# Chapter_1

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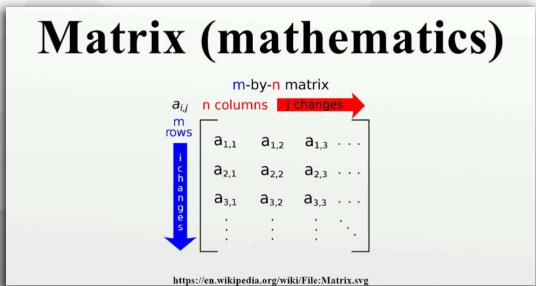


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# I Matrices

## 1. Introduction to Matrix

**Matrices** hold a pivotal position in the realm of linear algebra, serving as versatile mathematical entities. The term matrix was proposed by Sylvester in his 1850 *article*[1]* in the Philosophical Magazine. See *Muir*[2]*. Beyond their fundamental role in representing systems of linear equations, matrices also serve as adept representations of linear functions, commonly referred to as linear mappings. This chapter aims to delve into the multifaceted world of matrices, providing insights into various types of matrices, elucidating operations performed on them, and exploring the concept of matrix inverses. By the end of this exploration, readers will gain a comprehensive understanding of how matrices intricately weave through the fabric of linear algebra, offering indispensable tools for mathematical analysis and problem-solving.

## 2. Specific Objectives

The goal of this chapter is to :

1. Recall different types of matrices (e.g., square matrices, symmetric matrices, identity matrices).
2. Explain the concept of matrix addition and subtraction.
3. Solve systems of linear equations using a matrix method.
4. Analyze the properties of the inverse matrix imethod.

# 3. General

In this section, we delve into different types of matrices, which quotes from this *book*[3]*, to enhance your understanding see[2].

## Matrix

Let $m, n \in \mathbb{N}$, a real-valued $(m, n)$ matrix $A$ is an $m \times n$-tuple of elements $a_{ij} \in \mathbb{R}$, where $i = 1, \ldots, m$ and $j = 1, \ldots, n$, which is ordered according to a rectangular scheme consisting of $m$ rows and $n$ columns:

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}.$$

Also, can be written $A \in \mathbb{R}^{n \times m}$. By convention $(1, n)$-matrices are called rows and $(m, 1)$-matrices are called columns. These special matrices are also called row/column vectors.

***Example*** **: Matrix**

$A \in \mathbb{R}^{4 \times 5}$,

$$A = \begin{bmatrix} 2 & 3 & 1 & 3 & 7 \\ 3 & 9 & 4 & 9 & 5 \\ 4 & 2 & 0 & 4 & 1 \\ 2 & 1 & 3 & 2 & 3 \end{bmatrix}.$$

## Square matrix

A **square matrix** is a matrix where the number of rows $(m)$ is equal to the number of columns $(n)$, often denoted as $m = n$. It is represented as follows:

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix}$$

In this matrix $A \in \mathbb{R}^{n \times n}$.

***Example*** **: Square matrix**

Square matrix, $A \in \mathbb{R}^{4 \times 4}$,

$$A = \begin{bmatrix} 2 & 3 & 1 & 3 \\ 3 & 9 & 4 & 9 \\ 4 & 2 & 0 & 4 \\ 2 & 1 & 3 & 2 \end{bmatrix}.$$

2. https://www.geeksforgeeks.org/types-of-matrices/

## Lower triangular matrix

Let $A$ is square matrix, we can say $A$ **lower triangular** matrix if

$$A = \begin{bmatrix} a_{11} & 0 & \dots & 0 \\ a_{21} & a_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}.$$

In this matrix $A \in \mathbb{R}^{n \times n}$.

***Example*** **: Lower triangular matrix**

Lower triangular matrix, $A \in \mathbb{R}^{4 \times 4}$,

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 3 & 9 & 0 & 0 \\ 4 & 2 & 1 & 0 \\ 2 & 1 & 3 & 2 \end{bmatrix}.$$

## Upper triangular matrix

Let $A$ is square matrix, we can say $A$ **upper triangular** matrix if

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{bmatrix}.$$

***Example*** **: Upper triangular matrix**

Upper triangular matrix, $A \in \mathbb{R}^{4 \times 4}$,

$$A = \begin{bmatrix} 2 & 1 & 3 & 5 \\ 0 & 9 & 6 & 7 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

## Diagonal matrix

Let $A$ is square matrix, we can say $A$ is **diagonal matrix** if

$$A = \begin{bmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{bmatrix}.$$

In this matrix $A \in \mathbb{R}^{n \times n}$.

