Chapter_2

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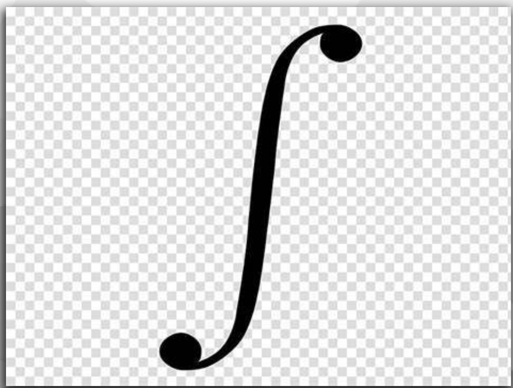

Dr. Walid Remili

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I Simple, double and triple Integrals

1. Introduction

In this chapter, we will not develop the general theory of the integral of a function of n variables over a bounded part of $\mathbb{R}^n$, sourced from *reference*[4]*, where $n \geq 1$. We will content ourselves with providing methods for calculating simple, double and triple integrals (i.e., $n = 2, n = 3$) over specific compacts, those whose boundaries can be delimited by continuous functions.

2. Specific Objectives

The goal of this chapter is to :

1. Identify the basic concepts of integration (simple, double and triple).
2. Explain the geometric interpretation of a definite integral as an area under a curve.
3. Application integration methods.
4. Synthesize the integration method

3. Simple integral

The indefinite integral is the inverse problem of finding the derivative of a given function. In calculus, the process of integration involves determining a function whose derivative is the original function. It plays a crucial role in various mathematical and scientific applications, providing a way to compute accumulated quantities such as area under a curve or total change in a quantity over an interval. Indefinite integrals are represented symbolically using the integral sign $(\int)$ and are characterized by the inclusion of an arbitrary constant of integration. This constant reflects the family of functions that differ only by a constant value. Therefore, the indefinite integral encapsulates a wide range of possible antiderivatives, each differing by this constant.

Definition :

Let f be a continuous function on an interval $I \subseteq \mathbb{R}$. We say that a function F is an antiderivative (primitives) of f if and only if $F'(x) = f(x)$ on I or,

$$\int f = F + C \text{ where } C \in \mathbb{R}.$$

$f(x)$	$\int f(x)dx = F(x) + C$	$f(x)$	$\int f(x)dx = F(x) + C$
a (cste)	$ax + C$	$\frac{1}{x^2 + a^2}$	$\frac{1}{a} arctg \frac{x}{a} + C \quad a \neq 0$
x^n	$\frac{x^{n+1}}{n+1} + C \quad n \neq -1$		$-\frac{1}{a} arcctg \frac{x}{a} + C \quad a \neq 0$
$\frac{1}{x}$	$ln\|x\| + C$	$\frac{1}{a^2 - x^2}$	$\frac{1}{2a} ln \left\| \frac{a+x}{a-x} \right\| + C \quad a \neq 0$
$\frac{1}{x+a}$	$ln\|x + a\| + C$	$\frac{1}{\sqrt{x^2 \pm a}}$	$ln \left\| x + \sqrt{x^2 \pm a} \right\| + C \quad a \neq 0$
a^x	$\frac{1}{lna} a^x + C \quad a > 0$		
e^x	$e^x + C$	$\frac{1}{\sqrt{a - x^2}}$	$arcsin \frac{x}{a} + C \quad a > 0$
			$-arccos \frac{x}{a} + C \quad a > 0$
e^{ax+b}	$\frac{1}{a} e^{ax+b} + C$	$\frac{1}{sinx}$	$ln \left\| tg \frac{x}{2} \right\| + C$
$sinx$	$-cosx + C$	$\frac{1}{cosx}$	$ln \left\| tg \frac{x}{2} + \frac{\pi}{4} \right\| + C$
$\sin(ax + b)$	$-\frac{1}{a} \cos(ax + b) + C$	$\frac{1}{sin^2 x}$	$-ctgx + C$
$cosx$	$sinx + C$	$\frac{1}{cos^2 x}$	$tgx + C$
$\cos(ax + b)$	$\frac{1}{a} \sin(ax + b) + C$		

