Chapter 2

Functions sequences

We consider the set $\mathscr{F}(I,\mathbb{R})$, of all functions defined on I (I interval of $\mathbb{R}$) with values in $\mathbb{R}$, namely,

$$\mathscr{F}(I,\mathbb{R}) = \{f \mid f : I \longrightarrow \mathbb{R},\ f \text{ function}\}.$$

Definition 1. We call *sequence of functions* on I any application

$$f : \mathbb{N} \longrightarrow \mathscr{F}(I,\mathbb{R})$$

$$n \longmapsto f(n)$$

We denote $f(n)$ by f_n and we denote the sequence by $(f_n)_{n\in\mathbb{N}}$.

2.1. Simple convergence (pointwise) of a sequence of functions

Definition 2. We say that a sequence of functions $(f_n)_{n\in\mathbb{N}}$ *simply converges on* I to a function f (or else converges point by point on I) if

$$\text{for all } x \in I, \text{the numerical sequence } (f_n(x))_{n\in\mathbb{N}} \text{ converges to} f(x),$$

In other words,

$$\forall x \in I, \forall \varepsilon > 0, \exists N_{\varepsilon,x} \in \mathbb{N}, \forall n \in \mathbb{N} : (n \geqslant N_{\varepsilon,x} \Rightarrow |f_n(x) - f(x)| < \varepsilon).$$

f is called the *simple limit* of the sequence $(f_n)_{n\in\mathbb{N}}$ and we write

$$f_n \overset{\text{SC}}{\longrightarrow} f \,\text{on} I.$$

This means that

$$\forall x \in I, \forall \varepsilon > 0, \exists N_{\varepsilon,x} \in \mathbb{N}, \forall n \in \mathbb{N} : (n \geqslant N_{\varepsilon,x} \Rightarrow |f_n(x) - f(x)| < \varepsilon).$$

Example 1.

1. Let be the sequence of functions $f_n(x) = \sin\left(x + \frac{1}{n}\right)$; $x \in \mathbb{R}$.

   For all $x \in \mathbb{R}$, we have, $f_n(x) = \sin\left(x + \frac{1}{n}\right) \xrightarrow[n\to+\infty]{} \sin(x)$. Then the sequence of functions $(f_n)_{n\in\mathbb{N}^*}$ converges simply to the function $f(x) = \sin(x)$ on $\mathbb{R}$.

2. The sequence of functions $f_n(x) = \dfrac{nx}{1+nx}$ converges simply on $\mathbb{R}^+$, because

   $$\lim_{n\to+\infty} f_n(x) = \begin{cases} 0 & \text{if} x = 0 \\ 1 & \text{if} x > 0 \end{cases},$$

   and then, $f_n \xrightarrow{\text{SC}} f$ on $\mathbb{R}^+$, where $f(x) = \begin{cases} 0 & \text{si } x = 0 \\ 1 & \text{si } x > 0. \end{cases}$

3. Let be the sequence of functions $\psi_n(x) = n\exp(-nx)$, $x \in \mathbb{R}^+$. It is clear that for any strictly positive real $\psi_n(x) \xrightarrow[n\to+\infty]{} 0$. So the sequence of functions $(\psi_n)_{n\in\mathbb{N}}$ simply converges to the identically zero function, on $\mathbb{R}^+$.

If $x < 0$, then $\lim\limits_{n\to+\infty} \psi_n(x) = -\infty$, so $(\psi_n)_{n\in\mathbb{N}}$ does not simply converge on $]-\infty, 0[$.

Remark 1. The previous example shows that the continuity of the functions f_n does not necessarily imply the continuity of the limit function f and the integral of the limit f is not necessarily equal to the limit of the integrals of the functions f_n.

Is there a concept of convergence of sequences of functions which allows us to ensure that,

a. The limit function f is continuous on the interval I if all the functions f_n are continuous.

B. Is the permutation between the limit and the integral correct?

We will study this question in the following paragraph.

2.2. Uniform convergence of a sequence of functions

Definition 3. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions defined from I into $\mathbb{R}$ and let $f : I \longrightarrow \mathbb{R}$ be a function.

