Chapter 4

Integers series

4.1. Integers series

Integer series are series of functions of a particular form. They are well suited to the derivation operation, and therefore to solving differential equations.

Definition 1. We call an integer series a series of functions $\sum_n f_n$ such that for all $n \in \mathbb{N}$, f_n is defined as follows,

$$f_n(z) = a_n z^n,$$

the variable z can be real or complex.

4.1.1. Radius of convergence

The convergence radius of an integer series approximately characterises the convergence modes of the series of functions $\sum_n a_n z^n$ and the analytical properties of the sum.

Lemma 4.1.1 (Abel's lemma). Let $\sum_n a_n z^n$ be an integer series. Assume that the sequence $(a_n z_0^n)_n$ is bounded for some $z_0 \in \mathbb{C}$. Then for all $z \in \mathbb{C}$, if $|z| < |z_0|$, the series with general term $a_n z^n$ is absolutely convergent.

Theorem 1. Let $\sum a_n x^n$ be an integer series. The set of positive reals r such that the sequence $(a_n r^n)_n$ is bounded is an interval of $\mathbb{R}_+$ containing 0.

Proof. If $(a_n r^n)_n$ is bounded and if $0 \leqslant s < r$, then the series with general term $a_n s^n$ is absolutely convergent according to the previous lemma, so the sequence $(a_n s^n)_n$ tends to 0 and is therefore bounded. ∎

Definition 2 (Convergence radius)**.** The radius of convergence of the integer series $\sum a_n z^n$ is the element

$$\sup\{r \geqslant 0 : (a_n r^n)_n \text{ is bounded}\} \in [0, +\infty]$$

If R is the radius of convergence of the integer series $\sum a_n z^n$, the set of $r \geqslant 0$ such that the sequence $(a_n r^n)_n$ is bounded is $[0, R]$.

Theorem 2.

1. If $R = +\infty$, then for all $z \in \mathbb{C}$, the series with general term $a_n z^n$ is absolutely convergent.

2. If $R = 0$, for all $z \in \mathbb{C} \setminus \{0\}$, the sequence $(a_n z^n)_n$ is not bounded, in particular, the series diverges.

3. If $R \neq 0$ and $R \neq +\infty$:

   - For all $z \in \mathbb{C}$ such that $|z| < R$, the series with general term $a_n z^n$ is absolutely convergent.
   - If $|z| > R$, the sequence $(a_n z^n)_n$ is not bounded, so the series diverges grossly.
   - If $|z| = R$, we can say nothing in general. We thus have a partition of the complex plane into three parts (in the last case).

Example 1. The series $\sum z^n$ is absolutely convergent if $|z| < 1$ and diverges if $|z| \geqslant 1$, so we conclude that the radius of convergence is $R = 1$.

4.1.2. Radius of convergence of a sum and a product

Theorem 3. Let R_a be the radius of convergence of an integer series $\sum a_n z^n$, R_b that of an integer series $\sum b_n z^n$. Then the Radius of Convergence of the sum and product series are greater than $\min(R_a, R_b)$. And for sum: If $R_a \neq R_b$, the radius of convergence of $\sum (a_n + b_n) z^n$ is equal to $\min(R_a, R_b)$.

Example 2.

1. $\forall n \in \mathbb{N}, a_n = -b_n = 1$.

   Then $R_a = R_b = 1$ but the radius of convergence of the sum series equals $+\infty$.

2. We take $(a_n)_n$ such that,

$$\frac{1-z}{1+z} = \sum_{n=0}^{+\infty} a_n z^n, \quad \text{and} \quad b_n = (-1)^n a_n.$$

   Thus,

$$\sum_{n=0}^{+\infty} b_n z^n = \frac{1+z}{1-z} \quad \text{and} \quad c_n = \sum_{k=0}^{n} a_k b_{n-k} = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

   We then have $R_a = R_b = 1$, but the radius of convergence of the product series equals $+\infty$.

   With

$$\sum_{n=0}^{+\infty} a_n z^n = \frac{1-\frac{z}{2}}{1+z}, \quad R_a = 1, \quad \sum_{n=0}^{+\infty} b_n z^n = \frac{1+\frac{z}{2}}{1-z}.$$

   $R_b = 2$ and the radius of convergence of the product series equals $+\infty$.

Definition 3. If R is the radius of convergence of an integer series $\sum_{n=0}^{+\infty} a_n z^n$, the open disk

$$D^{\circ}(0,R) = \{z \in \mathbb{C} \mid |z| < R\}$$

is called the disk of convergence of the integer series.