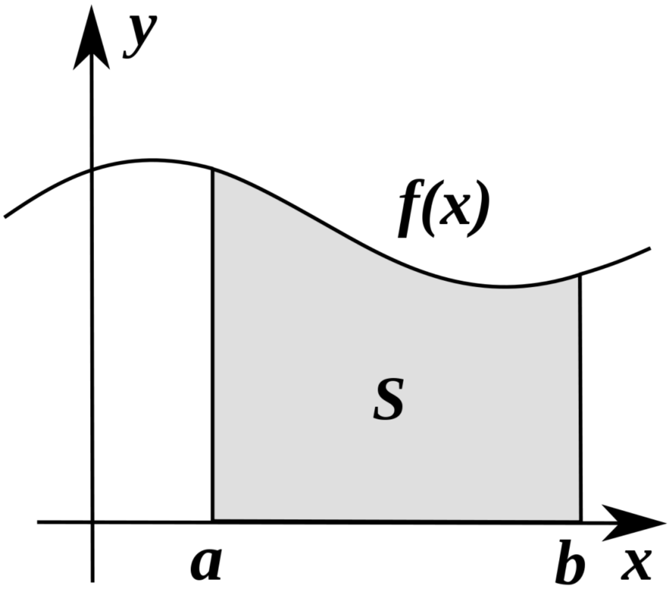
Table of primitives of elementary functions.

Theorem

Any continuous function on an interval has a primitive on that interval.

Definition :

The definite integral of a function f on the interval $[a, b]$ is a real number denoted as $\mathrm{S} = \int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$.



Integral of f from a to b.

Properties

Let f and g be two continuous functions on $[a, b]$, and let α and β be real numbers.

- $\int[\alpha f(x) + \beta g(x)]\,dx = \alpha \int f(x)\,dx + \beta \int g(x)\,dx$ for $a \leq x \leq b$.
- $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$ for $a \leq b$.
- $\int_a^a f(x)\,dx = 0$ for any a.
- $\int_{-a}^a f(x)\,dx = 0$ if f is an odd function.
- $\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$ if f is an even function.
- If f≥ 0 on $[a, b]$, then $\int_a^b f(x)\,dx \geq 0$.
- $Iff \leq g$ on $[a, b]$, then $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$.
- $|\int_a^b f(x)\,dx| \leq \int_a^b |f(x)|\,dx$.

3.1. Integration methods

Integration by Parts

Let u and v be two functions of class C^1 on $[a, b]$. The integration by parts formula is given by:

$$\int_a^b uv' = [uv]_a^b - \int_a^b u'v.$$

Example :

Evaluate the integral $\int_0^\pi x\sin(x)\,dx$

$$\begin{aligned}\int_0^\pi x\sin(x)\,dx &= [-x\cos(x)]_0^\pi + \int_0^\pi \cos(x)\,dx \quad \text{(Integration by parts)}\\ &= -\pi\cos(\pi) - (-0\cos(0)) + \int_0^\pi \cos(x)\,dx\\ &= \pi + \int_0^\pi \cos(x)\,dx\\ &= \pi + [\sin(x)]_0^\pi\\ &= \pi + (\sin(\pi) - \sin(0))\\ &= \pi + (0-0)\\ &= \pi.\end{aligned}$$

Therefore, $\int x\sin(x)\,dx = \pi$.

Integration by Change of Variable

Let f be a continuous function on $I \subseteq \mathbb{R}$, and let $u : J \subseteq \mathbb{R} \to I$ be a C^1 function on J with $u(t) = x$. Then,

$$\int_a^b f(x)dx\,dx = \int_{u^{-1}(a)}^{u^{-1}(b)} f(u(t))u'(t)\,dt.$$

Example :

Evaluate the integral $\int_0^1 \sqrt{1-x^2}\,dx$. Let $x = u(t) = \sin(t)$ such that $u'(t) = \cos(t)$ and $x = 0 \to t = 0$ and $x = 1 \to t = \dfrac{\pi}{2}$.

$$\int_0^1 \sqrt{1-x^2}\,dx = \int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2(t)}\cos(t)\,dt = \int_0^{\frac{\pi}{2}} \cos^2(t)\,dt.$$