We say that the sequence of functions $(f_n)_{n\in\mathbb{N}}$ *converges uniformly* to the function f on I if

$$\lim_{n\to+\infty} \sup_{x\in I} |f_n(x) - f(x)| = 0, \tag{2.2.1}$$

namely,

$$\forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N} / \forall n \in \mathbb{N}, \forall x \in E : \ (n \geqslant N_\varepsilon \Rightarrow |f_n(x) - f(x)| < \varepsilon).$$

By posing $\|f_n - f\| = \sup_{x\in I} |f_n(x) - f(x)|$, then (2.2.1) translates as

$$\forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N} / \forall n \in \mathbb{N} : \ (n \geqslant N_\varepsilon \Rightarrow \|f_n - f\| < \varepsilon).$$

We also say that f_n converges to f for the norm of uniform convergence and that f is the *uniform limit* on I of the sequence $(f_n)_n$.

And we note, $f_n \xrightarrow{\text{convergence uniformly}} f$ on I or else $f_n \xrightarrow{\text{UC}} f$ on I or else $f_n \xrightarrow{\|\cdot\|} f$.

2.2.1. Graphical interpretation of uniform convergence

If we plot the representative curves of the functions $f - \varepsilon$ and $f + \varepsilon$. To say that the sequence $(f_n)_{n\in\mathbb{N}}$ converges uniformly towards f is equivalent to saying that from a certain rank the curve of fn lies between the other two.

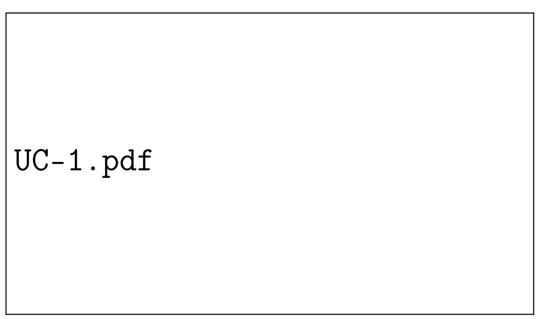

Figure 2.1: Illustration of uniform convergence

Example 2.

1. For any integer n, let
$$f_n : [0,1] \longrightarrow \mathbb{R}$$
$$x \longmapsto x^n(1-x).$$
The sequence of functions $(f_n)_n$ simply converges on $[0,1]$, to the zero function.

   Let us calculate $\sup_{x\in[0,1]} |f_n(x)-f(x)|$ on $[0,1]$. By studying the variations of the function $|f_n(x)-f(x)| = x^n(1-x)$ on $[0,1]$, we find that
$$\sup_{x\in[0,1]} |f_n(x)-f(x)| = f_n\left(\frac{n}{n+1}\right) = \left(\frac{n}{n+1}\right)^n \left(\frac{1-\frac{n}{n+1}}{n+1}\right).$$
Or
$$\forall n \geqslant 0, 0 \leqslant \left(\frac{n}{n+1}\right)^n \left(\frac{1-\frac{n}{n+1}}{n+1}\right) \leqslant 1-\frac{n}{n+1},$$
hence, according to the Three sequence theorem, $\lim_{n\to+\infty} \|f_n - f\| = 0$, thus, $f_n \xrightarrow{\text{UC}} 0$ on $[0,1]$.

2. The sequence of functions defined on $[0,1]$, by $f_n(x) = x^n$ is not uniformly convergent on $[0,1]$.

   In fact, it is clear that the sequence $(f_n)_n$ simply converges to the function f on $[0,1]$ such that,
$$f(x) = \begin{cases} 0 & \text{si } x \in [0,1[\\ 1 & \text{si } x = 1 \end{cases},$$
but,
$$\sup_{x\in[0,1]} |f_n(x)-f(x)| = \sup_{x\in[0,1]} |x^n - 0| = \sup_{x\in[0,1]} |x^n - 0| = \sup_{x\in[0,1]} x^n,$$
because $f_n(1)-f(1) = 0$. And since $\sup_{x\in[0,1]} x^n = 1$ (the function $x \mapsto x^n$ is increasing on $[0,1]$). Then $\sup_{x\in[0,1]} |f_n(x)-f(x)|$ does not tend to 0. Thus the sequence $(f_n)_n$ is not uniformly convergent on $[0,1]$.