Remark 1. If R is finite, we don't know a priori whether $\sum_{n=0}^{+\infty} a_n z^n$ will converge for $|z| = R$.

Theorem 4. Let $\sum a_n z^n$ and $\sum b_n z^n$ be two integer series of radii of convergence R_a and R_b respectively. Then we have

$$\exists n_0 \in \mathbb{N}, \forall n \geqslant n_0 : \ |a_n| \leqslant |b_n| \Rightarrow R_a \geqslant R_b,$$

and more generally,

$$\exists \alpha \in \mathbb{R}, \exists k > 0, \exists n_0 \in \mathbb{N}, \forall n \geqslant n_0 : (|a_n| \leqslant k|b_n| n^\alpha) \Rightarrow (R_a \geqslant R_b),$$

$$\bullet \exists \alpha \in \mathbb{R}, \left(\frac{a_n}{b_n n^\alpha}\right) \Rightarrow (R_a = R_b),$$

in particular,

$$\bullet \left(\frac{a_n}{b_n} \sim 1\right) \Rightarrow (R_a = R_b).$$

4.1.3. Methods for calculating the convergence radius

Theorem 5 (d'Alembert's rule)**.** Let $(a_n)_n$ be a sequence of complexes such that,

1. There exists $n_0 \in \mathbb{N}$ such that $\forall n \geqslant n_0, a_n \neq 0$.
2. The sequence $\left|\frac{a_{n+1}}{a_n}\right|$ tends to $l \in [0, +\infty]$.

Then the radius of convergence of the integer series $\sum_{n=0}^{+\infty} a_n z^n$ is $R = \frac{1}{l} \in [0, +\infty]$.

Example 3.

1. Let $\sum_{n=0}^{+\infty} \frac{(-1)^{n+1}}{(n+1)^n} z^n$, we calculate the radius of convergence. So, we have,

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)^n}{(n+2)^{n+1}}\right| = \left(\frac{n+1}{n+2}\right)^n \frac{1}{n+2} \underset{+\infty}{\overset{e}{}} \times 0 = 0.$$

   So the radius of convergence $R = +\infty$, so $\sum_{n=0}^{+\infty} \frac{(-1)^{n+1}}{(n+1)^n} z^n$ converges in $\mathbb{C}$.

2. $\displaystyle\sum_{n=0}^{+\infty} \frac{n!}{(n+1)\cdots(2n+1)} z^n$, radius of convergence, study in $\pm R$? Then, we have,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)n!}{(n+2)\cdots(2n+3)} \times \frac{(n+1)\cdots(2n+1)}{n!} = \frac{(n+1)^2}{(2n+2)(2n+3)} \underset{+\infty}{\to} .$$

So the radius of convergence $R = 4$.

- Study in ± 4. For $\displaystyle\sum_{n=0}^{+\infty} \frac{n!}{(n+1)\cdots(2n+1)} 4^n$

$$\left|\frac{f_{n+1}(4)}{f_n(4)}\right| = \left|4\frac{a_{n+1}}{a_n}\right| = 4\frac{(n+1)^2}{(2n+2)(2n+3)} = 2\frac{(n+1)}{(2n+3)} < 1,$$

so the sequence $(4^n a_n)_n$ decreases. We are looking for an equivalent of $4^n a_n$ in $+\infty$.

We will look for α and $k > 0$ such that $f_n \underset{+\infty}{\sim} kn^\alpha$. We are looking for α such that the sequence of general term $\ln\left(\frac{f_n}{n^\circ}\right)$ converges.

$$\begin{aligned}
\ln\left(\frac{f_n}{n^\alpha}\right) - \ln\left(\frac{f_{n-1}}{(n-1)^\alpha}\right) &= \ln\left(\frac{f_n}{f_{n-1}}\right) + \alpha\ln\left(\frac{n-1}{n}\right) \\
&= \ln\left(\frac{2n}{2n+1}\right) + \alpha\ln\left(\frac{n-1}{n}\right) \\
&= -\ln\left(1+\frac{1}{2n}\right) + \alpha\ln\left(\frac{n-1}{n}\right) \\
&= -\left(\alpha+\frac{1}{2}\right)\frac{1}{n} + 0\left(\frac{1}{n^2}\right).
\end{aligned}$$

Thus, if we take $\alpha = -\frac{1}{2}$, the series converges absolutely, so the sequence with general term $\ln\left(\frac{f_n}{a_n}\right)$ converges to $\lambda \in \mathbb{R}$ and $f_n \sim e^\lambda$. i.e., $f_n \sim e^{\lambda n^\alpha}$, we have therefore also found k such that

$$f_n \sim \frac{k}{\sqrt{n}}.$$

Thus, at $R = 4$, there is divergence because $4^n a_n \sim \frac{k}{\sqrt{n}}$. And at $R = -4$, there is convergence by the Leibniz criterion.

