuously with time at a rate proportional to the number present at that time. Suppose that $N(t)$ denotes the number in the population at time t and λ denotes the constant birth rate of the population. Then the population satisfies the differential equation

$$\frac{dN(t)}{dt} = \lambda N(t),$$

whose solution is $N(t) = N_0 e^{\lambda t}$,where N_0 denotes the initial population.



This exponential model is valid only when the population is isolated, with no immigration. If immigration is permitted at a constant rate v, then the differential equation becomes

$$\frac{dN(t)}{dt} = \lambda N(t) + v,$$

whose solution is

$$N(t) = N_0 e^{\lambda t} + \frac{v}{\lambda}\left(e^{\lambda t} - 1\right).$$

Suppose a certain population contains N(0) = 1,000,000 individuals initially, that 435,000 individuals immigrate into the community in the first year, and that N(1) =1,564,000 individuals are present at the end of one year. To determine the birth rate of this population, we need to findλ in the equation

$$1,564,000 = 1,000,000 e^{\lambda} + \frac{435,000}{\lambda}\left(e^{\lambda} - 1\right).$$

It is not possible to solve explicitly for λ in this equation, but numerical methods discussed in this chapter can be used to approximate solutions of equations of this type to an arbitrarily high accuracy.

3. The Bisection Method

In this chapter, we consider one of the most basic problems of numerical approximation, the root-finding problem. This process involves finding a root, or solution, of an equation of the form $f(x) = 0$, for a given function f . A root of this equation is also called a zero of the function f .

The problem of finding an approximation to the root of an equation can be traced back at least as far as 1700 B.C.E. A cuneiform table in the Yale Babylonian Collection dating from that period gives a sexigesimal (base-60) number equivalent to 1.414222 as an approximation to $\sqrt{2}$, a result that is accurate to within 10^{-5}.

As its name implies, to "bi-sect" is to divide (in this case an interval) into two (in this case even) parts.

In computer science, the process of dividing a set continually in half to search for the solution to a problem, as the bisection method does, is known as a binary search procedure.*Aho*[*Aho* p.30]

Desciption *Méthode*

The first technique, based on the Intermediate Value Theorem, is called the Bisection method. Suppose f is a continuous function defined on the interval [a, b], with f (a) and f (b) of opposite sign. By the Intermediate Value Theorem, there exists a number p in (a, b) with f (p) = 0. Although the procedure will work when there is more than one root in the interval (a, b), we assume for simplicity that the root in this interval is unique. The method calls for a repeated halving of subintervals of [a, b] and, at each step, locating the half containing p.*Brown, K. M., A quadratically convergent Newton-like method based upon Gaussian elimination, SIAM Journal on Numerical Analysis 6, No. 4 (1969), 560–569, QA297.A1S2 652*[*Brown, K. M., A quadratically convergent Newton-like method based upon Gaussian elimination, SIAM Journal on Numerical Analysis 6, No. 4 (1969), 560–569, QA297.A1S2 652* p.28]

To begin, set $a_1 = a$and $b_1 = b$, and let p_1 be the midpoint of $[a, b]$; that is,

$$p_1 = a_1 + \frac{b_1 - a_1}{2} = \frac{a_1 + b_1}{2}.$$

If $f(p_1) = 0$, then $p = p_1$, and we are done. If $f(p_1) \neq 0$, then $f(p_1)$ has the same sign as either $f(a_1)$ or $f(b_1)$. When $f(p_1)$ and $f(a_1)$ have the same sign, $p \in (p_1, b_1)$, and we set $a_2 = p_1$ and $b_2 = b_1$. When $f(p_1)$ and $f(a_1)$ have opposite signs, $p \in (a_1, p_1)$, and we set $a_2 = a_1$ and $b_2 = p_1$. We then reapply the process to the interval $[a_2, b_2]$. This produces the method described in Algorithm 2.1. (See Figure 2.1.)

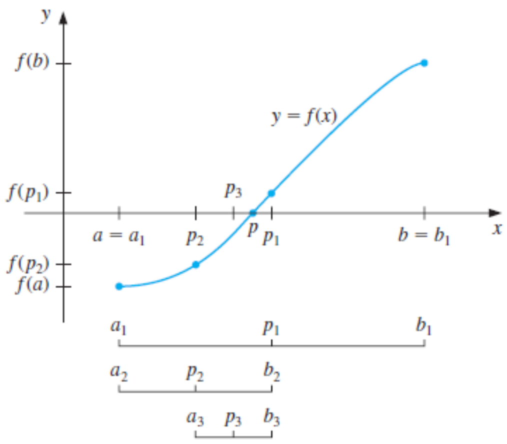


Figure 2.1

ALGORITHM 2.1

Bisection

To find a solution to $f(x) = 0$ given the continuous function f on the interval $[a, b]$, where $f(a)$ and $f(b)$ have opposite signs:

INPUT endpoints a, b; tolerance TOL; maximum number of iterations N_0.

OUTPUT approximate solution p or message of failure.

Step 1 Set $i = 1$;
$FA = f(a)$.

Step 2 While $i \leq N_0$ do Steps 3–6.

Step 3 Set $p = a + (b - a)/2$; (*Compute* p_i.)
$FP = f(p)$.

Step 4 If $FP = 0$ or $(b - a)/2 < TOL$ then
OUTPUT (p); (*Procedure completed successfully.*)
STOP.

Step 5 Set $i = i + 1$.

Step 6 If $FA \cdot FP > 0$ then set $a = p$; (*Compute* a_i, b_i.)
$FA = FP$
else set $b = p$.

Step 7 OUTPUT ('Method failed after N_0 iterations, N_0 =', N_0);
(*The procedure was unsuccessful.*)
STOP.

Algorithm 2.1

Other stopping procedures can be applied in Step 4 of Algorithm 2.1 or in any of the iterative techniques in this chapter. For example, we can select a tolerance $\varepsilon > 0$and generate $p_1, \ldots, p_N$ until one of the following conditions is met:

$$|p_N - p_{N-1}| < \varepsilon,$$
$$\frac{|p_N - p_{N-1}|}{|p_N|} < \varepsilon, \quad p_N \neq 0, \quad \text{or}$$
$$|f(p_N)| < \varepsilon.$$

Unfortunately, difficulties can arise using any of these stopping criteria. For example, sequences $pn_n^\infty = 0$ with the property that the differences $p_n - p_{n-1}$ can converge to zero while the sequence itself diverges.

When using a computer to generate approximations, it is good practice to set an upper bound on the number of iterations. This will eliminate the possibility of entering an infinite loop, a situation that can arise when the sequence diverges (and also when the program is incorrectly coded). This was done in Step 2 of Algorithm 2.1 where the bound N_0 was set and the procedure terminated if $i > N_0$.

3.1. Examples

Example 1 ? *Exemple*

The equation $f(x) = x^3 + 4x^2 - 10 = 0$ has a root in $[1, 2]$ since $f(1) = -5$ and $f(2) = 14$. The Bisection Algorithm gives the values in Table 2.1.

After 13 iterations, $p_{13}=1.365112305$ approximates the root p with an error

$$|p - p_{13}| < |b_{14} - a_{14}| = |1.365234375 - 1.365112305| = 0.000122070.$$

Since $|a_{14}| < |p|$,

$$\frac{|p - p_{13}|}{|p|} < \frac{|b_{14} - a_{14}|}{|a_{14}|} \leq 9.0 \times 10^{-5},$$

so the approximation is correct to at least four significant digits. The correct value of p, to nine decimal places, is $p = 1.365230013$. Note that p_9 is closer to p than is the final

n	a_n	b_n	p_n	$f(p_n)$
1	1.0	2.0	1.5	2.375
2	1.0	1.5	1.25	−1.79687
3	1.25	1.5	1.375	0.16211
4	1.25	1.375	1.3125	−0.84839
5	1.3125	1.375	1.34375	−0.35098
6	1.34375	1.375	1.359375	−0.09641
7	1.359375	1.375	1.3671875	0.03236
8	1.359375	1.3671875	1.36328125	−0.03215
9	1.36328125	1.3671875	1.365234375	0.000072
10	1.36328125	1.365234375	1.364257813	−0.01605
11	1.364257813	1.365234375	1.364746094	−0.00799
12	1.364746094	1.365234375	1.364990235	−0.00396
13	1.364990235	1.365234375	1.365112305	−0.00194

approximation p_{13}. You might suspect this is true since $|f(p_9)| < |f(p_{13})|$, but we cannot be sure of this unless the true answer is known.