Using the identity $\cos(2t) = \cos^2(t) - \sin^2(t) = 2\cos^2(t) - 1$, we get $\cos^2(t) = \frac{1}{2}(1+\cos(2t))$.

$$\int_0^{\frac{\pi}{2}} \cos^2(t)\,dt = \frac{1}{2}\int_0^{\frac{\pi}{2}} (1+\cos(2t))\,dt = \frac{1}{2}\left[t + \frac{1}{2}\sin(2t)\right]_0^{\frac{\pi}{2}} = \frac{\pi}{4}.$$

Therefore, $\int_0^1 \sqrt{1-x^2}\,dx = \frac{\pi}{4}$.

4. Simple integral test

Quiz 1

[solution n°1 p. 14]

Which integration technique is used to find the area under a curve?

- ☐ Integration by Substitution
- ☐ Integration by Parts
- ☐ Integration by Partial Fractions
- ☐ Definite Integration

Quiz 2

[solution n°2 p. 14]

Explain the meaning of the definite integral $\int_a^b f(x)dx$.

- ☐ It represents the area under the curve of f(x) from a to b
- ☐ It represents the antiderivative of f(x) from a to b.
- ☐ It represents the derivative of f(x) from a to b
- ☐ It represents the limit of f(x) as x approaches b.

Quiz 3

[solution n°3 p. 14]

Compute the integral: $\int_1^3 \frac{3}{x} dx$

- ○ $2\ln(3)$
- ○ $3\ln(3)$
- ○ $\frac{2}{3}$
- ○ $\frac{1}{3}$

5. Double Integral

The general form of a double integral *Wikipedia* Wikipedia*over a region D is written as follows:

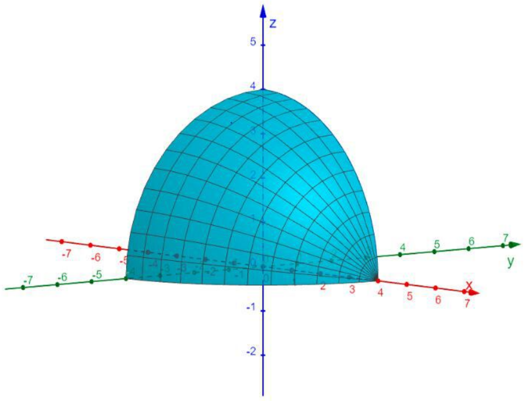
$$\iint_D f(x,y)\,dx\,dy,$$

1. D represents the domain of integration, specifying the region in the $xy-$plane over which the integration is performed.
2. $f(x,y)$ is the function being integrated over the specified domain.

Definition :

If $D = [a,b] \times [c,d]$ is a rectangle defined by $a \le x \le b$ and $c \le y \le d$, the double integral would be written as:

$\iint_D f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx.$
If $f(x,y) = f_1(x) f_2(y)$, we have
$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_a^b f_1(x)dx \int_c^d f_2(y)\,dy.$



Integral of f on D.

Example :

Let $I = \int \int (2x+y)\,dx\,dy = ?.$

- $I = \int_0^1 \left(\int_0^1 (2x+y)\,dx \right) dy = \int_0^1 (1+y)\,dy = \dfrac{3}{2}.$
- $I = \int_0^1 \left(\int_0^1 (2x+y)\,dy \right) dx = \int_0^1 (2x+12)\,dx = \dfrac{3}{2}.$

Double Integral over a Non-Rectangular Domain

Let f be a continuous function on a domain $D \subseteq \mathbb{R}^2$. The domain D can be represented in one of the following forms:

- Case 2: $D = \{(x,y) \in \mathbb{R}^2 \,|\, c \le y \le d \text{ and } \phi(y) \le x \le \psi(y)\}$. The double integral is given by: $\iint_D f(x,y)\,dx\,dy = \int_c^d \int_{\phi(y)}^{\psi(y)} f(x,y)\,dx\,dy.$
- Case 2: $D = \{(x,y) \in \mathbb{R}^2 \,|\, c \le y \le d \text{ and } \phi(y) \le x \le \psi(y)\}$. The double integral is $\iint_D f(x,y)\,dx\,dy = \int_c^d \int_{\phi(y)}^{\psi(y)} f(x,y)\,dx\,dy.$