The following proposition ensures the uniform convergence of a sequence of functions $(f_n)_{n\in\mathbb{N}}$ on an interval I, without knowing the limit function f.

Theorem 1 (Cauchy's theorem for uniform convergence). A sequence of functions $(f_n)_{n\in\mathbb{N}}$ converges uniformly on I if and only if

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}/\forall p, q \in \mathbb{N},\ \forall x \in I:\ (p > q \geqslant N \Rightarrow |f_p - f_q| < \varepsilon)$$

hence,

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}/\forall p, q \in \mathbb{N}:\ (p > q \geqslant N \Rightarrow \|f_p - f_q\| < \varepsilon)$$

Proposition 1 (Sufficient condition for uniform convergence). For a sequence of functions $(f_n)_{n\in\mathbb{N}}$ to converge uniformly on I to a function f, it suffices that there exists a numerical sequence $(u_n)_n$ such that,

$$|f_n(x) - f(x)| \leqslant u_n, \forall n \in \mathbb{N}, \forall x \in I \text{ and } \lim_{n\to+\infty} u_n = 0.$$

Example 3.

1. Study the simple and uniform convergence of the sequence of functions $(f_n)_n$, on $[0,1]$ such that
$$f_n(x) = \frac{ne^{-x} + x^2}{n + x}.$$
In fact, for any $x \in [0,1]$, $\lim_{n\to+\infty} f_n(x) = e^{-x}$ (i.e., $f_n \xrightarrow{SC} e^{-x}$ on $[0,1]$).

   For uniform convergence. We have for all $x \in [0,1]$,
$$\left|\frac{ne^{-x} + x^2}{n + x} - e^{-x}\right| = \left|\frac{xe^{-x} - x^2}{n + x}\right| = \frac{|x| \cdot |xe^{-x} - x|}{n + x} \leqslant \frac{x}{n + x} \leqslant \frac{2}{n} \xrightarrow{n\to+\infty} 0,$$

so according to the previous proposition, $(f_n)_{n\in\mathbb{N}}$ converges uniformly to e^{-x} on $[0,1]$.

2.2.2. Some operations

Proposition 2. Let $(f_n)_n$ and $(g_n)_n$ be two sequences of functions defined on I converging uniformly to f and g respectively. If λ and μ are two real numbers, then the sequence of functions $(\lambda f_n + \mu g_n)_n$ converges uniformly to the function $\lambda f + \mu g$ on I.

Proposition 3. Let $(f_n)_n$ and $(g_n)_n$ be two sequences of functions defined on I converging uniformly respectively to f and g on I. If the limit functions f and g are bounded on I, then the sequence of functions $(f_n g_n)_n$ converges uniformly to fg on I.

Uniform convergence implies simple convergence, and this is a consequence of the following proposition.

Proposition 4. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions defined on I. Then

$$f_n \overset{UC}{\longrightarrow} f \text{ sur } I \Rightarrow f_n \overset{SC}{\longrightarrow} f \text{ sur } I.$$

Remark 2. The converse of the previous proposition is generally false. Indeed, let us take the sequence of functions $(f_n)_n$ defined on $[0,1]$ with a value in $\mathbb{R}$, by $f_n(x) = x^n$. It is clear that $(f_n)_n$ simply converges on $[0,1]$ to the function

$$f(x) = \begin{cases} 0 & \text{if } x \in [0,1[\\ 1 & \text{if } x = 1 \end{cases}$$

but the sequence $(f_n)_n$ does not converge uniformly on $[0,1]$, since

$$\sup_{x\in[0,1]} |f_n(x) - f(x)| = 1 \underset{n\to+\infty}{\nrightarrow} 0.$$

2.3. Properties of sequences of uniformly convergent functions

2.3.1. Continuity

Theorem 2 (Seidel continuity)**.** Let $(f_n)_n$ be a sequence of functions defined from an interval I of $\mathbb{R}$ into $\mathbb{R}$. Let $a \in I$, if

i) for any integer n, the function f_n is continuous in a,

ii) the sequence of functions $(f_n)_n$ converges uniformly on I to a function f.