**Example : Diagonal matrix**

Diagonal matrix, $A \in \mathbb{R}^{4\times4}$,

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

## Identity matrix

The **identity matrix** $I_n$ of size n is the $n-$by$-n$ matrix in which all the elements on the main diagonal are equal to $1$ and all other elements are equal to $0$, for example,

$$I_1 = [1],\ I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \dots,\ I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}.$$

## Transpose matrix

The **transpose** of a matrix $A$, denoted as $A^T$ or $A'$, is obtained by interchanging its rows and columns. In other words, if $A$ is an $m \times n$ matrix, then its transpose $A^T$ is an $n \times m$ matrix, and the elements of $A^T$ are defined as:

$$A' = A^T = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{bmatrix}, \text{ or } (A^T)_{ij} = a_{ji}.$$

where $1 \leq i \leq n$ and $1 \leq j \leq m$.

**Example : Transpose matrix**

Transpose matrix , $A \in \mathbb{R}^{3\times4}$,

$$A = \begin{bmatrix} 2 & 7 & 8 & 1 \\ 4 & 9 & 3 & 7 \\ 1 & 4 & 1 & 2 \end{bmatrix}.$$

The transpose of matrix $A$:

$$A^T = \begin{bmatrix} 2 & 4 & 1 \\ 7 & 9 & 4 \\ 8 & 3 & 1 \\ 1 & 7 & 2 \end{bmatrix}.$$

### Square matrix

A square matrix $A$ is symmetric if and only if $A = A^T$,

$$A = \begin{bmatrix} 3 & 1 & 4 \\ 1 & 5 & 2 \\ 4 & 2 & 6 \end{bmatrix}, \quad A^T = \begin{bmatrix} 3 & 1 & 4 \\ 1 & 5 & 2 \\ 4 & 2 & 6 \end{bmatrix}.$$

To gain a more comprehensive insight, consider watching this video[3].

## 4. Quiz : Section test

[solution n°1 p. 16]

Let $A = a_{ij}$ is a square matrix

- If $i < j$ or $i > j$, then $A$ is ______.
- If $i = j$, then $A$ is ______.
- If $a_{ij} = 0$ for $i > j$, then $A$ is ______.
- If $a_{ij} = 0$ for $i < j$, then $A$ is ______.
- If $a_{ij} = a_{jI}$ , then $A$ is ______.

## 5. Operations on matrices

In this section, we defined matrix multiplication and matrix addition[4]. Also,we look at some properties of matrices.

### 5.1. Addition of matrices

### Sum

The **sum** of two matrices $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{m\times n}$ is defined as the elementwise sum, i.e.,

$$A + B := \begin{bmatrix} a_{11} + b_{11} & \dots & a_{1n} + b_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & \dots & a_{mn} + b_{mn} \end{bmatrix} \in \mathbb{R}^{m\times n}$$

***Example*** :

Let $A \in \mathbb{R}^{2\times 2}$ and $B \in \mathbb{R}^{2\times 2}$ defined as

- If

$$A = \begin{bmatrix} 3 & -2 \\ 1 & 7 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 5 \\ 2 & -1 \end{bmatrix} \text{ then } A + B = \begin{bmatrix} 3+0 & -2+5 \\ 1+2 & 7-1 \end{bmatrix} = B_0 = \begin{bmatrix} 2 \\ 8 \end{bmatrix} \begin{bmatrix} 3 & 3 \\ 3 & 6 \end{bmatrix}.$$

- However, if $B_0 = \begin{bmatrix} 2 \\ 8 \end{bmatrix}$, then $A + B_0$ is not defined.

3. video - https://www.youtube.com/watch?v=Kbv7rw6sUBo
4. https://www.youtube.com/watch?v=ilFJYjfKYjk&list=PLROOIV7hGpZjm4FMV3xYcSKmFcElLudLH

The product of a matrix $A$ of size $M_{n,m}(K)$ by a scalar $\lambda \in K$ is the matrix formed by multiplying each coefficient of $A$ by $\lambda$. It is denoted as $\lambda \cdot A$ (or simply $\lambda A$).

$$\lambda A = \begin{bmatrix} \lambda a_{11} & \lambda a_{12} & \dots & \lambda a_{1m} \\ \lambda a_{21} & \lambda a_{22} & \dots & \lambda a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda a_{n1} & \lambda a_{n2} & \dots & \lambda a_{nm} \end{bmatrix}.$$

✎**Note** :

The matrix $(-1)A$ is the negation of $A$ and is denoted as $-A$. The difference $A - B$ is defined as $A + (-B)$.