Example :

Consider the double integral: I = $\iint_D y\,dx\,dy$, where the domain D is defined as:

$$D = \{(x,y) \in \mathbb{R}^2 \,|\, 1 \ge x \ge 0, 1 \ge y \ge 0, \text{ and } x+y \le 1\}.$$

- Expression 1: $I = \int_0^1 \int_0^{1-x} y\,dy\,dx = \frac{1}{2}\int_0^1 (x^2 - 2x + 1)\,dx = \frac{1}{6}.$
- Expression 2: $I = \int_0^1 y \int_0^{1-y} dx\,dy = \int_0^1 (y - y^2)\,dy = \frac{1}{6}.$

5.1. Integration methods

a) Double Integration with Change of Variables

Let $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ be a transformation given by

$$\phi(u,v) = \begin{bmatrix} x(u,v) \\ y(u,v) \end{bmatrix}.$$

The Jacobian matrix of ϕ is given by the matrix of partial derivatives:

$$J_\phi(u,v) = \begin{bmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{bmatrix}.$$

Affine change

Consider a continuous function f on a domain $D \subseteq \mathbb{R}^2$. Let $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ be a bijective affine transformation defined by $\phi(u,v) = (x,y)$. We have the following expression for the double integral:

$$\iint_D f(x,y)\,dx\,dy = \iint_\Delta f(\phi(u,v),\phi(u,v))\,J\,du\,dv,$$

where $\Delta = \phi^{-1}(D)$ and $J = |J_\phi(u,v)|$.

Example :

Consider the double integral: $I = \iint_D (x+y)\,dx\,dy$, where the domain D is defined as:

$$D = \{(x,y) \in \mathbb{R}^2 \,|\, 1 \leq x - y \leq 2, -1 \leq x + 3y \leq 1\}.$$

Let's introduce the following change of variables:

$$\begin{cases} u = x - y \\ v = x + 3y \end{cases} \Rightarrow \begin{cases} x = \frac{1}{4}(3u+v) \\ y = \frac{1}{4}(-u+v) \end{cases}$$

The Jacobian determinant is given by $J = \dfrac{1}{4}$, where Δ satisfies undefined $1 \leq u \leq 2$ and $-1 \leq v \leq 1$. The integral is then transformed into:

$$I = \frac{1}{8}\iint_\Delta (u+v)\,du\,dv = \frac{1}{8}\left(\int_1^2 u\,du \int_{-1}^1 dv + \int_{-1}^1 dv \int_1^2 v\,du\right) = \frac{3}{8}.$$

Change to Polar Coordinates

Consider a change to polar coordinates given by:

$$\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \end{cases}$$

where the Jacobian determinant is $J = r$. Let Δ be the region in the $xy-$plane and D be its image under the transformation. The double integral is then transformed as follows:

$$\iint_D f(x,y)\,dx\,dy = \iint_\Delta f(r\cos(\theta), r\sin(\theta))\,r\,dr\,d\theta.$$

Example :

Consider the double integral: $I = \iint_D xy\,dx\,dy$, where the domain D is defined as:

$$D = \{(x,y) \in \mathbb{R}^2 \,|\; x \geq 0,\; y > 0,\; \text{and } x^2 + y^2 \leq 1\}.$$

Now, let's perform a change to polar coordinates:

$$\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \end{cases},$$

with Jacobian determinant $J = r$. The region Δ in polar coordinates is defined as with Jacobian determinant $J = r$. The region Δ in polar coordinates is defined as $0 \leq r \leq 1$ and $0 \leq \theta \leq \frac{\pi}{2}$. $0 \leq r \leq 1$ and $0 \leq \theta \leq \frac{\pi}{2}$. The integral becomes:

$$I = \int_0^{\frac{\pi}{2}} \int_0^1 r^3 \sin(\theta)\cos(\theta)\, dr\, d\theta = \left(\int_0^1 r^3\, dr\right)\left(\frac{1}{2}\int_0^{\frac{\pi}{2}} \sin(2\theta)\, d\theta\right) = \frac{1}{8}.$$

6. Double integral test

Quiz 1

[solution n°4 p. 14]

What does a double integral $\iint f(x,y)dxdy$. represent geometrically?

