Chapter 3

Function series

Definition 1. Let $(f_n)_n$ be a sequence of functions defined on $I \subset \mathbb{R}$. Then the series $\sum f_n(x)$ is called a series of functions.

3.1. Simple or point convergence

Definition 2 (Simple convergence).
The series of functions $\sum f_n(x)$ is said to be **simply convergent** on I, if the sequence of partial sums $(S_n)_n$ (i.e. $S_n(x) = \sum_{k=0}^{n} f_k(x)$) simply converges to a function S on I.

Remark 1.

i To study the simple convergence on I, of a series of functions amounts to fixing $x \in I$, and studying the numerical series $\sum f_n(x)$.

ii If the series of functions $\sum f_n(x)$ converges simply to a function S on a domain D, then

(a) the set D is called the domain of convergence of the series of functions $\sum f_n(x)$,

(b) the limit function S is called the sum of the series $\sum f_n(x)$.

3.2. Absolute, normal and uniform convergence

3.2.1. Absolute convergence

Definition 3. The series of functions $\sum f_n(x)$ is said to be absolutely convergent on I, if the series of general term $|f_n(x)|$ simply converges on I.

Remark 2. All the convergence criteria studied for numerical series with positive terms remain valid for studying the convergence of series of functions with positive terms, in particular the study of the absolute convergence of series of functions.

Example 1.

1. Let $\sum f_n(x)$ such that $\forall n \geqslant 1, \forall x \in \mathbb{R}:\ f_n(x) = \dfrac{\sin(nx)}{n\sqrt{n}}$.
   For all $x \in \mathbb{R}$, we have,
   $$\left|\frac{\sin(nx)}{n\sqrt{n}}\right| \leqslant \frac{1}{n\sqrt{n}},$$
   or the Riemann series $\sum \dfrac{1}{n^{\frac{3}{2}}}$ is convergent. Thus, for all $x \in \mathbb{R}$ the series of functions $\sum f_n(x)$ is convergent, that is to say it is simply convergent on $\mathbb{R}$, hence the domain of simple convergence is $\mathbb{R}$.

2. Let the series of functions $\sum f_n(x)$ be such that $f_n(x) = n!(x+1)^n$. We have
   $$\left|\frac{f_{n+1}(x)}{f_n(x)}\right| = (n+1)|x+1|,$$
   which tends to $+\infty$ if $x \neq -1$, and for all $n \in \mathbb{N}, f_n(-1) = 0$, thus the numerical series $\sum f_n(-1)$ converges to 0. Hence the domain of convergence of the series of functions $\sum n!(x+1)^n$ is the set $\{-1\}$.

3.2.2. Normal convergence

Definition 4. The series of functions $\sum f_n(x)$ is said to be **normally convergent** on I, if the numerical series with general term $\|f_n\|$ (where $\|f_n\| = \sup_{x\in I} |f_n(x)|$) is convergent.

Example 2. Let the series $\sum f_n(x)$ on $\mathbb{R}_+$ be defined by

$$\forall x \in \mathbb{R}_+, \forall n \in \mathbb{N}^* : \ f_n(x) = \frac{e^{-nx}}{n^2}.$$

It is clear that for all $n \in \mathbb{N}^*$, the function f_n is decreasing on $\mathbb{R}_+$. So

$$\|f_n\| = \sup_{x\in\mathbb{R}_+} |f_n(x)| = \frac{1}{n^2},$$

or the Riemann series $\sum \frac{1}{n^2}$ is convergent, so the series of functions $\sum f_n(x)$ converges normally on $\mathbb{R}_+$.

Condition suffisante de la convergence normale

Theorem 1 (Weierstrass). Let the series of functions $\sum f_n(x)$ be defined on I. If there exists a numerical sequence $(u_n)_n$ such that

$$|f_n(x)| \leqslant u_n, \forall n \in \mathbb{N}, \forall x \in I.$$

and the series $\sum u_n$ converges then the series of functions $\sum f_n(x)$ is normally convergent on I.

Example 3. We take the previous example, namely

$$\forall x \in \mathbb{R}_+, \forall n \in \mathbb{N}^* : \ f_n(x) = \frac{e^{-nx}}{n^2}.$$

We have,

$$\forall n \in \mathbb{N}^* : \ f_n(x) = \frac{e^{-nx}}{n^2} \leqslant \frac{1}{n^2} \quad \forall x \in \mathbb{R}_+,$$

or the Riemann series $\sum \frac{1}{n^2}$ converges, so according to the previous Theorem, the series $\sum \frac{e^{-nx}}{n^2}$ is normally convergent on $\mathbb{R}_+$.