### Properties

Let $A$, $B$, and $C$ be three matrices belonging to $M_{n,p}(K)$. Let $\lambda \in K$ and $\mu \in K$ be two scalars.

- $A + B = B + A$: Addition is commutative.
- $A + (B + C) = (A + B) + C$ : Addition is associative.
- $A + 0 = A$: The zero matrix is the additive identity.
- $(\lambda + \mu)A = \lambda A + \mu A$: Scalar addition distributes over matrix addition.
- $\lambda(A + B) = \lambda A + \lambda B$: Scalar multiplication distributes over matrix addition.
- $(A + B)^T = A^T + B^T$.

## 5.2. Multiplication of matrices

Here, we will call this the dot product of the corresponding row and column. In cases, where we need to be explicit that we are performing multiplication, we use the notation $A.B$ to denote multiplication (explicitly showing" . ").

If $A$is a matrix in $\mathbb{R}^{I\times m}$, and $B$ is a matrix in $\mathbb{R}^{m\times n}$ , then the product matrix $C$ in$\mathbb{R}^{I\times n}$ matrix, and it is defined as:$C = A \cdot B$, where each element of $C$ is given by:

$$C = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{I1} & c_{I2} & \dots & c_{In} \end{bmatrix}, \quad c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj} \text{ for } 1 \leq i \leq I \text{ and } 1 \leq j \leq n$$

The fundamental matrix operation is multiplication of a $m \times n$ matrix A with a column vector of length $n$ to yield a column vector of length $m$. Here is the allimportant formula:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \dots & a_{in} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1j} & \dots & b_{1p} \\ b_{21} & b_{22} & \dots & b_{2j} & \dots & b_{2p} \\ b_{31} & b_{32} & \dots & b_{3j} & \dots & b_{3p} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nj} & \dots & b_{np} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1j} & \dots & c_{1p} \\ c_{21} & c_{22} & \dots & c_{2j} & \dots & c_{2p} \\ \vdots & \vdots & & \vdots & & \vdots \\ c_{i1} & c_{i2} & \dots & c_{ij} & \dots & c_{ip} \\ \vdots & \vdots & & \vdots & & \vdots \\ c_{m1} & c_{m2} & \dots & c_{mj} & \dots & c_{mp} \end{bmatrix}$$

$$a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{in}b_{nj} = c_{ij}$$

**Example :**

Let's consider two matrices, $A$ and $B$, and their product $C$.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 4 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \\ 2 & 1 \end{bmatrix}$$

The product matrix $C$ is calculated as follows:

$$C = A \cdot B = \begin{bmatrix} 1\times 5 + 2\times 7 + 1\times 2 & 1\times 6 + 2\times 8 + 1\times 1 \\ 3\times 5 + 4\times 7 + 0\times 2 & 3\times 6 + 4\times 8 + 0\times 1 \end{bmatrix} = \begin{bmatrix} 21 & 23 \\ 43 & 50 \end{bmatrix}.$$

### Remark

- If $A$ is an $n \times p$ matrix, then $A \cdot I_p = A \text{ and } I_n \cdot A = A$, where $I_n$ and $I_p$are the identity matrices of size $n \times n$ and $p \times p$, respectively.
- The product of two matrices is generally not commutative. If $A$ and $B$ are $n \times n$ matrices, $A \cdot B \neq B \cdot A$.

### Properties

Let $A \in \mathbb{R}^{n\times p}$, $B \in \mathbb{R}^{p\times q}$, and $C \in \mathbb{R}^{q\times s}$ be three matrices that satisfy the following:

- $A(BC) = (AB)C$.
- $A(B + C) = AB + AC$.
- $(A + B)C = AC + BC$.
- If $n = p = q$, we have $(A + B)^2 = A^2 + B^2 + AB + BA$.
- $(AB)^T = B^T A^T$.