4.2. Functional properties of an integer series

Theorem 6. Let $\sum_{n=0}^{+\infty} a_n z^n$ be an integer series of radius of convergence R. The series converges normally on any compact included in the open disk of convergence (in the case of a complex variable) or the open interval of convergence (in the case of a real variable). In the real case, there is in particular normal convergence of the integer series on any segment of type $[a,b]$ or $[-a,a]$ included in $]-R,R[$.

4.2.1. Continuity of the sum of an integer series

Theorem 7 (Continuity of the sum of an integer series of real variables). Let $\sum_{n=0}^{+\infty} a_n x^n$ be an integer series with a real variable, a convergence radius R and a sum S. The function S is continuous on the open interval of convergence. $]-R,R[$.

Proof. The continuity of the functions. $\forall n \in \mathbb{N}, x \mapsto a_n x^n$ on any interval $[a,b] \subset]-R,R[$ and the normal convergence on $[a,b]$ of the series of these functions (**??**), means that the sum of this series (i.e. the sum of the integer series) is continuous on any interval $[a,b] \subset]-R,R[$, and therefore on $]-R,R[$ itself. ■

Theorem 8 (Continuity of the sum of an integer series of complex variables). Let $\sum_{n=0}^{+\infty} a_n z^n$ be an integer series of complex variable, radius of convergence R and sum S. The function S is continuous on the open disc $D(0,R)$.

4.2.2. Primitives of the sum of an integer series of real variables

Theorem 9.
Let $\sum_{n=0}^{+\infty} a_n x^n$ be an integer series of real variables, of convergence radius R and of sum

$$S(x) = \sum_{n=0}^{+\infty} a_n x^n.$$

We can integrate S term by term on any segment contained in $]-R,R[$. In particular, S has primitives on $]-R,R[$ which are equal to

$$c+\sum_{n=0}^{+\infty} a_n \frac{x^{n+1}}{n+1}, \text{ where } c \in \mathbb{C}.$$

These primitives have the same convergence radius R as $\sum_{n=0}^{+\infty} a_n x^n$.

Proof. Since S is continuous on $]-R,R[$, it has primitives there. Moreover, for $0 \leqslant a < R$, the integer series $\sum_{n=0}^{+\infty} a_n x^n$ converges normally sur$[-a,a]$ so we can calculate the primitive term by term.

Finally the primitives of S on $[-a,a]$ (which are all equal up to an additive constant) are:

$$\int S(x)dx = c+\sum_{n=0}^{+\infty}\int a_n x^n dx = c+\sum_{n=0}^{+\infty} a_n \frac{x^{n+1}}{n+1}$$

where c is a real or complex constant. These new integer series have a convergence radius R_p.

- For x non-zero, the convergence of $\sum_{n=0}^{+\infty} a_n \frac{x^{n+1}}{n+1}$ is equivalent to that of $\sum_{n=0}^{+\infty} a_n \frac{x^n}{n+1}$, since they are equal up to a multiplicative constant. But

  $$\forall n \in \mathbb{N}, \left|\frac{a_n}{n+1}\right| \leqslant |a_n|,$$

  and we deduce that $R_p \geqslant R$.

- Then $\forall z \in \mathbb{C} : |z| < R_p, \exists \rho \in \mathbb{R}^*, |z| < \rho < R_p$. We can then write

  $$\forall n \in \mathbb{N} :\ |a_n z^n| = |a_n| \frac{\rho^n}{n+1}\left[(n+1)\frac{|z^n|}{\rho^n}\right],$$

  and as the sequence $\left((n+1)\frac{|z^n|}{\rho^n}\right)_n$ tends to 0, due to the theorem of comparative growths, we deduce that

  $$\exists n_0 \in \mathbb{N}, \forall n \geqslant n_0 : (n+1)\frac{|z^n|}{\rho^n} \leqslant 1 \text{and} \, |a_n z^n| \leqslant |a_n| \frac{\rho^n}{n+1},$$

  or the series $\sum_{n=0}^{+\infty} |a_n| \frac{\rho^n}{n+1}$ converges (since $0 < \rho < R_p$), and therefore the series $\sum_{n=0}^{+\infty} a_n x^n$ converges absolutely. We deduce that $|z| \leqslant R$, therefore $\left[0,R_p\right[\subset [0,R]$. and finally $R_p = R$, and the primitive series have the same radius of convergence as the initial series.

