Analysis 1

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Analysis 1 Tutorial

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Objectives

- Understand the properties of real functions and their behaviors.
- Discover how we define the derivative using limits.
- Learn about a bunch of very useful rules (like the power, product, and quotient rules) that help us find derivatives quickly
- The students will learn how to study the differentiability of a function, that is to say, whether a function is differentiable or not. In addition, we will see the relationship between differentiability and continuity of a function.

Chapter 1 : Differentiable Functions

1. Introduction

The group of functions is one of the focus points of Calculus, and you should already be familiar with many aspects of those functions.

In our setting, these functions will play a rather minor role and we will only briefly review the main topics of that theory.

In mathematics[1] a differentiable function of one real[2] variable is a function[3] whose derivative exists at each point in its domain[4]. In other words, the graph[5] of a differentiable function has a non-vertical tangent line at each interior point in its domain. A differentiable function is smooth (the function is locally well approximated as a linear function[6] at each interior point) and does not contain any break, angle, or cuspo.

2. Specific Objectives

After successful completion of this lesson, you should be able to:

1- *recall the exact definition of differentiation.*

2- *derive exact derivatives of a few functions.*

3. Prerequisite Test

Limits

Question 1: Find $\lim_{x\to 3} f(x)$:

$$f(x) = \frac{x^2 - 9}{x - 3}$$

(A) $+\infty$

(B) -6

(C) 6

(D) Does not exist!

(E) None of the above

Question 2: Find $\lim_{x\to 2} f(x)$:

$$f(x) = 1776$$

(A) $+\infty$

(B) 1770

(C) $-\infty$

(D) Does not exist!

(E) None of the above

1. https://en.wikipedia.org/wiki/Mathematics
2. https://en.wikipedia.org/wiki/Real_number
3. https://en.wikipedia.org/wiki/Function_(mathematics)
4. https://en.wikipedia.org/wiki/Domain_of_a_function
5. https://en.wikipedia.org/wiki/Graph_of_a_function
6. https://en.wikipedia.org/wiki/Linear_function

Question 5: Find $\lim_{x \to 2} f(x)$:

$$f(x) = \frac{3x^2 - 4x + 6}{x^2 + 8x - 15}$$

Continuity and Differentiability

Question 7: Which of the following functions are *NOT* everywhere continuous:

(A) $f(x) = \frac{x^2-4}{x+2}$

(B) $f(x) = (x+3)^4$

(C) $f(x) = 1066$

Question 8: Which of the following functions are continuous:

(A) $f(x) = |x|$

(B) $f(x) = \begin{cases} 3 & x < 4 \\ \frac{1}{2}x + 3 & x \geq 4 \end{cases}$

(C) $f(x) = \frac{1}{x}$

(D) $f(x) = \begin{cases} \ln x & x < 0 \\ 0 & x = 0 \end{cases}$

(E) None of the above

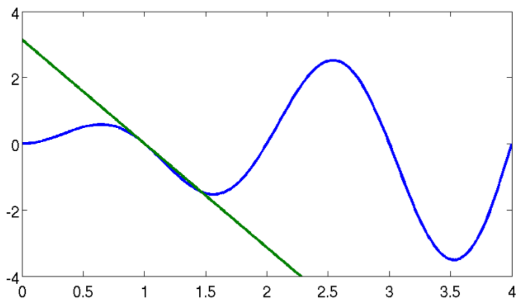
4. Differentiable Functions

Reminder **: Summary**

- A function $f(x)$ has limit l as $x \to a$ if and only if $f(x)$ has a left-hand limit at $x = a$, has a right-hand limit at $x = a$, and the left- and right-hand limits are equal. Visually, this means that there can be a hole in the graph at $x = a$, but the function must approach the same single value from either side of $x = a$.
- A function $f(x)$ is continuous at $x = a$ wheneve $f(a)$ is defined, f has a limit as $x \to a$, and the value of the limit and the value of the function agree. This guarantees that there is not a hole or jump in the graph of f at $x = a$.
- A function f is differentiable a $x = a$ whenever $f\prime(a)$ exists, which means that f has a tangent line at $(a, f(a))$ and thus f is locally linear at the value $x = a$. Informally, this means that the function looks like a line when viewed up close at $(a, f(a))$ and that there is not a corner point or at $(a, f(a))$.
- Of the three conditions discussed (having a limit at $x = a$, being continuous at $x = a$, and being differentiable at $x = a$, the strongest condition is being differentiable, and the next strongest is being continuous. In particular, if f is differentiable at $x = a$, then f is also continuous at $x = a$, and if f is continuous at $x = a$, then f has a limit at $x = a$.

Example **:**

The tangent to the function *f(x) = x \sin(\pi x)* (blue curve) through x_{0} = 1.



5. Exercise sheet

5.1. Excercise 1

Find the derivatives of the following functions :

$(a) f(x) = (x + \frac{1}{x})^2, x \in \mathbb{R}, x \neq 0,$

$(b) f(x) = \sum_{k=0}^{10} 2^k x^k, x \in \mathbb{R},$

$(c) f(x) = x^{(a^x)}, a > 0, x \in (0, \infty).$

Solution

$(a) f(x) = x^2 + 2 + \frac{1}{x^2} \Rightarrow f'(x) = 2x - \frac{2}{x^3}$

$(b) f'(x) = (\sum_{k=0}^{10} 2^k x^k)' = \sum_{k=1}^{10} 2^k k x^{k-1}$

(c) We have $(ax)' = a^x \ln a$. Also:

$(x^{a^x})' = (e^{a^x \ln x})' = x^{a^x \ln x}(a^x \ln a \ln x + \frac{a^x}{x}) = x^{a^x} a^x (\ln a \ln x + \frac{1}{x}).$

5.2. Excercise 2

Let a, b be real numbers and let the function $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} a \cos x + b \sin x, x < \pi, \\ x, x \geq \pi. \end{cases}$$

$i)$ Find all $a, b \in \mathbb{R}$ such that f is a continuous function.

$ii)$ Find the left-hand derivative and the right-hand derivative of f at the point $x_0 = \pi$.

$iii)$ Find all $a, b \in \mathbb{R}$ such that f is differentiable for all $x \in \mathbb{R}$. In this case, check whether the derivative f' of f is a continuous or a differentiable function.

Solution

It is clear that f is continuous at all points except π. So that f is also continuous in π, the following must apply:

$n : \lim_{x \to \pi^-} f(x) = f(\pi),$

also

$\lim_{x \to \pi^-} a \cos x + b \sin x = -a = \pi$

f is continuous for any $b$and $a = -\pi$.

Left-hand derivation: We calculate using the addition theorems

$$\lim_{x \to \pi^-} \frac{f(x) - f(\pi)}{x - \pi} = \lim_{x \to \pi^-} \frac{-\pi \cos x + b \sin x - \pi}{x - \pi}$$

$$= \pi \lim_{x \to \pi^-} \frac{-1 - \cos x}{x - \pi} + b \lim_{x \to \pi^-} \frac{\sin x}{x - \pi}$$

$$= -\pi \lim_{x \to \pi^-} \frac{1 + \cos(x - \pi + \pi)}{x - \pi} + b \lim_{x \to \pi^-} \frac{\sin(x - \pi + \pi)}{x - \pi}$$

$$= -\pi \lim_{x \to \pi^-} \frac{1 + \cos(x - \pi) \cos \pi - \sin(x - \pi) \sin \pi}{x - \pi}$$

$$+ b \lim_{x \to \pi^-} \frac{\sin(x - \pi) \cos \pi + \sin \pi \cos(x - \pi)}{x - \pi}$$