3.2.3. Uniform convergence

Definition 5. The series of functions $\sum f_n(x)$ **uniformly converges** to the function S on I, if its sequence of partial sums $(S_n)_n$ uniformly converges to the function S on I. That is, the numerical sequence with general term

$$\sup_{x\in I}\left|\sum_{k=0}^{n} f_k(x) - S(x)\right|$$

converges to 0.

Remark 3. A series of functions $\sum f_n(x)$ simply convergent on I to a function S, converges uniformly on I if and only if, the sequence $(R_n)_n$ with remainder of order n (i.e., $R_n(x) = \sum_{k=n+1}^{+\infty} f_k(x)$) converges uniformly to 0.

Proposition 1. Let be the series of functions $\sum f_n(x)$ such that for all $n \in \mathbb{N}$, $f_n \in \mathscr{F}(I, \mathbb{R})$. If the series of functions $\sum f_n(x)$ converges uniformly on I, then the series of functions $(f_n)_n$ converges uniformly to the null function on I.

Remark 4. The previous proposition is useful because of its contrapositive. If $(f_n)_n$ does not converge uniformly to 0 on I, then the series of functions $\sum f_n(x)$ does not converge uniformly on I.

Example 4. Let $f_n(x)$ be the series of functions such that for all $n \in \mathbb{N}^*$, and for all $x \in \mathbb{R}_+$: $f_n(x) = nx^2 e^{-x\sqrt{n}}$.

- **Study of the simple convergence of the series, $\sum f_n(x)$ on $\mathbb{R}_+$.**

  If $x = 0$. $\forall n \in \mathbb{N}^*, f_n(0) = 0$, so the series $\sum f_n(0)$ converges to 0.

If $x > 0$, we have,

$$\lim_{n\to+\infty} \frac{nx^2e^{-x\sqrt{n}}}{\frac{1}{n^2}} = \lim_{n\to+\infty} x^2\frac{n^3}{e^{x\sqrt{n}}} = 0,$$

so the Riemann series $\sum \frac{1}{n^2}$ converges, and therefore for all $x > 0$, the numerical series $\sum f_n(x)$ is convergent. Thus the series of functions converges simply on $\mathbb{R}_+$.

- **Study of the uniform convergence of the series $\sum f_n(x)$ on $\mathbb{R}_+$.**

  For all $x \in \mathbb{R}_+$, we have

  $$f_n'(x) = nx(2 - x\sqrt{n})e^{-x\sqrt{n}},$$

  then we deduce that

  $$\sup_{x\in\mathbb{R}_+} |f_n(x)| = \sup_{x\in\mathbb{R}_+} |nx^2e^{-x\sqrt{n}}| = f_n\left(\frac{2}{\sqrt{n}}\right) = \frac{4}{e^2}\frac{n}{\sqrt{n}} \underset{n\to+\infty}{\not\to} 0,$$

  hence the sequence of functions $(f_n)_n$ does not converge uniformly to the zero function on $\mathbb{R}_+$. Thus $\sum f_n(x)$ does not converge uniformly on $\mathbb{R}_+$.

Sometimes it is not easy to calculate the limit function $S(x)$ of such a series of functions $\sum f_n(x)$, so to study the uniform convergence of this series we can use the following theorem.

Theorem 2 (**Cauchy's criterion**).

The series of functions $\sum f_n(x)$ converges uniformly on I if and only if

$$\forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N} / \forall p, q \in \mathbb{N} : \left(p > q \geqslant N_\varepsilon, \forall x \in I \Rightarrow |S_p(x) - S_q(x)| = \left| \sum_{k=q+1}^{p} f_k(x) \right| < \varepsilon \right). \tag{3.2.1}$$

In other words, (3.2.1) is equivalent to the following logical proposition

$$\forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N} / \forall p, q \in \mathbb{N} : \left(p > q \geqslant N_\varepsilon \Rightarrow \sup_{x\in I} |S_p(x) - S_q(x)| = \sup_{x\in I} \left| \sum_{k=q+1}^{p} f_k(x) \right| < \varepsilon \right).$$

Theorem 3 (**Abel's theorem for uniform convergence**).

Let be the series $\sum f_n(x)g_n(x)$ which satisfies

i) $\exists M > 0 :\ \forall n \in \mathbb{N}, \|f_0 + f_1 + \cdots + f_n\| \leqslant M$ (i.e., the partial sums of the series $\sum f_n(x)$ are uniformly bounded),

ii) The series $\sum |g_{n+1} - g_n|$ is convergent,

iii) The sequence of functions $(g_n(x))_n$ converges uniformly to 0 on I (i.e., $\lim_{n\to+\infty} \|g_n\| = 0$).