Table 2.1

The Bisection method, though conceptually clear, has significant drawbacks. It is slow to converge (that is, N might become quite large before $|p - p_N|$is sufficiently small), and a good intermediate approximation can be inadvertently discarded. However, the method has the important property that it always converges to a solution, and for that reason it is often used as a starter for the more efficient methods we will present later in this chapter

Example 2 ***Exemple***

Find a root of the function $f(x) = x^3 + x - 1$ by using the Bisection Method on the interval $[0, 1]$.

As noted, $f(a_0)f(b_0) = (-1)(1) < 0$, so a root exists in the interval. The interval midpoint is $c_0 = 1/2$. The first step consists of evaluating $f(1/2) = -3/8 < 0$ and choosing the new interval $[a_1, b_1] = [1/2, 1]$, since $f(1/2)f(1) < 0$. The second step consists of

evaluating $f(c_1) = f(3/4) = 11/64 > 0$, leading to the new interval $[a_2, b_2] = [1/2, 3/4]$. Continuing in this way yields the following intervals:

i	a_i	$f(a_i)$	c_i	$f(c_i)$	b_i	$f(b_i)$
0	0.0000	−	0.5000	−	1.0000	+
1	0.5000	−	0.7500	+	1.0000	+
2	0.5000	−	0.6250	−	0.7500	+
3	0.6250	−	0.6875	+	0.7500	+
4	0.6250	−	0.6562	−	0.6875	+
5	0.6562	−	0.6719	−	0.6875	+
6	0.6719	−	0.6797	−	0.6875	+
7	0.6797	−	0.6836	+	0.6875	+
8	0.6797	−	0.6816	−	0.6836	+
9	0.6816	−	0.6826	+	0.6836	+

3.2. Number of iterations

Number of iterations for the bisection method ***Méthode***

uppose that $f \in C[a, b]$ and $f(a) \cdot f(b) < 0$. The Bisection method generates a sequence $p_n{}_{n=1}^{\infty}$ approximating a zero p of f with :

$$|p_n - p| \leq \frac{b-a}{2^n}, \quad \text{when} \quad n \geq 1.$$

Proof ***Complément***

For each $n \geq 1$, we have

$$b_n - a_n = \frac{1}{2^{n-1}}(b - a) \text{ and } p \in (a_n, b_n).$$

Since $p_n = \frac{1}{2}(a_n + b_n)$ for all $n \geq 1$, it follows that

$$|p_n - p| \leq \frac{1}{2}(b_n - a_n) = \frac{b-a}{2^n}.$$

Since

$$|p_n - p| \leq (b - a)\frac{1}{2^n},$$

Example ***Exemple***

To determine the number of iterations necessary to solve $f(x)=x^3+4 x^2-10=0$ with accuracy undefined0^{-3}$ using $a_1=1$ and $b_1=2$ requires finding an integer N that satisfies

$|p_N - p| \leq 2^{-N}(b - a) = 2^{-N} < 10^{-3}$.

To determine N we will use logarithms. Although logarithms to any base would suffice, we will use base-10 logarithms since the tolerance is given as a power of 10 . Since $2^{-N} < 10^{-3}$ implies that $log_{10}(2)^{-N} < log_{10}(10)^{-3} = -3$, we need to

have

$$-Nlog_{10}(2) < -3, \text{ so } N > \frac{3}{log_{10}(2)} \approx 9.96.$$

Hence, ten iterations will ensure an approximation accurate to within 10^{-3}. Table 2.1 on page 49 shows that the value of $p_9 = 1.365234375$ is accurate to within 10^{-4}. Again, it is important to keep in mind that the error analysis gives only a bound for the number of iterations, and in many cases this bound is much larger than the actual number required.

Links to a practicals examples *Conseil*

We give the following links to illustrate the proposed method by giving some numericals examples with videos :

- **Bisection method - an example** :[2]https://www.youtube.com/watch?v=14etsIN_2Fs[3]
- **Bisection Method | Solved Examples | Easiest Tricks** : https://www.youtube.com/watch?v=iSkxy13h6NQ
- **Bisection Method Example Numerical Analysis Root Finding** : https://www.youtube.com/watch?v=ECUAvaTnQ2M

4. False Position Method (Regula Falsi Method)

Each successive pair of approximations in the Bisection method brackets a root p of the equation; that is, for each positive integer n, a root lies between an and bn. This implies that, for each n, the Bisection method iterations satisfy

$|p_n - p| < \frac{1}{2}|a_n - b_n|,$

In mathematics, an ancient method of solving an equation in one variable is the ***false position method*** *(method of false position) or* ***regula falsi method****. In simple words, the method is described as the trial and error approach of using "false" or "test" values for the variable and then altering the test value according to the result. In this article, you will learn how to solve an equation in one variable using the false position method. Also, get solved examples on the regula falsi method here*[False Position Method (or) Regula Falsi Method p.31]. https://byjus.com/maths/false-position-method/

The method of False Position (also called Regula Falsi) generates approximations in the same manner as the Secant method, but it includes a test to ensure that the root is always bracketed between successive iterations. Although it is not a method we generally recommend, it illustrates how bracketing can be incorporated.

First choose initial approximations p0 and p1 with f (p0) · f (p1) < 0. The approximation p2 is chosen in the same manner as in the Secant method, as the x-intercept of the line joining (p0, f (p0)) and (p1, f (p1)). To decide which secant line to use to compute p3, consider f (p2) · f (p1), or more correctly sgn f (p2) · sgn f (p1).

• If sgn f (p2) · sgn f (p1) < 0, then p1 and p2 bracket a root. Choose p3 as the x-intercept

of the line joining (p1, f (p1)) and (p2, f (p2)).

• If not, choose p3 as the x-intercept of the line joining (p0, f (p0)) and (p2, f (p2)), and

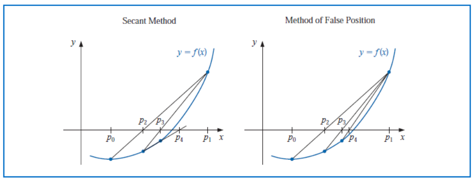
then interchange the indices on p0 and p1.

In a similar manner, once p3 is found, the sign of f (p3) · f (p2) determines whether we use p2 and p3 or p3 and p1 to compute p4. In the latter case a relabeling of p2 and p1 is performed. The relabeling ensures that the root is bracketed between successive iterations.

The process is described in Algorithm 2.5, and Figure 2.11 shows how the iterations can differ from those of the Secant method. In this illustration, the first three approximations are the same, but the fourth approximations differ.

2. Bisection method - an example - https://www.youtube.com/watch?v=14etsIN_2Fs
3. ssss - Bisection method - an example

Figure 2.11



Méthode

The term Regula Falsi, literally a false rule or false position, refers to a technique that uses results that are known to be false, but in some specific manner, to obtain convergence to a true result. False position problems can be found on the Rhind papyrus, which dates from about 1650 b.c.e.

False Position

To find a solution to $f(x) = 0$ given the continuous function f on the interval $[p_0, p_1]$ where $f(p_0)$ and $f(p_1)$ have opposite signs:

INPUT initial approximations p_0, p_1; tolerance TOL; maximum number of iterations N_0.

OUTPUT approximate solution p or message of failure.

Step 1 Set $i = 2$;
$q_0 = f(p_0)$;
$q_1 = f(p_1)$.

Step 2 While $i \leq N_0$ do Steps 3–7.

Step 3 Set $p = p_1 - q_1(p_1 - p_0)/(q_1 - q_0)$. (*Compute* p_i.)

Step 4 If $|p - p_1| < TOL$ then
OUTPUT (p); (*The procedure was successful.*)
STOP.

Step 5 Set $i = i + 1$;
$q = f(p)$.