$= -\pi \lim_{x\to\pi^-} \frac{1-\cos(x-\pi)}{x-\pi} + b \lim_{x\to\pi^-} \frac{\sin(x-\pi)}{x-\pi} = -b;$

Where:

$\lim_{x\to\pi^-} \frac{1-\cos(x-\pi)}{x-\pi} = \lim_{x\to\pi^-} \frac{1-\sum_{n=0}^{\infty}(-1)^n \frac{(x-\pi)^{2n}}{(2n)!}}{x-\pi}$

$= -\lim_{x\to\pi^-} \frac{1}{x-\pi} \sum n = 1\infty (-1)^n \frac{(x-\pi)^{2n}}{(2n)!} = -\lim_{x\to\pi^-} \sum n = 1\infty (-1)^n \frac{(x-\pi)^{2n-1}}{(2n)!}$

$= \sum n = 1\infty (-1)^n \frac{0}{(2n)!} = 0.$

Analogously we get

$\lim_{x\to\pi^-} \frac{sin(x-\pi)}{x-\pi} = 1.$

Right-side derivation:

$\lim_{x\to\pi^+} \frac{f(x)-f(\pi)}{x-\pi} = \lim_{x\to\pi^-} \frac{x-\pi}{x-\pi} = 1.$

f is differentiable if the left-hand and right-hand derivatives are the same, i.e. if $b = -1$ (and of course if

$f'(x) = \begin{cases} \pi \sin x - \cos x, x < \pi, \\ 1, x \geq \pi. \end{cases}$

f' is also continuous in this case because

$\lim_{x\to\pi^-} f'(x) = 1 = f'(\pi).$

Left-sided derivative (of f'):

$\lim_{x\to\pi^-} \frac{f'(x)-f'(\pi)}{x-\pi} = \lim_{x\to\pi^-} \frac{\pi \sin x - \cos x - 1}{x-\pi} = \pi \lim_{x\to\pi^-} \frac{\sin x}{x-\pi} - \lim_{x\to\pi^-} \frac{\cos x + 1}{x-\pi}$

$= \pi \lim_{x\to\pi^-} \frac{\sin(x-\pi+\pi)}{x-\pi} + \lim_{x\to\pi^-} \frac{\cos(x-\pi+\pi)+1}{x-\pi}$

$= -\pi \lim_{x\to\pi^-} \frac{\sin(x-\pi)}{x-\pi} + \lim_{x\to\pi^-} \frac{-\cos(x-\pi)+1}{x-\pi} = -\pi.$

Right-side derivation:

$\lim_{x\to\pi^+} \frac{f'(x)-f'(\pi)}{x-\pi} = \lim_{x\to\pi^-} \frac{1-1}{x-\pi} = 0.$

$0 \neq -\pi$, therefore f' is not differentiable

5.3. Excercise 3

Consider the function

$f(x) = \begin{cases} \exp x, if x \leq 0, \\ \cos(x) + x, if x > 0. \end{cases}$

(a) Show that f is continuously differentiable at $x \neq 0$ arbitrarily many times.

(b) Show that f is (once) continuously differentiable at $x = 0$.

(c) Is f twice continuously differentiable at $x = 0$?

Solution

(a) The functiom $\exp x$ and $x + \cos x$ are arbitrary often continuously differentiable on the whole of $\mathbb{R}$ and thus also f as an arbitrarily often static combination of both.

(b) Further is for $x \neq 0$

$f'(x) = \begin{cases} \exp x, if x < 0, \\ 1 - \sin x, if x > 0. \end{cases}$

So it applies

$\lim_{x \to -0} f'(x) = 1 = \lim_{x \to +0} f'(x)$.

The derivation f' is therefore in $x = 0$, f is then continuously differentiable in $x_0 = 0$.

(c) One calculates for $x \neq 0$

$f''(x) = \begin{cases} e^x, if x < 0, \\ -\cos x, if x > 0. \end{cases}$

Here is

$\lim_{x \to -0} f'(x) = 1 \neq -1 = \lim_{x \to +0} f''(x)$.