Then, the series $\sum f_n(x)g_n(x)$ is uniformly convergent on I.

Example 5. Study the nature of the series of functions $\sum_{n=1}^{\infty} \frac{e^{-nx}}{n}$, $x \geqslant \alpha > 0$ (i.e., on $[\alpha, +\infty[$ with $\alpha > 0$).

Indeed. We put $f_n(x) = e^{-nx}$ and $g_n(x) = \frac{1}{n}$.

On the one hand, it is clear that for any $n \in \mathbb{N}^*$,

$$|f_1(x) + f_2(x) + \ldots + f_n(x)| = \left| \sum_{k=1}^{n} \left(e^{-x}\right)^k \right| = \left| \frac{e^{-x} - e^{-(n+1)x}}{1 - e^{-x}} \right| \leqslant \left| \frac{e^{-x} - e^{-(n+1)x}}{|1 - e^{-x}|} \right| \leqslant \frac{e^{-x} - e^{-(n+1)x}}{1 - e^{-x}},$$

because for any $n \in \mathbb{N}^*$ and for any $x \in [\alpha, +\infty[$: $e^{-x} - e^{-(n+1)x} \geqslant 0$ and $1 - e^{-x} \geqslant 0$ (the function e^{-x} is decreasing on $\mathbb{R}_+$). Thus,

$$|f_1(x) + f_2(x) + \ldots + f_n(x)| \leqslant \frac{e^{-x}}{1 - e^{-x}} \leqslant \frac{e^{-\alpha}}{1 - e^{-\alpha}} = M.$$

The last inequality comes from the decrease of the function $\frac{e^{-x}}{1 - e^{-x}}$ on the domain $[\alpha, +\infty[$. Thus

$$\|f_1 + f_2 + \ldots + f_n\| \leqslant M.$$

On the other hand, we have

$$\|g_{n+1} - g_n\| = \sup_{x \in [\alpha, +\infty[} |g_{n+1}(x) - g_n(x)| = \sup_{x \in [\alpha, +\infty[} \frac{1}{n(n+1)} = \frac{1}{n(n+1)} \overset{+\infty}{\sim} \frac{1}{n^2}$$

so the series of functions $\sum_{n=1}^{\infty} \|g_{n+1} - g_n\|$ is convergent, and

$$\|g_n\| = \sup_{x \in [\alpha, +\infty[} |g_n(x)| = \sup_{x \in [\alpha, +\infty[} \left| \frac{1}{n} \right| = \frac{1}{n} \to \frac{1}{n} \underset{n\to+\infty}{\longrightarrow} 0.$$

Hence, according to the previous theorem, the series of functions $\sum_{n=1}^{\infty} \frac{e^{-nx}}{\sqrt{n}}$ is uniformly convergent on $[\alpha, +\infty[, (\alpha > 0)$.

Proposition 2. Let be the series $\sum_n f_n(x)g_n(x)$ which satisfies

1. $\exists M > 0: \ \forall n \in \mathbb{N}, \|f_0 + f_1 + \ldots + f_n\| \leqslant M$, (i.e., the partial sums of the series $\sum_n f_n(x)$ are uniformly bounded),
2. For all $x \in I$, the sequence of functions $(g_n(x))_n$ is monotone,
3. The sequence of functions $(g_n(x))_n$ converges uniformly to 0 on I.

Then the series $\sum_n f_n(x)g_n(x)$ is uniformly convergent on I.

Example 6. Study the nature of the series of functions $\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}, \quad |x| \leqslant \alpha < 1$.

In fact. Let $f_n(x) = x^n$ and $g_n(x) = \frac{1}{\sqrt[7]{n}}$. On the one hand, it is clear that for any $n \in \mathbb{N}^*$,

$$|f_1(x) + f_2(x) + \ldots + f_n(x)| = \left|\sum_{k=1}^{n} x^k\right| = \left|\frac{x - x^{n+1}}{1-x}\right| \leqslant \frac{|x| + |x|^{n+1}}{|1-x|} \leqslant \frac{2\alpha}{1-\alpha},$$

so,

$$\|f_1 + f_2 + \ldots + f_n\| \leqslant \frac{2\alpha}{1-\alpha} = M.$$

On the other hand, the series of functions g_n is decreasing and converges uniformly to 0. Thus, according to the previous proposition, the series of functions $\sum_{n=1}^{\infty} \frac{x^n}{\sqrt[7]{n}}$ converges uniformly on $[-\alpha, \alpha]$ $(\alpha < 1)$.