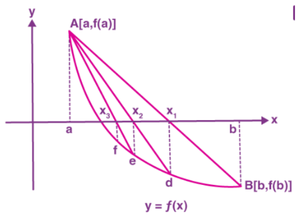
Step 6 If $q \cdot q_1 < 0$ then set $p_0 = p_1$;
$q_0 = q_1$.

Step 7 Set $p_1 = p$;
$q_1 = q$.

Step 8 OUTPUT ('Method failed after N_0 iterations, N_0 =', N_0);
(*The procedure unsuccessful.*)
STOP.

Remarque

Geometrical representation of the roots of the equation f(x) = 0 can be shown as:



Example 1 **Exemple**

To make a reasonable comparison we will use the same initial approximations as in the Secant method, that is, p0 = 0.5 and p1 = π/4. Table 2.6 shows the results of the method of False Position applied to f (x) = cos x−x

Table 2.6

	False Position
n	p_n
0	0.5
1	0.7853981635
2	0.7363841388
3	0.7390581392
4	0.7390848638
5	0.7390851305
6	0.7390851332

Example 2 **Exemple**

Find the positive root of the equation 3x+sinx-e^x using Regula Falsi method and correct upto 4 decimal places.

f(1) = -7

f(2) = 16

Therefore, root lies between 1 and 2

a = 1; f(a) = -7

b = 2; f(b) = 16

Substituting the values in the formula,

$x = \frac{bf(a)-af(b)}{f(a)-f(b)}$,

we get $x_1 = 2(-7) - 16 - 7 - 16 = 1.304347826; f(x1) = -1.334757952$

Therefore,x_1 becomes a to find the next point.

$X_2 = \frac{2(-1.334757952)-(1.304347826)16}{-1.334757952-16} = 1.357912305; f(x2) = -0.229135731$

Therefore, X_2 becomes a to find the next point.

$X_3 = \frac{2(-0.229135731)-(1.357912305)16}{-0.229135731-16} = 1.366977805; f(x3) = -0.038591868$

Therefore, X_3 becomes a to find the next point.

$X_4 = \frac{2(-0.038591868)-16(1.366977805)}{-0.038591868-16} = 1.368500975; f(x4) = -6.478731338 * 10^{-3}$

Therefore, X_4 becomes a to find the next point.

$X_5 = \frac{2(-6.478731338*10-3)-16(1.368500975)}{(-6.478731338*10-3)-16} = 1.368756579; f(x5) = -1.087052822 * 10^{-3}$

Therefore X_5 becomes a to find the next point.

$X_6 = \frac{2(-1.087052822*10-3)-16(1.368756579)}{(-1.087052822*10-3)-16} = 1.368799463; f(x6) = -1.823661977 * 10^{-4*}$

Therefore X_6 becomes a to find the next point.

$X_7 = \frac{2(-1.823661977*10-4)-16(1.368799463)}{(-1.823661977*10-4)-16} = 1.368806657; f(x7) = -3.0601008 * 10^{-5}$

Therefore X_7 becomes a to find the next point.

$X_8 = \frac{2(-3.0601008*10-5)-1.368806657(16)}{(-3.0601008*10-5)-16} = 1.368807864.$

Therefore, the positive root corrected to 4 decimal places is 1.3688.

5. Fixed Point iteration Method

Rappel

Fixed-point results occur in *many* *Davis, P.J.; Rabinowitz, P. (2007). Methods of numerical integration. Courier Corporation. ISBN 978-0-486-45339-2.* p.28 areas of mathematics, and are a major tool of economists for proving results concerning equilibria. Although the idea behind the technique is old, the terminology was first used by the Dutch mathematician L.E.J. Brouwer (1881–1966) in the early 1900s.*Davis, P.J.; Rabinowitz, P. (2007). Methods of numerical integration. Courier Corporation. ISBN 978-0-486-45339-2.* *Davis, P.J.; Rabinowitz, P. (2007). Methods of numerical integration. Courier Corporation. ISBN 978-0-486-45339-2.* p.28

5.1. Fixed Point Iteration

A number p is a fixed point for a given function g if $g(p) = p$. In this section we consider the problem of finding solutions to fixed-point problems and the connection between the fixed-point problems and the root-finding problems we wish to solve.

Root-finding problems and fixed-point problems are equivalent classes in the following sense:

Given a root-finding problem $f(p) = 0$, we can define functions g with a fixed point at p in a number of ways, for example, as $g(x) = x - f(x)$ or as $g(x) = x + 3f(x)$. Conversely, if the function g has a fixed point at p, then the function defined by $f(x) = x - g(x)$ has a zero at$p$.

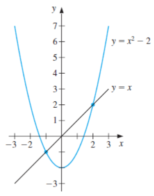
Although the problems we wish to solve are in the root-finding form, the fixed-point form is easier to analyze, and certain fixed-point choices lead to very powerful root-finding techniques.*Bultheel, Adhemar; Cools, Ronald, eds. (2010). The Birth of Numerical Analysis. Vol. 10. World Scientific. ISBN 978-981-283-625-0.*[*Bultheel, Adhemar; Cools, Ronald, eds. (2010). The Birth of Numerical Analysis. Vol. 10. World Scientific. ISBN 978-981-283-625-0.* p.28]

We first need to become comfortable with this new type of problem and to decide when a function has a fixed point and how the fixed points can be approximated to within a specified accuracy.*Ames, W.F. (2014). Numerical methods for partial differential equations (3rd ed.). Academic Press. ISBN 978-0-08-057130-0.*[*Ames, W.F. (2014). Numerical methods for partial differential equations (3rd ed.). Academic Press. ISBN 978-0-08-057130-0.* p.27]

Exemple

The function $g(x) = x^2 - 2$, for $-2 \leq x \leq 3$, has fixed points at$x = -1$and $x = 2$ since $g(-1) = (-1)2 - 2 = -1$ and $g(2) = 22 - 2 = 2$.

This can be seen in the following Figure.



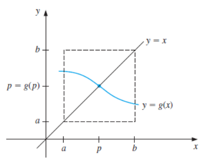
Graphique 1

The following theorem gives sufficient conditions for the existence and uniqueness of a fixed point.

a) Theorem 1

- a. If $g \in C[a, b]$ and $g(x) \in [a, b]$ for all $x \in [a, b]$, then g has a fixed point in $[a, b]$.
- If, in addition, $g^{'}(x)$ exists on $[a, b]$ and a positive constant $k < 1$ exists with $|g^{'}(x)| \leq k$, for all $x \in (a, b)$,

then the fixed point in $[a, b]$ is unique. (See the following figure)



i) ALGORITHM

ALGORITHM

Fondamental

We give the following algoritme for the fixed point iteration method as follos :

To find a solution to $p = g(p)$ given an initial approximation p_0:

INPUT initial approximation p_0; tolerance TOL; maximum number of iterations N_0.

OUTPUT approximate solution p or message of failure.

Step 1 Set $i = 1$.

Step 2 While $i \leq N_0$ do Steps 3–6.

Step 3 Set $p = g(p_0)$. (*Compute* p_i.)

Step 4 If $|p - p_0| < TOL$ then
OUTPUT (p); (*The procedure was successful.*)
STOP.

Step 5 Set $i = i + 1$.

Step 6 Set $p_0 = p$. (*Update* p_0.)

Step 7 OUTPUT ('The method failed after N_0 iterations, N_0 =', N_0);
(*The procedure was unsuccessful.*)
STOP.

1 Example

The equation $x^3 + 4x^2 - 10 = 0$ has a unique root in $[1, 2]$. There are many ways to change the equation to the fixed-point form $x = g(x)$ using simple algebraic manipulation. For example, to obtain the function g described in previous theorem, we can manipulate the equation

$x^3 + 4x^2 - 10 = 0$as follows:

$4x^2 = 10 - x^3$, so $x^2 = 14(10 - x^3)$, and $x = \pm\frac{1}{2}(10 - x^3)^{\frac{1}{2}}$.

To obtain a positive solution, $g_3(x)$ is chosen. It is not important to derive the functions shown here, but you should verify that the fixed point of each is actually a solution to the original equation, $x^3 + 4x^2 - 10 = 0$.