- Volume under the surface $f(x,y)$.
- Area under the curve $f(x,y)$.
- Slope of the tangent plane to $f(x,y)$.
- Length of the curve $f(x,y)$.

Quiz 2

[solution n°5 p. 15]

Match the correct concept or definition with its corresponding term or symbol.

The region in the xy-plane over which the double integral is evaluated.

$\iint_R f(x,y)dA$

$\int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} f(r,\theta) r dr d\theta$

$dxdy$

Double Integral	Region of Integration	Polar Coordinates	Rectangular Coordinates

Quiz 3

[solution n°6 p. 15]

Consider the region R in the xy-plane bounded by the curves $x^2 < y < 4$ and $0 < x < 2$. What is the double integral $\iint_R f\,(x+y)dA >$?

- ☐ 18
- ☐ 15
- ☐ 8
- ☐ 19

7. Triple Integral

In this subsection, we present the definition and properties of triple integrals in measurable sets *Paper*$^{\mathit{Paper}}{}_{*}$, or more understanding see .

Definition **:**

Let f be a continuous function on a domain $D \subseteq \mathbb{R}^3$. The triple integral of f over D is denoted by I and is defined as:

$$I = \iiint_D f(x,y,z)\,dx\,dy\,dz.$$

Fubini's Theorem

Let $D = [a\;b] \times [c\;d] \times [p\;q] \subseteq \mathbb{R}^3$. The triple integral of f over D can be expressed as iterated integrals:

$$\begin{aligned}\iiint_D f(x,y,z)\,dx\,dy\,dz &= \int_a^b \int_c^d \int_p^q f(x,y,z)\,dz\,dy\,dx \\ &= \int_a^b \int_c^d \int_p^q f(x,y,z)\,dx\,dz\,dy \\ &= \int_a^b \int_c^d \int_p^q f(x,y,z)\,dy\,dz\,dx.\end{aligned}$$

7.1. Triple Integration with Change of Variables

Let $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ be a transformation given by

$$\phi(u,v,w) = \begin{bmatrix} x(u,v,w) \\ y(u,v,w) \\ z(u,v,w) \end{bmatrix}.$$

The Jacobian matrix of ϕ is given by the matrix of partial derivatives:

$$J_\phi(u,v,w) = \begin{bmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{bmatrix}$$

Cylindrical Coordinates

The cylindrical coordinates of a point (x, y, z) in $\mathbb{R}^3$ are obtained by representing the x and y coordinates using polar coordinates (or potentially the y and z coordinates or x and z coordinates) and letting the third coordinate remain unchanged

$$\phi := \begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \\ z = z \end{cases}, \quad \Delta = \phi(D), \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq r < \infty, \quad J = |J_\phi(u, v, w)| = r,$$

Integral of f on D:

$$\iiint_D f(x, y, z)\, dx\, dy\, dz. = \iiint_\Delta f(r\cos(\theta), r\sin(\theta), z) J\, dr\, d\theta\, dz.$$

Example :

Let

$$I = \iiint_D z dx dy dz, \quad D = \{(x, y, z) \in \mathbb{R}^3, 0 \leq z \leq 1, x^2 + y^2 \leq z^2\},$$

Switch to cylindrical coordinates:

$$\{x = r\cos(\theta), y = r\sin(\theta), z = z, J = r, 0 \leq r \leq z \leq 1, 0 \leq \theta \leq 2\pi\},$$

then

$$I = \int_0^1 \int_0^{2\pi} \int_0^z zr\, dr\, d\theta\, dz = 2\pi \int_0^1 z^3\, dz = \frac{\pi}{4}.$$

Spherical Coordinates

To evaluate the triple integral of a function f over D using spherical coordinates, we need to express the integral in terms of the spherical coordinates (r, ϕ, θ). The spherical coordinates are related to Cartesian coordinates by the following transformations:

$$\phi : \begin{cases} x = r\sin(\phi)\cos(\theta) \\ y = r\sin(\phi)\sin(\theta) \\ z = r\cos(\phi) \end{cases}, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \pi,$$

$$\iiint_D f(x, y, z)\, dx\, dy\, dz = \iiint_\Delta f(r\sin(\phi)\cos(\theta), r\sin(\phi)\sin(\theta), r\cos(\phi)) J\, dr\, d\theta\, d\phi.$$

The Jacobian determinant of the spherical coordinate transformation is $J = r^2\sin(\phi)$.