3.2.4. Link between different types of convergence

Theorem 4. Let be the series of functions $\sum_n f_n(x)$. Then

$$\sum_n f_n(x) \text{ normally converges on } I \Rightarrow \sum_n f_n(x) \text{ uniformly converges on } I.$$

Remark 5. The reciprocal of the previous theorem is false.

Example 7. The series of functions $\sum_{n\geqslant 1}^{\infty} \frac{(-1)^n}{x+n}$ converges uniformly on $\mathbb{R}_+$ because it is a Leibniz series, but

$$\sup_{x\in\mathbb{R}_+} \left| \frac{(-1)^n}{x+n} \right| = \frac{1}{n}$$

and the series $\sum_{n\geqslant 1}^{\infty} \frac{1}{n}$ is divergent. Thus, the series of functions $\sum_{nge1}^{\infty} \frac{(-1)^n}{x+n}$ is not normally convergent on $\mathbb{R}_+$.

Proposition 3. Let be the series of functions $\sum_n f_n(x)$, then

$$\sum_n f_n(x) \text{normally converges on} I \Rightarrow \sum_n f_n(x) \text{absolutely converges on} I.$$

The following diagram shows the relationship between the different types of convergence for series of functions.

$$\begin{array}{ccc} \text{Normal convergence} & \Rightarrow & \text{Uniform convergence} \\ \Downarrow & & \Downarrow \\ \text{Absolute convergence} & \Rightarrow & \text{Simple convergence} \end{array}$$

Example 8.

1. We have already seen that the series of functions $\sum_{n=1}^{\infty} \frac{(-1)^n}{x+n}$ converges uniformly on $\mathbb{R}_+$.

   However, $\left| \frac{(-1)^n}{x+n} \right| = \frac{1}{x+n} \to \frac{1}{n}$, $\forall x \geqslant 0$, and the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{x+n}$ does not converge absolutely on $\mathbb{R}_+$.

   On the other hand, $\sup_{x\geqslant 0} \left| \frac{(-1)^n}{x+n} \right| = \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{x+n}$ does not converge normally on $\mathbb{R}_+$.

   Thus the series $\sum_{n\geqslant 1} \frac{(-1)^n}{x+n}$ converges uniformly, but not normally, nor absolutely on $\mathbb{R}_+$.

2. Soit la série de fonctions $\sum_{n\geqslant 1} \frac{(-1)^n}{x^2n^2+n}$, $f_n(x) = \frac{(-1)^n}{x^2n^2+n}$ avec $x \in [0,1]$.

We have,

$$\left|\frac{(-1)^n}{x^2n^2+n}\right| = \frac{1}{x^2n^2+n} \leqslant \frac{1}{n^2n^2+n} = \frac{1}{x^2n^2}, \forall x \in [0,1].$$

Or, the series $\sum_{n\geqslant 1} \frac{1}{x^2n^2}$ converges on $]0,1]$ (Riemann series), so the series $\sum_{n\geqslant 1} \frac{(-1)^n}{x^2n^2+n}$ converges absolutely on $]0,1]$.

According to theorem **??** (Leibniz's Theorem), we have

$$\forall x \in [0,1], |R_n(x)| \leqslant |f_{n+1}(x)| = \frac{1}{x^2(n+1)^2+n+1} \implies \forall x \in [0,1], |R_n(x)| \leqslant \frac{1}{n+1}.$$

Hence,

$$0 \leqslant \sup_{x\in[0,1]} |R_n(x)| \leqslant \frac{1}{n+1},$$

i.e., $R_n \underset{\text{UC}}{\to} 0$ sur $]0,1]$, the series is therefore uniformly convergent on $]0,1]$, but

$$\sup_{x\in[0,1]} \left|\frac{(-1)^n}{x^2n^2+n}\right| = \sup_{x\in[0,1]} \frac{1}{x^2n^2+n} = \frac{1}{n},$$

which means that the series does not converge normally, even though it is absolutely and uniformly convergent.

3.3. Continuity, integration and derivation of series of functions

3.3.1. Continuity

Theorem 5 (Seidel Continuity).

Let the series of functions $\sum f_n(x)$ be uniformly convergent on I and at $a \in I$.