- $x = g_1(x) = x - x^3 - 4x^2 + 10$
- $x = g_2(x) = \left(\frac{10}{x} - 4x\right)^{1/2}$
- $x = g_3(x) = \frac{1}{2}(10 - x^3)^{\frac{1}{2}}$
- $x = g_4(x) = \left(\frac{10}{4+x}\right)^{\frac{1}{2}}$
- $x = g_5(x) = x - \frac{x^3+4x^2-10}{3x^2+8x}$

With $p_0 = 1.5$ Table 2.2 lists the results of the fixed-point iteration for all five choices of g.

The actual root is 1.36523001. Comparing the results to the Bisection Algorithm given in that example, it can be seen that excellent results have been obtained for choices (c), (d), and (e), since the Bisection method requires 27 iterations for this accuracy. It is interesting to note that choice (a) was divergent and that (b) became undefined because it involved the square root of a negative number

Table 2.2

n	(a)	(b)	(c)	(d)	(e)
0	1.5	1.5	1.5	1.5	1.5
1	−0.875	0.8165	1.286953768	1.348399725	1.373333333
2	6.732	2.9969	1.402540804	1.367376372	1.365262015
3	−469.7	$(-8.65)^{1/2}$	1.345458374	1.364957015	1.365230014
4	1.03×10^8		1.375170253	1.365264748	1.365230013
5			1.360094193	1.365225594	
6			1.367846968	1.365230576	
7			1.363887004	1.365229942	
8			1.365916734	1.365230022	
9			1.364878217	1.365230012	
10			1.365410062	1.365230014	
15			1.365223680	1.365230013	
20			1.365230236		
25			1.365230006		
30			1.365230013		

How can we find a fixed-point problem that produces a sequence that reliably and rapidly converges to a solution to a given root-finding problem?

The following theorem and its corollary give us some clues concerning the paths we should pursue and, perhaps more importantly, some we should reject.

5.2. Theorem 2 (Fixed-Point Theorem)

Let $g \in C[a, b]$ be such that $g(x) \in [a, b]$, for all $x \in [a, b]$. Suppose, in addition, that g exists on $[a, b]$ and that a constant $0 < k < 1$ exists with :

$$|g^{'}(x)| \leq k, \forall x \in [a, b].$$

Then, for any number p_0 in$[a, b]$, the sequence defined by

$$p_n = g(p_n - 1), n \geq 1,$$

converges to the unique fixed point p in$[a, b]$.

a) Corollary (Number of iterations)

Complément

If g satisfies the hypotheses of Theorem 2 (Fixed-Point Theorem), then bounds for the error involved in using p_n to approximate p are given by

$$|p_n - p| \leq k^n \max(p_0 - a, b - p_0)$$

and

$$|p_n - p| \leq \frac{k^n}{1-k}|p_1 - p_0|, \text{ for all } n \geq 1.$$

Remarque

Both inequalities in the corollary relate the rate at which $(p_n)_{n=0}^{\infty}$ converges to the bound k on the first derivative. The rate of convergence depends on the factor k^n. The smaller the value of$k$, the faster the convergence, which may be very slow if k is close to 1.

5.3. Exercices

a) Exercice 1

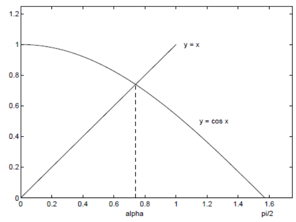
Consider the equation $x = \cos x$.

- (a) Show graphically that there exists a unique positive root α. Indicate, approximately, where it is located.
- (b) Prove local convergence of the iteration $x_{n+1} = \cos x_n$.
- (c) For the iteration in (b) prove: if $x_n \in \left[0, \frac{\pi}{2}\right]$, then

$|x_{n+1} - \alpha| < \left(\sin \frac{\alpha+\pi/2}{2}\right)|x_n - \alpha|$.

In particular, one has global convergence on $\left[0, \frac{\pi}{2}\right]$.

i) Solution 1



- From the graph below one sees that $\alpha \approx \frac{\pi}{4}$.
- The iteration function is $\varphi(x) = \cos x$. Since $\varphi'(x) = -\sin x$, we have $|\varphi'(\alpha)| = \sin \alpha < 1$, implying local convergence.
- If $x_0 \in \left[0, \frac{\pi}{2}\right]$, then clearly $x_n \in [0, 1]$ for all $n \geq 1$. Furthermore,

$$\begin{aligned} |x_{n+1} - \alpha| &= |\cos x_n - \cos \alpha| = 2\left|\sin \tfrac{1}{2}(x_n + \alpha) \sin \tfrac{1}{2}(x_n - \alpha)\right| \\ &< \sin \tfrac{1}{2}(x_n + \alpha) \cdot |x_n - \alpha| < \sin\left(\tfrac{\alpha+\pi/2}{2}\right)|x_n - \alpha|. \end{aligned}$$

1 Exercice 2

Consider the equation

$x = \mathrm{e}^{-x}$.

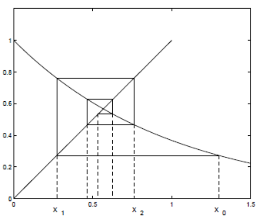
- (a) Show that there is a unique real root α and determine an interval containing it.
- (b) Show that the fixed point iteration $x_{n+1}=\mathrm{e}^{-x_n}, n=0,1,2, \ldots$, converges locally to α and determine the asymptotic error constant.
- (c) Illustrate graphically that the iteration in (b) actually converges globally, that is, for arbitrary $x_0>0$. Then prove it.
- An equivalent equation is

$x = \ln \frac{1}{x}$

Does the iteration $x_{n+1} = \ln \frac{1}{x_n}$ also converge locally? Explain.

1 Solution 2

- (a) Letting $f(x) = x - \mathrm{e}^{-x}$, we have $f'(x) = 1 + \mathrm{e}^{-x} \geq 1$ for all real x. Consequently, f increases monotonically on $\mathbb{R}$ from $-\infty$ to $+\infty$, hence has exactly one real zero, α. Given that $f(0) = -1$ and $f(1) = 1 - \mathrm{e}^{-1} > 0$, we have (a) Letting $f(x) = x - \mathrm{e}^{-x}$, we have $f'(x) = 1 + \mathrm{e}^{-x} \geq 1$ for all real x. Consequently, f increases monotonically on $\mathbb{R}$ from $-\infty$ to $+\infty$, hence has exactly one real zero, α. Given that $f(0) = -1$ and $f(1) = 1 - \mathrm{e}^{-1} > 0$, we have $0 < \alpha < 1$..
- (b) The fixed point iteration is $x_{n+1} = \varphi(x_n)$ with $\varphi(x) = \mathrm{e}^{-x}$. Clearly, $\alpha = \varphi(\alpha)$, and $\varphi'(\alpha) = -\mathrm{e}^{-\alpha}$, hence (b) The fixed point iteration is $x_{n+1} = \varphi(x_n)$ with $\varphi(x) = \mathrm{e}^{-x}$. Clearly, $\alpha = \varphi(\alpha)$, and $\varphi'(\alpha) = -\mathrm{e}^{-\alpha}$, hence $0 < |\varphi'(\alpha)| = \mathrm{e}^{-\alpha} < 1$. Therefore, we have local convergence, the asymptotic error constant being $c = -\mathrm{e}^{-\alpha}$. $0 < |\varphi'(\alpha)| = \mathrm{e}^{-\alpha} < 1$. Therefore, we have local convergence, the asymptotic error constant being $c = -\mathrm{e}^{-\alpha}$.
- (c) For definiteness, assume $x_0 > \alpha$. Then the fixed point iteration behaves as indicated in the figure below: the iterates "spiral" clockwise around, and into, the fixed point α. The same spiraling takes place if (c) For definiteness, assume $x_0 > \alpha$. Then the fixed point iteration behaves as indicated in the figure below: the iterates "spiral" clockwise around, and into, the fixed point α. The same spiraling takes place if $0 < x_0 < \alpha$ (simply relabel x_1 in the figure as x_0). $< x_0 < \alpha$ (simply relabel x_1 in the figure as x_0).