It means thatf'' in$x_0 = 0$ cannot be continuously supplemented and therefore $f$has a continuous second derivative!

5.4. Excercise 4

The inverse function of the tangent is denoted by $\arctan : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$.

(a)Find the derivative of $\arctan$ using the rules for differentiation of the inverse function.

(b) The function $\arctan$ has a power series expansion for all $x \in \mathbb{R}$ with $|x| < 1$:

$\arctan x = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$.

Calculate $(\arctan x)'$by differentiating this power series.

Solution

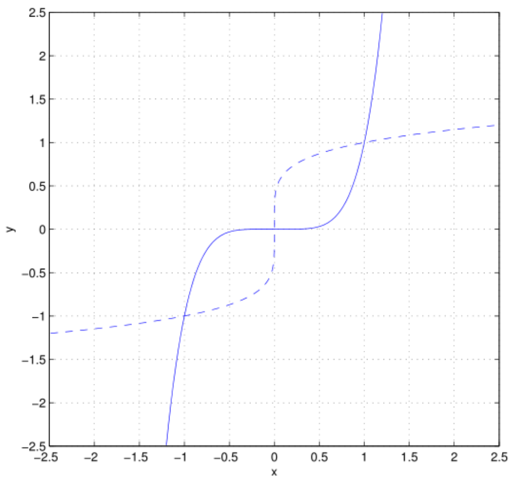


Fig.: Graph of the function f on[-1.2, 1.2]and graph of the inverse function g.

It applies

$(\arctan x)' = \frac{1}{(\tan y)'} \Big|_{y=\arctan x}$.

It is $(\tan y)' = (\cos 2y) - 1$. Using the identity undefined $1 = \cos^2 y + \sin^2 y$ we receive

$\frac{1}{(\tan y)'} = \cos^2 y = \frac{\cos^2 y}{\sin^2 y + \cos^2 y} = \frac{1}{\tan^2 y + 1}$.

Also

$(\arctan x)' = \frac{1}{\tan^2 y + 1} \Big|_{y=\arctan x} = \frac{1}{1 + x^2}$.

6. Quiz : Final test

Quiz

Find the derivative of the following function: $f(x) = (7x - 4)(3x - 8)^4$

Quiz

Find the derivative of the following function: $f(x) = 5x^2(x + 47)$

- $f'(x) = 15x^2 + 470x$
- $f'(x) = 5x^2 + 470x$
- $f'(x) = 10x$

Quiz

Let $f(x) = |x - 1|$. We note $f'_d(a)$ *(resp.* $f'_g(a)$) to designate the derivative on the right (resp. on the left) in a. What are the correct answers?

- $f'_d(1) = 1 and f'_g(1) = 1$
- f is differentiable at 0 and $f'(0) = -1$.
- $f$is differentiable at 1 and $f'(1) = 1$.
- f is not differentiable in 1 because $f' f'_d(1) = 1$ and $f' f'_g(1) = -1$.

Quiz

What are the correct answers?

- The derivative of $f(x) = \arcsin(1 - 2x^2)$ is $f'(x) = \frac{-2x}{|x|p\sqrt{1-x^2}}$.
- The derivative of $f(x) = \arcsin(1 - 2x^2)$ is $f'(x) = \frac{1}{\sqrt{1-2x^2}}$.
- The derivative of $f(x) = \arccos(x^2 - 1)$ is $f'(x) = \frac{2x}{\sqrt{x^2-1}}$.
- The derivative of $f(x) = \arccos(x^2 - 1)$ is $f'(x) = \frac{-2x}{|x|p\sqrt{2-x^2}}$.

Quiz

Let $f(x) = x^2 - e^{x^2-1}$. What are the correct answers?

- f admits a local minimum at 0.
- f admits a local maximum at 0.
- f admits an inflection point at 0.
- The tangent to C_f at 0 is a vertical line.