## 6. Quiz : Section test

[solution n°2 p. 16]

Let $A \in \mathbb{R}^{n\times m}$ and $B \in \mathbb{R}^{p\times q}$ give us the relation between $n, m$ and $p, q$ to do these operations:

- Addition
- Multiplication

## 7. Determinant of a square matrix

The determinant is **a special number that can be calculated from a matrix[5].** The matrix has to be square (same number of rows and columns) . The determinant of a square matrix $A \in \mathbb{R}^{n\times n}$ is denoted as $\det(A)$, and it is defined as follows:

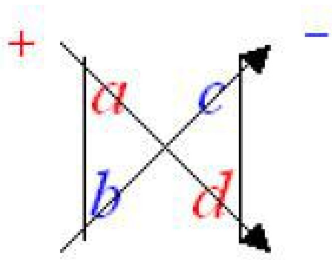
$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}.$$

5. Determinant of a matrix - https://www.mathsisfun.com/algebra/matrix-determinant.html

## 7.1. Determinant of a matrix with $n = 2$

Let $A$ is a matrix in $\mathbb{R}^{2\times 2}$ , $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb.$$




Let us take an example to understand this very clearly,

- If $A = \begin{pmatrix} 2 & 5 \\ 3 & 7 \end{pmatrix}$, then $\det(A) = \begin{vmatrix} 2 & 5 \\ 3 & 7 \end{vmatrix} = 2 \times 7 - 3 \times 5 = -1$.

## 7.2. Determinant of a matrix with $n = 3$

Let $A$ is a matrix in $\mathbb{R}^{3\times 3}$, $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{13} \end{pmatrix}$, the determinant of the matrix $A$ is defined as

$$\det(A) = (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Use the formula for a $2 \times 2$ determinant, then

$\det(A) = a_{11}(a_{22}a_{33} - a_{32}a_{23})| - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$.

If $A = \begin{pmatrix} 2 & 4 & 1 \\ 3 & 2 & 5 \\ 1 & 2 & 4 \end{pmatrix}$, then

$\det(A) = 2(2 \times 4 - 2 \times 5) - 4(3 \times 4 - 1 \times 5) + 1(3 \times 2 - 1 \times 2) = -30.$

## 7.3. Determinant of a matrix with $n > 3$

Let $A$ is a matrix in $\mathbb{R}^{3\times 3}$ ,$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$,the determinant of the matrix $A$ is defined as

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det(A_{1j}),$$

where

$$A_{11} = \begin{bmatrix} a_{22} & a_{23} & \dots & a_{2n} \\ a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n2} & a_{n2} & \dots & a_{nn} \end{bmatrix}, \quad A_{12} = \begin{bmatrix} a_{21} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n3} & \dots & a_{nn} \end{bmatrix}, \dots, \quad A_{1n} = \begin{bmatrix} a_{21} & a_{22} & \dots & a_{2n-1} \\ a_{31} & a_{32} & \dots & a_{3n-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn-1} \end{bmatrix}$$

Let us see an example to find out the determinant of a $n \times n$ matrix, If $A = \begin{pmatrix} 2 & 4 & 1 & 2 \\ 3 & 2 & 5 & 3 \\ 1 & 2 & 4 & 4 \\ 1 & 2 & 8 & 3 \end{pmatrix}$, then the sub-matrices defined as

$$A_{11} = \begin{bmatrix} 2 & 5 & 3 \\ 2 & 4 & 4 \\ 2 & 8 & 3 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 3 & 5 & 3 \\ 1 & 4 & 4 \\ 1 & 8 & 3 \end{bmatrix}, \quad A_{13} = \begin{bmatrix} 3 & 2 & 3 \\ 1 & 2 & 4 \\ 1 & 2 & 3 \end{bmatrix} \text{ and } A_{14} = \begin{bmatrix} 3 & 2 & 5 \\ 1 & 2 & 4 \\ 1 & 2 & 8 \end{bmatrix}.$$

The determinants of the matrices $A_{11}$, $A_{12}$, $A_{13}$, and $A_{14}$ as:
$\det(A_{11}) = 22, \quad \det(A_{12}) = -59, \quad \det(A_{13}) = -14 \text{ and } \det(A_{14})$= -2.
Substituting the calculated values:

$$\begin{aligned} \det(A) &= 2\det(A_{11}) - 4\det(A_{12}) + 1\det(A_{13}) - 2\det(A_{14}) \\ &= 2 \times 22 - 4 \times (-59) + 1 \times (-14) - 2 \times (-2) \\ &= 270 \end{aligned}$$

So, the determinant of matrix $A$ is$270$.