Example :

Let

$$\iiint_D z dx dy dz, \quad D = \{(x, y, z) \in \mathbb{R}^3, x^2 + y^2 + z^2 \leq 1\}.$$

Using spherical coordinates, we get

$$\iiint_\Delta r\cos(\phi)r^2\sin(\phi)\,dr\,d\theta\,d\phi = \int_0^1\int_0^{2\pi}\int_0^{\pi} r^3\sin(\phi)\cos(\phi)\,d\phi\,d\theta\,dr$$
$$= \frac{1}{2}\int_0^1\int_0^{2\pi}\int_0^{\pi} r^3\sin(2\phi)\,d\phi\,d\theta\,dr = \frac{\pi}{8}.$$

8. Triple integral test

Quiz 1

[solution n°7 p. 15]

The difference between a double integral and a triple integral are the number of variables and the region of integration

the number of ______.

The ______ of integration.

Quiz 2

[solution n°8 p. 15]

Which of the following coordinate systems is commonly used to simplify triple integrals?

- [ ] Spherical coordinates
- [ ] Cartesian coordinates
- [ ] Cylindrical coordinates
- [ ] Polar coordinates

Quiz 3

[solution n°9 p. 15]

Calculate this triple integral $\int_0^1\int_0^1\int_0^1 8xyzdxdydz$

______

Exercise solutions

Solution n°1

[exercice p. 7]

Which integration technique is used to find the area under a curve?

- [x] Integration by Substitution
- [x] Integration by Parts
- [ ] Integration by Partial Fractions
- [ ] Definite Integration

Solution n°2

[exercice p. 7]

Explain the meaning of the definite integral $\int_a^b f(x)dx$.

- [x] It represents the area under the curve of f(x) from a to b
- [x] It represents the antiderivative of f(x) from a to b.
- [ ] It represents the derivative of f(x) from a to b
- [ ] It represents the limit of f(x) as x approaches b.

Solution n°3

[exercice p. 7]

Compute the integral: $\int_1^3 \frac{3}{x}dx$

- ○ $2\ln(3)$
- ◉ $3\ln(3)$
- ○ $\frac{2}{3}$
- ○ $\frac{1}{3}$

Solution n°4

[exercice p. 10]

What does a double integral $\iint f(x,y)dxdy$. represent geometrically?

- ◉ Volume under the surface $f(x,y)$.
- ○ Area under the curve $f(x,y)$.
- ○ Slope of the tangent plane to $f(x,y)$.

- ○ Length of the curve $f(x, y)$.

Solution n°5

[exercice p. 10]

Match the correct concept or definition with its corresponding term or symbol.

Double Integral	Region of Integration	Polar Coordinates	Rectangular Coordinates
$\iint_R f(x,y)dA$	The region in the xy-plane over which the double integral is evaluated.	$\int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} f(r,\theta) r dr d\theta$	$dxdy$

Solution n°6

[exercice p. 11]

Consider the region R in the xy-plane bounded by the curves $x^2 < y < 4$ and $0 < x < 2$. What is the double integral $\iint_R f\,(x+y)dA >$?

- [x] 18
- [x] 15
- [x] 8
- [ ] 19

Solution n°7

[exercice p. 13]

The difference between a double integral and a triple integral are the number of variables and the region of integration

the number of `variables`.

The `region` of integration.

Solution n°8

[exercice p. 13]

Which of the following coordinate systems is commonly used to simplify triple integrals?

- [x] Spherical coordinates
- [ ] Cartesian coordinates
- [x] Cylindrical coordinates
- [ ] Polar coordinates

Solution n°9

[exercice p. 13]

Calculate this triple integral $\int_0^1 \int_0^1 \int_0^1 8xyzdxdydz$

1

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