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Example 4. Determine the radius of convergence and the sum of the real integer series,

$$\sum_{n=1}^{+\infty} n x^{n-1}, \quad \sum_{n=0}^{+\infty} \frac{x^{n+1}}{n+1}, \quad \sum_{n=1}^{+\infty} n^2 x^{n-1}.$$

1. $\sum_{n=1}^{+\infty} n x^{n-1}$ is the derivative series of the integer series $\sum_{n=0}^{+\infty} x^n$ with radius of convergence $R=1$ and sum

$$S(x) = \frac{1}{1-x} \text{ on }]-1,1[\,,$$

and

$$S'(x) = \sum_{n=1}^{+\infty} n x^{n-1} = \frac{1}{(1-x)^2}, \forall x \in]-1,1[$$

2. $\sum_{n=0}^{+\infty} \frac{x^{n+1}}{n+1}$ is the primitive of the integer series $\sum_{n=0}^{+\infty} x^n$ with radius of convergence $R=1$ and

$$\sum_{n=0}^{+\infty} \frac{x^{n+1}}{n+1} = \int \sum_{n=0}^{+\infty} a_n x^n dx = \int \frac{1}{1-x} dx = -\ln(1-x), \forall x \in]-1,[\,.$$

3.
$$\sum_{n=1}^{+\infty} n^2 x^{n-1} = \sum_{n=1}^{+\infty} \left(n^2 - n + n\right) x^{n-1} = \sum_{n=1}^{+\infty} n(n-1) x^{n-1} + \sum_{n=1}^{+\infty} n x^{n-1}$$
$$= x \sum_{n=1}^{+\infty} n(n-1) x^{n-2} + \sum_{n=1}^{+\infty} n x^{n-1}.$$

$\sum_{n=1}^{+\infty} n(n-1) x^{n-2}$ est la série dérivée d'ordre 2 de la série entière $\sum_{n=0}^{+\infty} x^n$, donc

$$\sum_{n=1}^{+\infty} n^2 x^{n-1} = x S''(x) + S'(x) = \frac{2x}{(1-x)^2} + \frac{1}{(1-x)^2} = \frac{1+x}{(1-x)^3}, \forall x \in]-1,1[\,.$$

4.2.3. Derivability and C^∞ nature of the sum of an integer series

Theorem 10.
Let $\sum_{n=0}^{+\infty} a_n x^n$ be an integer series of real variable, of radius of convergence R and of sum

$$S(x) = \sum_{n=0}^{+\infty} a_n x^n.$$

On $]R;R[$, the function S is of class C^∞ and we obtain its successive derivatives by term-by-term derivation of the function S.

All the integer series derived from S have the same radius of convergence R as $\sum_{n=0}^{+\infty} a_n x^n$.

Moreover

- $\forall x \in]-R,R[\,, S'(x) = \sum_{n=1}^{+\infty} n a_n x^{n-1} = \sum_{n=0}^{+\infty} (n+1) a_{n+1} x^n$
- $\forall p \in \mathbb{N}, \forall x \in]-R,R[\,, S^{(p)}(x) = \sum_{n=p}^{+\infty} \frac{n!}{(n-p)!} a_n x^{n-p} = \sum_{n=0}^{+\infty} \frac{(n+p)!}{n!} a_{n+p} x^n.$

The coefficients of the integer series $\sum_{n=0}^{+\infty} a_n x^n$ then check

$$\forall n \in \mathbb{N}, a_n = \frac{S^{(n)}(0)}{n!}.$$

Proof. First, let's show that $S \in C^{infty}(]-R,R[)$. We have

$$\forall x \in]-R,R[\,, S(x) = \sum_{n=0}^{+\infty} a_n x^n.$$

$\sum_{n=0}^{+\infty} a_n x^n$ converges normally and therefore uniformly on any interval $[-r,r]$ such that $0 < r < R$. On the other hand, the function $f_n : x \mapsto f_n(x)$ is of class C^∞ on $\mathbb{R}$ and therefore in particular on $[-r,r]$.