Then f is continuous at a.

We can immediately deduce the following Corollary.

Corollary 1. Let $(f_n)_n$ be a sequence of functions defined from an interval I of $\mathbb{R}$ to $\mathbb{R}$. If,

i) for any integer n, the function f_n is continuous on I,

ii) the sequence $(f_n)_n$ converges uniformly on I to a function f.

Then f is continuous on I.

One method of showing that a sequence of functions $(f_n)_n$ does not converge uniformly to its simple limit f on a domain I, is the contrapositive of the previous Corollary, more precisely we have,

Proposition 5. Let $(f_n)_n$ be a sequence of functions which converges simply on I to f. If for any integer n, the function f_n is continuous on I, then

$$f \text{ is discontinuous at } x_0 \in I \Rightarrow f_n \overset{UC}{\nrightarrow} f \text{ on } I.$$

Example 4. Let the sequence of functions $(f_n)_n$ be defined on $[0,+\infty[$ with a value in $\mathbb{R}$, by $f_n(x) = \dfrac{1}{1+nx}$.

We note that for any integer n, the function f_n is continuous on $[0,+\infty[$. Furthermore, it is easy to see that $f_n \overset{SC}{\longrightarrow} f$ on $[0,+\infty[$ with

$$f(x) = \begin{cases} 0 & \text{if } x > 0, \\ 1 & \text{if} x = 0. \end{cases}$$

The function f is not continuous at $x_0 = 0$, hence according to the previous Proposition the sequence of functions $(f_n)_n$ does not converge uniformly on $[0,+\infty[$.

We have seen that simple convergence does not imply uniform convergence, but under certain conditions it does.

Theorem 3 (Dini's theorem)**.** Let $(f_n)_n$ be a sequence of functions that converges simply on $[a,b], (\subset \mathbb{R})$ to a function f continuous on $[a,b]$. If the sequence of functions $(f_n)_n$ is monotone on $[a,b]$, then $(f_n)_n$ converges uniformly to f on $[a,b]$.

2.3.2. Integration

Theorem 4 (Integration theorem)**.** Let $(f_n)_n$ be a sequence of functions such that, for all $n \in \mathbb{N}$, the function f_n is integrable on $[a,b]$.

If $(f_n)_n$ converges uniformly to f on $[a,b]$, then the function f is integrable on $[a,b]$. Moreover

$$\lim_{n\to+\infty} \int_a^b f_n(x)\,dx = \int_a^b \lim_{n\to+\infty} f_n(x)\,dx = \int_a^b f(x)\,dx.$$

Corollary 2. Let $(f_n)_n$ be a sequence of functions such that, for all $n \in \mathbb{N}$, the function f_n is integrable on $[a,b]$.

If $(f_n)_n$ converges uniformly to f on $[a,b]$ then, for all $\alpha \in [a,b]$, the sequence of functions $(F_n)_n$ such that $F_n(x) = \int_\alpha^x f_n(t)dt$, converges uniformly to the function $F(x) = \int_\alpha^x f(t)dt$ on $[a,b]$. Moreover, for all $x \underset{\alpha}{\in} [a,b]$

$$\lim_{n\longrightarrow+\infty} F_n(x) = \lim_{n\longrightarrow+\infty} \int_\alpha^x f_n(t)dt = \int_\alpha^x \lim_{n\longrightarrow+\infty} f_n(t)dt = \int_\alpha^x f(t)dt = F(x).$$

2.3.3. Derivation

Theorem 5 (Derivative theorem)**.** Let $(f_n)_n$ be a sequence of functions such that, for all $n \in \mathbb{N}$ the function f_n is continuously differentiable on I (i.e., $f_n \in C^1$ on I) and converges simply to a function f on I. If the sequence of functions $(f_n')_n$ converges uniformly to a function g on I, then the function f is continuously differentiable on I (i.e., $f \in C^1$ on I) and

$$f'(x) = g(x) \quad \forall x \in I,$$

in other words

$$\left(\lim_{n\to+\infty} f_n(x)\right)' = \lim_{n\to+\infty} f_n'(x) \quad \forall x \in I.$$