Proof of global convergence. From the mean value theorem of calculus, applied to the function e^{-x}, one has

$$|x_{n+1} - \alpha| = |\mathrm{e}^{-x_n} - \mathrm{e}^{-\alpha}| = \mathrm{e}^{-\xi_n}|x_n - \alpha|,$$

where ξ_n is strictly between α and x_n. Letting $\mu = \min(x_0, x_1)$, it is clear from the graph above that $\mu > 0$ and

$$x_n \geq \mu \quad (\text{ all } n \geq 0), \quad \alpha > \mu.$$

Therefore, ξ_n being stricly between α and x_n, we have that $\xi_n > \mu$ for all n, hence

$$|x_{n+1} - \alpha| < \mathrm{e}^{-\mu}|x_n - \alpha|.$$

Applying this repeatedly gives $|x_n - \alpha| < \mathrm{e}^{-\mu n}|x_0 - \alpha| \rightarrow 0$ as $n \rightarrow \infty$.

1 Exercice 3

The equation $x^2 - 2 = 0$ can be written as a fixed point problem in different ways, for example,

- (a) $x = \frac{2}{x}$$
- (b) $x = x^2 + x - 2$
- (c) $x = \frac{x+2}{x+1}$.

How does the fixed point iteration perform in each of these three cases? Be as specific as you can.

1 Solution 3

This is a fixed point iteration with iteration function

$$\varphi(x) = \frac{x(x^2+3a)}{3x^2+a}.$$

Clearly, $\varphi(\alpha) = \alpha$. Differentiating repeatedly the identity

$$(3x^2 + a)\varphi(x) = x^3 + 3ax,$$

one gets first

$$6x\varphi(x) + (3x^2 + a)\varphi'(x) = 3x^2 + 3a,$$

hence $4a\varphi'(\alpha) = 6a - 6\alpha\varphi(\alpha) = 6a - 6\alpha^2 = 0$, then

$$6\varphi(x) + 12x\varphi'(x) + (3x^2 + a)\varphi''(x) = 6x,$$

hence $4a\varphi''(\alpha) = 6\alpha - 6\alpha = 0$, and finally

$$18\varphi'(x) + 18x\varphi''(x) + (3x^2 + a)\varphi'''(x) = 6,$$

hence $4a\varphi'''(\alpha) = 6$, that is, $\varphi'''(\alpha) \neq 0$. This shows that the iteration converges with order $p = 3$.

The asymptotic error constant, is

$$c = \frac{1}{3!}\varphi'''(\alpha) = \frac{1}{6} \cdot \frac{6}{4a} = \frac{1}{4a}.$$

6. Newoton's Method

Isaac Newton and Joseph Raphson

Isaac Newton $(1642\text{– }1727)$ was one of the most brilliant scientists of all time. The late 17th century was a vibrant period for science and mathematics and Newton's work touched nearly every aspect of mathematics. His method for solving equations was introduced to find a root of $x^3 - 2x - 5 = 0$, a problem we consider in Exercise 5(a). Although he demonstrated the method only for polynomials, it is clear that he realized its broader applications.

Joseph Raphson $(1648\text{– }1715)$ gave a description of the method attributed to Isaac Newton in 1690, acknowledging Newton as the source of the discovery. Neither Newton nor Raphson explicitly used the derivative in their description since both considered only polynomials. Other mathematicians, particularly James Gregory (1636–1675), were aware of the underlying process at or before this time.

6.1. Description of Newton's Method

Newton's (or the Newton–Raphson) method is one of the most powerful and well-known numerical methods for solving a root-finding problem. There are many ways of introducing Newton's method.*Watson, G.A. (2010). "The history and development of numerical analysis in Scotland: a personal perspective" (PDF). The Birth of Numerical Analysis. World Scientific. pp. 161–177. ISBN 9789814469456.*[*Watson, G.A. (2010). "The history and development of numerical analysis in Scotland: a personal perspective" (PDF). The Birth of Numerical Analysis. World Scientific. pp. 161–177. ISBN 9789814469456.* p.29]

If we only want an algorithm, we can consider the technique graphically, as is often done in calculus. Another possibility is to derive Newton's method as a technique to obtain faster convergence than offered by other types of functional iteration. A third means of introducing Newton's method, discussed next, is based on Taylor polynomials.*Ezquerro Fernández, J.A.; Hernández Verón, M.Á. (2017). Newton's method: An updated approach of Kantorovich's theory. Birkhäuser. ISBN 978-3-319-55976-6.*[*Ezquerro Fernández, J.A.; Hernández Verón, M.Á. (2017). Newton's method: An updated approach of Kantorovich's theory. Birkhäuser. ISBN 978-3-319-55976-6.* p.28]

Suppose that $f \in C^2[a, b]$. Let $p_0 \in [a, b]$ be an approximation to the solution p of $f(x) = 0$ such that $f(p0) = 0$ and $|p - p_0|$ is “small.” Consider the first Taylor polynomial for $f(x)$expanded about $p0$, and evaluated at $x = p$,

$$f(p) = f(p_0) + (p - p_0)f'(p_0) + \frac{(p-p_0)^2}{2} f''(\epsilon(p)),$$

where $\epsilon(p)$ lies between p and p_0. Since $f(p) = 0$, this equation gives

$$0 = f(p_0) + (p - p_0)f'(p_0) + \frac{(p-p_0)^2}{2} f''(\epsilon(p)),$$

Newton'smethod is derived by assuming that since $|p - p_0|$ is small, the terminvolving $(p - p_0)^2$ is much smaller, so

$$0 \approx f(p_0) + (p - p_0)f(p_0).$$

Solving for p gives

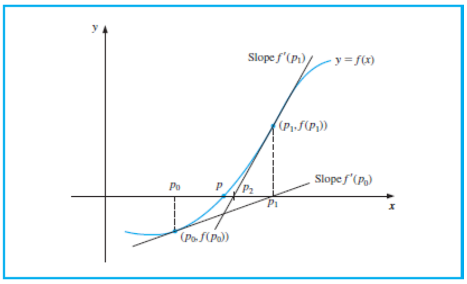
$$p \approx p_1 \equiv p_0 - \frac{f(p_0)}{f'(p_0)}$$

This sets the stage for Newton's method, which starts with an initial approximation p_0 and generates the sequence $(p_n)_{n=0}^{\infty}$ by

$$p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)} . \forall n \geq 0$$

Figure 2.7 illustrates how the approximations are obtained using successive tangents. Starting with the initial approximation p_0, the approximation p_1 is the x-intercept of the tangent line to the graph of f at $(p_0, f(p_0))$. The approximation p_2 is the x-intercept of the tangent line to the graph of f at$(p_1, f(p_1))$ and so on. Algorithm 2.3 follows this procedure*Quarteroni, A.; Saleri, F.; Gervasio, P. (2014). Scientific computing with MATLAB and Octave (4th ed.). Springer. ISBN 978-3-642-45367-0.*[*Quarteroni, A.; Saleri, F.; Gervasio, P. (2014). Scientific computing with MATLAB and Octave (4th ed.). Springer. ISBN 978-3-642-45367-0.* p.28]

Figure 2.7



6.2. Algorithm and procedure of Newton method

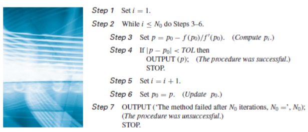
Algorithm

Méthode

To find a solution to $f(x) = 0$ given an initial approximation$p0$:

INPUT initial approximation $p0$; tolerance TOL; maximum number of iterations $N0$.

OUTPUT approximate solution p or message of failure.



Step 1 Set $i = 1$.

Step 2 While $i \leq N_0$ do Steps 3–6.

Step 3 Set $p = p_0 - f(p_0)/f'(p_0)$. *(Compute p_i.)*

Step 4 If $|p - p_0| < TOL$ then
OUTPUT (p); *(The procedure was successful.)*
STOP.

Step 5 Set $i = i + 1$.

Step 6 Set $p_0 = p$. *(Update p_0.)*

Step 7 OUTPUT ('The method failed after N_0 iterations, $N_0 =$', N_0);
(The procedure was unsuccessful.)
STOP.