If for all $n \in \mathbb{N}$, the function f_n is continuous at a (resp. on I), then the sum $S(x) = \sum_{n=0}^{+\infty} f_n(x)$ of the series is continuous at a (resp. on I). That is,

$$\lim_{x\to a} S(x) = \lim_{x\to a} \sum_{n=0}^{+\infty} f_n(x) = \sum_{n=0}^{+\infty} \lim_{x\to a} f_n(x) = \sum_{n=0}^{+\infty} f_n(a) = S(a).$$

$$\text{(resp. } \forall x_0 \in I : \lim_{x\to x_0} S(x) = \lim_{x\to x_0} \sum_{n=0}^{+\infty} f_n(x) = \sum_{n=0}^{+\infty} \lim_{x\to x_0} f_n(x) = \sum_{n=0}^{+\infty} f_n(x_0) = S(x_0)\text{).}$$

Example 9. Let the series of functions be $\sum_{n\geqslant 0}\frac{e^{-nx}}{1+n^2}$.

For any integer $n\geqslant 1$, the functions $f_n(x)=\frac{e^{-nx}}{1+n^2}$ are continuous on $\mathbb{R}_+$. Moreover we have

$$\forall x\in\mathbb{R}_+, |f_n(x)|=\frac{e^{-nx}}{1+n^2}\leqslant\frac{1}{1+n^2},$$

and since $\sum_{n\geqslant 0}\frac{1}{1+n^2}$ is a convergent series, then the series of functions $\sum_{n\geqslant 0}\frac{e^{-nx}}{1+n^2}$ converges normally and therefore uniformly on $\mathbb{R}_+$.

Thus according to the previous Theorem, the function $S(x)=\sum_{n=0}^{+\infty}\frac{e^{-nx}}{1+n^2}$ is continuous on $\mathbb{R}_+$, therefore,

$$\lim_{x\to+\infty}S(x)=\sum_{n=0}^{+\infty}\lim_{x\to+\infty}\frac{e^{-nx}}{1+n^2}=0.$$

Proposition 4 (The contrapositive of the previous theorem).

Let the series of functions $\sum f_n(x)$ with sum $S(x)$ be such that for all $n\in\mathbb{N}$, the function f_n is continuous on I.

If the sum S of the series $\sum f_n(x)$ is discontinuous at a point $x_0\in I$, then the series $\sum f_n(x)$ is not uniformly convergent on I.

3.3.2. Integration

Theorem 6 (Integration theorem).

Let the series of functions $\sum f_n(x)$ be, which converges uniformly to $S(x)$ on $[a,b]$.

If for all $n\in\mathbb{N}$, the function f_n is integrable on $[a,b]$. Then, the sum S of the series is integrable on $[a,b]$, and we have

$$\int_a^b S(x)=\int_a^b\sum_{n=0}^{+\infty}f_n(x)=\sum_{n=0}^{+\infty}\int_a^b f_n(x).$$

Example 10. Let be the series of functions $\sum_{n\geqslant 0}x^n$ avec $|x|\leqslant r<1$.

We have $|x|^n\leqslant r^n$, $\forall x\in[-r,r]$, with $r<1$, which means that the geometric series $\sum_{n\geqslant 0}x^n$ converges normally and therefore uniformly on $[-r,r]$.

Furthermore, the functions $f_n(x) = x^n$ are integrable on $[-r, r]$ (because they are continuous on $[a, b]$), so according to the previous the previous Theorem, the function $S(x) = \sum_{n=0}^{+\infty} x^n$ is integrable on $[-r, r]$, and we have

$$\forall x \in [-r, r], 0 < r < 1 : \int_0^x \sum_{n=0}^{+\infty} t^n dt = \sum_{n=0}^{+\infty} \int_0^x t^n dt$$

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$$\int_0^x \frac{dt}{1-t} = \sum_{n=0}^{+\infty} \frac{x^{n+1}}{n+1} \Rightarrow -\ln(1-x) = \sum_{n=0}^{+\infty} \frac{x^{n+1}}{n+1} = \sum_{n=1}^{+\infty} \frac{x^n}{n}, \forall x \in [-r, r], 0 < r < 1$$

3.3.3. Derivation

Theorem 7 (Derivation theorem).

Let be the series of functions $\sum f_n(x)$ of sum $S(x)$, such that for all $n \in \mathbb{N}$, the function f_n is continuously derivable on $[a, b]$. If

i) $\exists x_0 \in [a, b] / \sum f_n(x_0)$ converges.

ii) $\sum f_n'(x)$ uniformly converges on $[a, b]$.

Then the series $\sum f_n(x)$ uniformly converges on $[a, b]$, moreover the function S is derivable on $[a, b]$ and we have

$$S'(x) = \left(\sum_{n=0}^{+\infty} f_n(x) \right)' = \sum_{n=0}^{+\infty} f_n'(x)$$