The following properties quotes from Schwartz *book*[3]*.

## Properties

Properties of a determinant:

- Determinant of a Scalar Multiple: $\det(kA) = k^n \cdot \det(A)$.
- Determinant of a Product of Matrices: $\det(AB) = \det(A) \cdot \det(B)$.
- Determinant of the Transpose: $\det(A^T) = \det(A)$.
- Determinant of a Sum of Matrices: $\det(A + B) \neq \det(A) + \det(B)$.
- Determinant of a Diagonal Matrix: $\det(A) = a_{11} \times a_{22} \times \ldots \times a_{nn}$.
- Determinant of the Identity Matrix:$\det(I) = 1$.
- Determinant of an Upper Triangular Matrix: $\det(A) = a_{11} \times a_{22} \times \ldots \times a_{nn}$.
- Determinant of a Lower Triangular Matrix: $\det(L) = a_{11} \times a_{22} \times \ldots \times a_{nn}$.

## 8. Quiz : Section test

[solution n°3 p. 16]

Consider a square matrix $A$ with a determinant of zero. What can you conclude about the matrix $A$?

- $A$ is not invertible.
- $A$ is a diagonal matrix.
- $A$ has all its eigenvalues equal to zero.
- $A$ is a symmetric matrix.

## 9. Inverse of Matrix

The inverse of a matrix can only be defined for square matrices if the determinant of that matrix is non-zero $\det(A) \neq 0$.

***Definition*** **:**

Let $A$ be the matrix belonging to $\mathbb{R}^{n\times n}$ with $\det(A) \neq 0$, we can say $A$ is invertible matrix if there exists a matrix $B$ in $\mathbb{R}^{n\times n}$ such as $BA = AB = I_n$. We notice $A^{-1} = B$.

***Example*** **:**

Let , $A, B \in \mathbb{R}^{2\times 2}$,

$A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -5 \\ 1 & 2 \end{bmatrix}$.

We need to show that the product of these two matrices is the identity matrix, $I$. In other words, we need to demonstrate that:

$AB = BA = I$.

Let's calculate the product $AB$:

$$\begin{aligned} AB = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} &= \begin{bmatrix} (2\times 3 + 5\times -1) & (2\times -5 + 5\times 2) \\ (1\times 3 + 3\times -1) & (1\times -5 + 3\times 2) \end{bmatrix} \\ &= \begin{bmatrix} (6-5) & (-10+10) \\ (3-3) & (-5+6) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2, \end{aligned}$$

and

$$\begin{aligned} BA = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} &= \begin{bmatrix} (3\times 2 + (-5)\times 1) & (3\times 5 + (-5)\times 3) \\ ((-1)\times 2 + 2\times 1) & ((-1)\times 5 + 2\times 3) \end{bmatrix} \\ &= \begin{bmatrix} (6-5) & (15-15) \\ (-2+2) & (-5+6) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2. \end{aligned}$$

Since the product of matrices $A$ and $B$ yields the identity matrix, we can conclude that $A^{-1} = B$ as desired.

## 9.1. Inverse of a matrix with $n = 2$

Let , $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. with $\det(A) \neq 0$, then the inverse of $A$ is defined as

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} a_{22} & -a_{12} \\ a_{21} & a_{11} \end{bmatrix} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ a_{21} & a_{11} \end{bmatrix}$$

***Example*** :

Let's find the inverse of the $2 \times 2$ matrix $A = \begin{bmatrix} 1 & 4 \\ 7 & 2 \end{bmatrix}$. Therefore, the inverse of matrix $A$ is:

$$\begin{bmatrix} 2 & -4 \\ -7 & 1 \end{bmatrix} = \frac{1}{-26} \begin{bmatrix} 2 & -4 \\ -7 & 1 \end{bmatrix}.$$

## 9.2. Inverse of a matrix with $n = 3$

Let, $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{13} \end{pmatrix}$ with $\det(A) \neq 0$,

the cofactor of $A$ denoted by $C_A$ and defined as

$$C_A = \begin{pmatrix} (-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & (-1)^{1+2}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} & (-1)^{1+3}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ (-1)^{2+1}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} & (-1)^{2+2}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & (-1)^{2+3}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \\ (-1)^{3+1}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{32} \end{vmatrix} & (-1)^{3+2}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & (-1)^{3+3}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{pmatrix}$$