The stopping-technique inequalities given with the Bisection method are applicable to Newton's method. That is, select a tolerance $\varepsilon > 0$, and construct $p_1, \ldots p_N$, until

$$|p_N - p_{N-1}| < \epsilon,$$

$$\frac{|p_N - p_{N-1}|}{|p_N|} < \epsilon, p_N = 0,$$

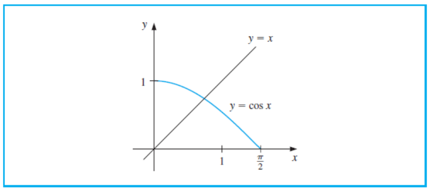
$$|f(p_N)| < \epsilon$$

Example Of Newton's method **Exemple**

Suppose we would like to approximate a solution to $f(x) = cosx - x = 0$. A solution to this root-finding problem is also a solution to the fixed-point problem $x = cosx$, and the graph in Figure 2.8 implies that a single fixed-point p lies in $[0, \pi/2]$. Table 2.3 shows the results of fixed-point iteration with $p_0 = \pi/4$. The best we could conclude from these results is that $p \approx 0.74$.

To approach this problem differently, define $f(x) = cosx - x$ and apply Newton's method. Since $f(x) = -sinx - 1$, the sequence is generated by

$$p_n = p_{n-1} - \frac{\cos p_{n-1} - p_{n-1}}{-\sin p_{n-1} - 1}, \qquad \text{for } n \geq 1$$



With $p0 = \pi/4$, the approximations in Table 2.4 are generated. An excellent approximation is obtained with $n = 3$. We would expect this result to be accurate to the places listed because of the agreement of p_3 and p_4.

n	p_n.
0	0.7853981635
1	0.7395361337
2	0.7390851781
3	0.7390851332
4	0.7390851332

Table 2.4

Remarque

The Taylor series derivation of Newton's method at the beginning of the section points out the importance of an accurate initial approximation. The crucial assumption is that the term involving (p − p0)2 is, by comparison with $|p - p_0|$, so small that it can be deleted. This will clearly be false unless p_0 is a good approximation to p. If p_0 is not sufficiently close to the actual root, there is little reason to suspect that Newton's method will converge to the root. However, in some instances, even poor initial approximations will produce convergence

Conseil

The following convergence theorem for Newton's method illustrates the theoretical importance of the choice of p_0.

6.3. Convergence theorem for Newton's method

Fondamental

Let $f \in C^2[a,b]$. If $p \in [a,b]$ is such that $f(p)=0$ and $f'(p) \neq 0$, then there exists a $\delta > 0$ such that Newton's method generates a sequence $\{p_n\}_{n=1}^{\infty}$ converging to p for any initial approximation $p_0 \in [p-\delta, p+\delta]$.

a) Proof

The proof is based on analyzing Newton's method as the functional iteration scheme $p_n = g(p_{n-1})$, for $n \geq 1$, with

$g(x) = x - \frac{f(x)}{f'(x)}$.

Let k be in $[0,1]$. We first find an interval $[p-\delta, p+\delta]$ that g maps into itself and for which $|g'(x)| \leq k$, for all $x \in (p-\delta, p+\delta)$.

Since f' is continuous and $f'(p) \neq 0$, implies that there exists a \$$\delta_1 > 0$\$, such that $f'(x) \neq 0\$ for \$x \in [p-\delta_1, p+\delta_1] \subseteq [a,b]$. Thus, g is defined and continuous on $[p-\delta_1, p+\delta_1]$. Also,

$$g'(x) = 1 - \frac{f'(x)f'(x) - f(x)f''(x)}{[f'(x)]^2} = \frac{f(x)f''(x)}{[f'(x)]^2},$$

for $x \in [p-\delta_1, p+\delta_1]$, and, since $f \in C^2[a,b]$, we have $g \in C^1[p-\delta_1, p+\delta_1]$.

By assumption, $f(p)=0$, so

$$g'(p) = \frac{f(p)f''(p)}{[f'(p)]^2} = 0\,.$$

Since g' is continuous and Since g' is continuous and $0 < k < 1$, we will have $\delta\$, with \$0 < \delta < \delta_1$, and $0 < k < 1$, implies that there exists a δ, with $0 < \delta < \delta_1$, and

$$|g'(x)| \leq k, \qquad \text{for all } x \in [p-\delta, p+\delta].$$

We now need to show that g maps $[p-\delta, p+\delta]$ into $[p-\delta, p+\delta]$. If $x \in [p-\delta, p+\delta]$, then the Mean Value Theorem implies that for some number ξ between x and p, we have $|g(x)-g(p)| = |g'(\xi)||x-p|$. So

$$|g(x)-p| = |g(x)-g(p)| = |g'(\xi)||x-p| \leq k|x-p| < |x-p|.$$

Since $x \in [p-\delta, p+\delta]$, it follows that $|x-p| < \delta$ and that $|g(x)-p| < \delta$. Hence, g maps $[p-\delta, p+\delta]$ into $[p-\delta, p+\delta]$.

All the hypotheses of the Fixed-Point Theorem are now satisfied, so the sequence $\{p_n\}_{n=1}^{\infty}$, defined by

$$p_n = g(p_{n-1}) = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}, \qquad \text{for } n \geq 1,$$

converges to p for any $p_0 \in [p-\delta, p+\delta]$.

Remarque

Newton's method is an extremely powerful technique, but it has a major weakness: the need to know the value of the derivative of f at each approximation. Frequently, $f(x)$is far more difficult and needs more arithmetic operations to calculate than $f(x)$.

6.4. Exercices

a) Exercice 1

Use Newton's method to find solutions accurate to within undefined 10^{-5} for the following problems.

- $e^{x}+2^{-x}+2 \cos x-6=0 \$ for undefined \leq x \leq 2$
- $\ln (x-1)+\cos (x-1)=0 \$ for undefined .3 \leq x \leq 2$
- $2 x \cos 2 x-(x-2)^{2}=0 \quad \$ for \$ 2 \leq x \leq 3 \$ and \$ 3 \leq x \leq 4$
- $(x-2)^{2}-\ln x=0 \quad \$ for undefined \leq x \leq 2 \$ and \$ e \leq x \leq 4$
- $e^{x}-3 x^{2}=0 \$ for e. \$ e^{x}-3 x^{2}=0 \$ for \$ 0 \leq x \leq 1 \$ and \$ 3 \leq x \leq 5 \$ \leq x \leq 1 \$ and \$ 3 \leq x \leq 5$
- $\sin x-e^{-x}=0 \quad \$ for f. \$ \quad \sin x-e^{-x}=0 \quad \$ for \$ 0 \leq x \leq 1,3 \leq x \leq 4 \$ and \$ 6 \leq x \leq 7 \$ \leq x \leq$

i) Solution 1

(a) For $p_{0}=1$, we have $p_{8}=1.829384$.

(b) For $p_{0}=1.5$, we have $p_{4}=1.397748$.

(c) For $p_{0}=2$, we have $p_{4}=2.370687$; and for $p_{0}=4$, we have $p_{4}=3.722113$.

(d) For $p_{0}=1$, we have $p_{4}=1.412391$; and for $p_{0}=4$, we have $p_{5}=3.057104$.

(e) For $p_{0}=1$, we have $p_{4}=0.910008$; and for $p_{0}=3$, we have $p_{9}=3.733079$.

(f) For $p_{0}=0$, we have $p_{4}=0.588533$; for $p_{0}=3$, we have $p_{3}=3.096364$; and for $p_{0}=6$, we have $p_{3}=6.285049$.

1 Exercice 2

The following describes Newton's method graphically: Suppose that $f^{\prime}(x)$ exists on $[a, b]$ and that $f^{\prime}(x) \neq 0$ on $[a, b]$. Further, suppose there exists one $p \in[a, b]$ such that $f(p)=0$, and let $p_{0} \in[a, b]$ be arbitrary. Let p_{1} be the point at which the tangent line to f at $\left(p_{0}, f\left(p_{0}\right)\right)$ crosses the x-axis. For each $n \geq 1$, let p_{n} be the x-intercept of the line tangent to f at $\left(p_{n-1}, f\left(p_{n-1}\right)\right.$\$). Derive the formula describing this method.

1 Solution 2

The equation of the tangent line is

$y-f\left(p_{n-1}\right)=f^{\prime}\left(p_{n-1}\right)\left(x-p_{n-1}\right)$.

To complete this problem, set $y=0$ and solve for $x=p_{n}$.

1 Exercice 3

Use Newton's method to approximate, to within \$10^{-4}\$, the value of \$x\$ that produces the point on the graph of \$y=1 / x\$ that is closest to \$(2,1)\$.

1 Solution 3

For $p_{0}=2$, we have $p_{2}=1.866760$. The point is $(1.866760,0.535687)$.

1 Exercice 4

The accumulated value of a savings account based on regular periodic payments can be determined from the annuity due equation,

$A=\frac{P}{i}\left[(1+i)^{n}-1\right]$.

In this equation, A is the amount in the account, P is the amount regularly deposited, and i is the rate of interest per period for the n deposit periods. An engineer would like to have a savings account valued at $750,000$ upon retirement in 20 years and can afford to put 1500 per month toward this goal. What is the minimal interest rate at which this amount can be invested, assuming that the interest is compounded monthly?

1 Solution 4

$\text{The minimal annual interest rate is } 6.67\%.$

1 Exercice 5

Let $f(x)=3^{3x+1}-7\cdot 5^{2x}$

a. Use the Maple commands solve and fsolve to try to find all roots of f.

b. Plot $f(x)$ to find initial approximations to roots of f.

c. Use Newton's method to find roots of f to within undefined 10^{-16}.

d. Find the exact solutions of $f(x)=0$ without using Maple.

1 Solution 5

(a) solve $\left(3^{-}(3*x+1)-75^{-}(2x),x\right)$ and fsolve $\left(3^{-}(3x+1)-75^{-}(2*x),x\right)$ both fail.

(b) $\operatorname{plot}\left(3^{\wedge}(3x+1)-75^{\wedge}(2*x),x=a\ldots b\right)$ generally yields no useful information. However, $a=10.5$ and $b=11.5$ in the plot command show that $f(x)$ has a root near $x=11$.

(c) With $p_0=11,p_5=11.0094386442681716$ is accurate to undefined 10^{-16}.

(d) $p=\frac{\ln(3/7)}{\ln(25/27)}$

1 Exercice 6

Repeat the previous exercise 29 using $f(x)=2x^2-3^{7x+1}$.

1 Solution 6

(a) solve $\left(2^{(x^2)}-3*7^{(x+1)},x\right.$ fails and fsolve $\left(2^{(x^2)}-3*7^{(x+1)},x\right.$ returns -1.118747530.

(b) $\operatorname{plot}\left(2^{(\mathrm{x}^\wedge 2)}-3*7^{(\mathrm{x}+1)},\mathrm{x}=-2.4\right)$ shows there is also a root near $x=4$.

(c) With $p_0=1,p_4=-1.1187475303988963$ is accurate to 10^{-16}; with $p_0=4,p_6=3.9261024524565005$ is accurate to $^{-16}$

(d) The roots are : $\frac{\ln(7)\pm\sqrt{[\ln(7)]^2+4\ln(2)\ln(4)}}{2\ln(2)}$

1 Exercice 7

Consider the equation

$$f(x)=0,\ \text{where } f(x)=\tan x-cx,\quad 0<c<1.$$

- (a) Show that the smallest positive root α is in the interval $\left(\pi, \frac{3}{2} \pi\right)$.

- (b) Show that Newton's method started at $x_0 = \pi$ is guaranteed to converge to α if c is small enough. Exactly how small does c have to be?

1 Solution 7

- (a) This is readily seen by plotting the graphs of $y = \tan x$ and $y = cx$ for $x > 0$, and observing that they intersect for the first time in the interval $\left(\pi, \frac{3\pi}{2}\right)$.
- (b) Note that

$f'(x) = 1 + \tan^2 x - c > 0,$
$f''(x) = 2\tan x\left(1 + \tan^2 x\right),$

so that f on the interval $\left(\pi, \frac{3\pi}{2}\right)$ is convex and monotonically increasing from $-c\pi$ to ∞. Therefore, Newton's method converges with $x_0 = \pi$ if $x_1 < \frac{3\pi}{2}$, which translates to

$\pi + \frac{c\pi}{1-c} < \frac{3\pi}{2},$

that is, \$$c < \frac{1}{3}$\$.

1 Exercice 8

Let us suppose that the equation

$\cos x \cosh x - 1 = 0$

has exactly two roots $\alpha_n < \beta_n$ in each interval $\left[-\frac{\pi}{2} + 2n\pi, \frac{\pi}{2} + 2n\pi\right]$ $n = 1, 2, 3 \ldots$. Show that Newton's method applied to $(^*)$ converges to α_n when initialized by $x_0 = -\frac{\pi}{2} + 2n\pi$, and to β_n when initialized by $x_0 = \frac{\pi}{2} + 2n\pi$.

1 Solution 8

We have

$f(x) = \cos x \cosh x - 1,$
$f'(x) = -\sin x \cosh x + \cos x \sinh x,$
$f''(x) = -2\sin x \sinh x.$

Clearly, $f''(x) > 0$ on $\left[-\frac{\pi}{2} + 2n\pi, 2n\pi\right]$ and $f''(x) < 0$ on $\left[2n\pi, \frac{\pi}{2} + 2n\pi\right]$. Furthermore, $f\left(-\frac{\pi}{2} + 2n\pi\right) = f\left(\frac{\pi}{2} + 2n\pi\right) = -1$ and $f(2n\pi) = \cosh(2n\pi) > 1$. Since f is convex on the first half of the interval $\left[-\frac{\pi}{2} + 2n\pi, \frac{\pi}{2} + 2n\pi\right]$, Newton's method started at the left endpoint converges monotonically decreasing (except for the first step) to α_n, provided the first iterate is to the left of the midpoint. This is the case since, with $x_0 = -\frac{\pi}{2} + 2n\pi$, we have, for $n \geq 1$,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = -\frac{\pi}{2} + 2n\pi + \frac{1}{\cosh\left(-\frac{\pi}{2}+2n\pi\right)}$$
$$< -\frac{\pi}{2} + 2n\pi + \frac{1}{\cosh\left(\frac{3\pi}{2}\right)} = 2n\pi - 1.55283\ldots < 2n\pi.$$

Since f is concave on the second half of the interval, Newton's method started at the right endpoint converges monotonically decreasing to β_n.

7. Exercice

What is the main objective of the fixed-point iteration method?

- ☐ To solve algebraic equations directly
- ☐ To find the root of a function
- ☐ To find a fixed point of a function
- ☐ To approximate the derivative of a function

8. Exercice

Which of the following is a requirement for the convergence of the fixed-point iteration method?

- ☐ Stability of the domain of definition by the function
- ☐ The function must have a root
- ☐ The function must have a Lipschitz and contractant continuous derivative

Exercice : Selected exercices

Exercice 1

Suppose a positive sequence $\{\varepsilon_n\}$ converges to zero with order $p > 0$. Does it then also converge to zero with order p' for any $0 < p' < p$?

Solution

By assumption,

$$\lim n \to \infty \frac{\varepsilon n+1}{\varepsilon_n^p} = c, \quad c \neq 0$$

which implies

$$\frac{\varepsilon_{n+1}}{\varepsilon_n^{p'}} = \frac{\varepsilon_{n+1}}{\varepsilon_n^p}\varepsilon_n^{p-p'} \sim c\varepsilon_n^{p-p'} \to 0 \text{ as } n \to \infty \text{ when } p' < p.$$

Exercice 2

The sequence $\varepsilon_n = \mathrm{e}^{-\mathrm{e}^n}, n = 0, 1, \ldots,$ clearly converges to zero as $n \to \infty$. What is the order of convergence?

Solution 2

We have

$$\frac{\varepsilon_{n+1}}{\varepsilon_n^p} = \frac{\mathrm{e}^{-\mathrm{e}^{n+1}}}{\mathrm{e}^{-p\mathrm{e}^n}} = \mathrm{e}^{-\mathrm{e}^n(\mathrm{e}-p)}.$$

As $n \to \infty$, this tends to a nonzero constant if and only if $p = \mathrm{e}$. Hence, the order of convergence is $\mathrm{e} = 2.71828\ldots$.