The inverse matric of $A$ defined as

$$A^{-1} = \frac{1}{\det(A)} C_A^T = \frac{1}{\det(A)} \begin{bmatrix} (-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & (-1)^{2+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & (-1)^{3+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \\ (-1)^{1+2}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} & (-1)^{2+2}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} & (-1)^{3+2}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \\ (-1)^{1+3}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & (-1)^{2+3}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & (-1)^{3+3}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \end{bmatrix}.$$

***Example*** :

For given matrix $A$ defined as $A = \begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & 1 \\ 1 & 3 & 2 \end{bmatrix}$, the Cofactor of $A$ is $C_A = \begin{bmatrix} 2 & -2 & -3 \\ 3 & 0 & -2 \\ 1 & 3 & 0 \end{bmatrix}$.

Then the adjoint of $A$ is

$$C_A^T = \begin{bmatrix} 2 & -3 & 1 \\ -2 & 0 & 3 \\ -3 & -2 & 0 \end{bmatrix}.$$

This implis, the inverse of $A$ is

$$A^{-1} = \frac{1}{|A|} C_A^T = \frac{1}{3} \begin{bmatrix} 2 & -3 & 1 \\ -2 & 0 & 3 \\ -3 & -2 & 0 \end{bmatrix}.$$

## 10. Quiz : Section test

[solution n°4 p. 17]

Let $A$ is matrice defined as

*(see mathTex_74.mtex)* *(cf. p.15)*

- Determine the inverse of $A$.

○ *(see mathTex_75.mtex)* *(cf. p.15)*

○ *(see mathTex_76.mtex)* *(cf. p.15)*

# Complementary resources

Mathematical equation

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 3 \\ -1 & 2 & -1 \end{bmatrix}.$$

Mathematical equation

$$A^{-1} = \begin{bmatrix} 2 & -1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ -1 & 1/2 & -1/2 \end{bmatrix}.$$

Mathematical equation

$$A^{-1} = \begin{bmatrix} 2 & -1 & 1/2 \\ 1/2 & 0 & 1/2 \\ -1 & 1/2 & -1/2 \end{bmatrix}.$$

# Exercise solutions

## Solution n°1

[exercice p. 7]

Let $A = a_{ij}$ is a square matrix

- If $i < j$ or $i > j$, then $A$ is `a rectangular matrix`.
- If $i = j$, then $A$ is `a square matrix`.
- If $a_{ij} = 0$ for $i > j$, then $A$ is `upper triangular matrix`.
- If $a_{ij} = 0$ for $i < j$, then $A$ is `Lower triangular matrix`.
- If $a_{ij} = a_{jI}$ , then $A$ is `Symmetric matrix`.

if you didn't give the correct answers you need to come back to study the previous section.

## Solution n°2

[exercice p. 9]

Let $A \in \mathbb{R}^{n \times m}$ and $B \in \mathbb{R}^{p \times q}$ give us the relation between $n, m$ and $p, q$ to do these operations:

- Addition
- Multiplication

For addition n=p and m=q while for multiplication m=q.

To enhance your skills in matrix operations, watch this video[6]

## Solution n°3

[exercice p. 12]

Consider a square matrix $A$ with a determinant of zero. What can you conclude about the matrix $A$?

- (●) $A$ is not invertible.
- ( ) $A$ is a diagonal matrix.
- ( ) $A$ has all its eigenvalues equal to zero.
- ( ) $A$ is a symmetric matrix.

if you didn't give the correct answers you need to come back to study the previous section.

6. https://www.youtube.com/watch?v=p48uw2vFWQs

# Solution n°4

[exercice p. 14]

Let $A$ is matrice defined as

*(see mathTex_74.mtex)* *(cf. p.15)*

- Determine the inverse of $A$.

◉ *(see mathTex_75.mtex)* *(cf. p.15)*

○ *(see mathTex_76.mtex)* *(cf. p.15)*

If you didn't provide the correct answers, you should revisit the previous section for further study. You can also visit this website for additional information https://www.mathsisfun.com/algebra/matrix-inverse.html[7].

7. Inverse - https://www.mathsisfun.com/algebra/matrix-inverse.html

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