Exercice 3

Give an example of a positive sequence $\{\varepsilon_n\}$ converging to zero in such a way that $\lim_{n\to\infty} \frac{\varepsilon_{n+1}}{\varepsilon_n^p} = 0$ for some $p > 1$, but not converging (to zero) with any order $p' > p$.

Solution 3

Take, for example, $\varepsilon_n = \exp\left(-np^n\right)$. Then

$$\frac{\varepsilon_{n+1}}{\varepsilon_n^p} = \frac{\exp\left(-(n+1)p^{n+1}\right)}{\exp\left(-np^{n+1}\right)} = \exp\left(-p^{n+1}\right) \to 0 \text{ as } n \to \infty,$$

but

$$\frac{\varepsilon_{n+1}}{\varepsilon_n^{p'}} = \exp\left(-(n+1)p^{n+1} + np'p^n\right) = \exp\left(p^n[(p'-p)n - p]\right) \to \infty$$

for any $p' > p$.

Exercice 4

- (a) Consider the iteration $x_{n+1}=x_n^3$. Give a detailed discussion of the behavior of the sequence $\{x_n\}$ in dependence of x_0.
- (b) Do the same as (a), but for $x_{n+1}=x_n^{1/3}, x_0>0$.

Solution 4

- (a) If $|x_0|>1$, then $x_n \rightarrow \operatorname{sgn}(x_0) \cdot \infty$. If $|x_0|=1$, then trivially $x_n=\operatorname{sgn}(x_0)$ for all n. If $|x_0|<1$, then $x_n \rightarrow 0$ monotonically decreasing, if $x_0>0$, and monotonically increasing, if $x_0<0$, the order of convergence being $p=3$ in either case.
- (b) Now, x_n converges to $\alpha=1$, monotonically increasing if (b) Now, x_n converges to $\alpha=1$, monotonically increasing if $0<x_0<1$, monotonically decreasing if $x_0>1$, and trivially if $x_0=1$. Since for the iteration function $\varphi(x)=x^{1/3}$ we have $\varphi'(\alpha)=1/3$, convergence is linear with asymptotic error constant equal to $1/3$.

Exercice 5

Consider the quadratic equation $x^2-p=0, p>0$. Suppose its positive root $\alpha=\sqrt{p}$ is computed by the method of false position starting with two numbers a, b satisfying $0<a<\alpha<b$. Determine the asymptotic error constant c as a function of b and α. What are the conditions on b for $0<c<\frac{1}{2}$ to hold, that is, for the method of false position to be (asymptotically) faster than the bisection method?

Solution 5

When $f(x)=x^2-p$, we have

$$
\begin{aligned}
c & =1-(b-\alpha) \frac{f'(\alpha)}{f(b)}=1-(b-\alpha) \frac{2 \alpha}{b^2-p} \\
& =1-\frac{(b-\alpha) \cdot 2 \alpha}{(b-\alpha)(b+\alpha)}=1-\frac{2 \alpha}{b+\alpha} \\
& =\frac{b-\alpha}{b+\alpha} .
\end{aligned}
$$

Thus, there holds Thus, there holds $0<c<\frac{1}{2}$ precisely if <c<\frac{1}{2}$ precisely if

$\alpha<b<3 \alpha$.

Exercice 6 (home work)

Consider the iteration

$$x_{n+1}=\varphi(x_n), \quad \varphi(x)=\sqrt{2+x}.$$

- (a) Show that for any positive x_0 the iterates x_n remain on the same side of $\alpha=2$ as x_0 and converge monotonically to α.
- (b) Show that the iteration converges globally, that is, for any $x_0>0$, and not faster than linearly (unless $x_0=2$).
- (c) If (c) If $0<x_0<2$, how many iteration steps are required to obtain α with an error less than 10^{-10} ? $0<x_0<2$, how many iteration steps are required to obtain α with an error less than 10^{-10} ?

Exercice 7 (home work)

Consider "Kepler's equation"

$$f(x) = 0, \quad f(x) = x - \varepsilon \sin x - \eta, \quad 0 < |\varepsilon| < 1, \quad \eta \in \mathbb{R},$$

where ε, η are parameters constrained as indicated.

- (a) Show that for each ε, η there is exactly one real root $\alpha = \alpha(\varepsilon, \eta)$. Furthermore, $\eta - |\varepsilon| \leq \alpha(\varepsilon, \eta) \leq \eta + |\varepsilon|$.
- (b) Writing the equation in fixed point form

$$x = \varphi(x), \quad \varphi(x) = \varepsilon \sin x + \eta,$$

show that the fixed point iteration $x_{n+1} = \varphi(x_n)$ converges for arbitrary starting value x_0.

- (c) Let m be an integer such that $m\pi < \eta < (m+1)\pi$. Show that Newton's method with starting value

$$x_0 = \begin{cases} (m+1)\pi \text{ if } (-1)^m \varepsilon > 0, \\ m\pi \text{ otherwise} \end{cases}$$

is guaranteed to converge (monotonically) to $\alpha(\varepsilon, \eta)$.

- (d) Estimate the asymptotic error constant c of Newton's method.

Exam

1. Final Exam Of Numerical Analysis

Final Exam Of Numerical Analysis

1.1. Exercice 1

Among the following functions, which ones are contracting and on which intervals if it is not indicated:

- $g(x) = 5 - \frac{1}{4}\cos(3x), \quad 0 \leq x \leq \frac{2\pi}{3}$
- $g(x) = 2 + \frac{1}{2}|x|, \quad -1 \leq x \leq 1$
- $g(x) = \frac{1}{x}, \quad x \in [2, 3]$
- $g(x) = \sqrt{x+2}$

1.2. Exercice 2

Among the following functions, which ones are contracting and on which intervals if it is not indicated:

(a) $g(x) = \frac{1}{\sqrt{x}}$.
(b) $g(x) = e^{-x}$.
(c) $g(x) = (x-2)^2 + x - \frac{e^x}{\pi}$.
(d) $g(x) = x + (x-2)^3$.

1.3. Exercice 3

Consider the problem of calculating $\sqrt{2}$. This amounts to finding the positive zero $\alpha = \sqrt{2}$of the function $f(x) = x^2 - 2$, i.e., solving a nonlinear equation. Verify that $\alpha = \sqrt{2}$ is a fixed point of the function $g(x) = -\frac{1}{4}x^2 + x + \frac{1}{2}$ Then, prove that for $x^{(0)} \in [1, 2]$, there exists a constant $C > 0$ such that $|x^{(k)} - \alpha| \leq C^k |x^{(0)} - \alpha| \quad \forall k > 0$ What is the behavior of the sequence $\{x^{(k)}\}$as $k \to \infty$? How many iterations of the fixed point method are necessary to find an approximate value of $\sqrt{2}$ that is accurate to the tenth decimal place? (Hint: an estimation of the constant C is needed).

1.4. Exercice 4

Let α be a double root of the function f, i.e., $f(\alpha) = f'(\alpha) = 0$.

- Taking into account that we can write the function f as $f(x) = (x-\alpha)^2 h(x) \quad \text{where} \quad h(\alpha) \neq 0$, verify that Newton's method for approximating the root α is only of order 1.
- Consider the following modified Newton's method: $x^{(k+1)} = x^{(k)} - 2\frac{f(x^{(k)})}{f'(x^{(k)})}$ Verify that this method is of order 2 when approaching α.

Conclusion

Numerical Analysis is very important in numerous engineering and scientific problems. Through different types of numerical methods, we will understand the meaning and its applications.

Numbers play an important role in solving different problems. Whenever we want to make a rational decision in the stock market or something that is very unpredictable, we try to analyze the past numbers to get a clear picture of stocks and the variables. This analysis is possible because there is a specialized branch of mathematics that is dedicated to the analysis of these numerical methods and algorithms. This branch is called numerical analysis.

In this lecture, we looked at one of the most important branches of mathematics used for approximations and calculations, called Numerical Analysis. From the **introduction of Numerical Analysis** to its application, we understood how it plays a major role in scientific, engineering, and other disciplines. We also looked at different types of Numerical computation methods like linear and non linear algebraic equations,

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