Let us show by recurrence that,

$$f_n^{(p)}(x) = \begin{cases} 0 & \text{if } p > n \\ \frac{n!}{(n-p)!} a_n x^{n-p} & \text{if } p \leqslant n. \end{cases}$$

This expression is true for $p = 1$, in fact

$$f_n'(x) = n a_n x^{n-1}, \forall n \geqslant 1,$$

and

$$f_n'(x) = 0 \mathrm{for} n = 0.$$

Assume that the hypothesis is true for (p) and prove it for $(p+1)$. We have

$$f_n^{(p+1)}(x) = \left(f_n^{(p)}(x)\right)' = \begin{cases} 0 & \text{if } p > n, \\ \dfrac{n!}{(n-p)!}(n-p)a_n x^{n-p-1} & \text{if } p < n, \\ 0 & p = n, \end{cases}$$

$$= \begin{cases} 0 & \text{if } p > n-1, \\ \dfrac{n!}{(n-p-1)!}a_n x^{n-p-1} & \text{if } p \leqslant n-1, \\ 0 & \text{if } p+1 > n, \\ \dfrac{n!}{(n-(p+1))!}a_n x^{n-(p+1)} & \text{if } p+1 \leqslant n, \end{cases}$$

so the relation is true for $(p+1)$. It remains to show that $\sum_{n=0}^{+\infty} \dfrac{n!}{(n-p)!}a_n x^{n-p}$ converges uniformly on $[-r, r], 0 \leqslant r \leqslant R$.

We have the series $\sum_{n=0}^{+\infty} a_n x^n$ and the derivative series having the same radius of convergence R, from which $\sum_{n=0}^{+\infty} f_n^{(p)}(x)$ converges uniformly on $[-r, r] \subset]-R, R[$. Therefore, given the derivation theorem for series of functions (Theorem **??**), we conclude that,

$$S \in C^p([-r, r]) \text{ for all } p \geqslant 1,$$

so $S \in C^\infty(]-R, R[)$. Let's show that $\forall n \in \mathbb{N}: \ a_n = \dfrac{S^{(n)}(0)}{n!}$. We have,

$$f_k^{(p)}(x) = \begin{cases} 0 & \text{if } p > k, \\ \dfrac{k!}{(k-p)!}a_k x^{k-p} & \text{if } p \leqslant k, \end{cases}$$

$$\begin{aligned} S^{(n)}(x) &= \left(\sum_{n=0}^{+\infty} f_k(x)\right)^{(n)} = \sum_{n=0}^{+\infty} f_k^{(n)}(x) = \sum_{k \geqslant n}^{+\infty} \frac{k!}{(k-n)!}a_k x^{k-n} \\ &= \frac{n!}{0!}a_n x^0 + \sum_{k \geqslant n+1}^{+\infty} \frac{k!}{(k-n)!}a_k x^{k-n} \\ &= a_n n! + x \sum_{k \geqslant n+1}^{+\infty} \frac{k!}{(k-n)!}a_k x^{k-n-1}, \end{aligned}$$

and consequently

$$S^{(n)}(0) = a_n n! \Rightarrow a_n = \frac{S^{(n)}(0)}{n!}.$$

■

Theorem 11. If the series $\sum_{n=0}^{+\infty} a_n x^n$ converges for $x = R$ (or $x = -R$), then the sum $S(x) = \sum_{n=0}^{+\infty} a_n x^n$ is continuous on $]-R, R[$ (or $[-R, R]$).

4.3. Applications of integer series

4.3.1. Functions that can be developed into integer series

Definition 4.

Let I be an interval of $\mathbb{R}$ containing 0 and let f be a function of I in $\mathbb{R}$. We say that f is developable as an integer series at 0 if and only if there exists an integer series $\sum_{n=0}^{+\infty} a_n x^n$ of non-zero radius of convergence R, and $0 < r \leqslant R$ such that

$$\forall x \in]-r, r[\cap I, f(x) = \sum_{n=0}^{+\infty} a_n x^n.$$

4.3.2. Necessary existence conditions for the development of integer series

Theorem 12.

Let $r > 0$, and let f be a function of $]-r, r[$ in $\mathbb{R}$, developable as an integer series in 0 such that

$$\forall x \in]-r, r[, f(x) = \sum_{n=0}^{+\infty} a_n x^n.$$

Then f is of class C^∞ on $]-r, r[$ and $\forall n \in \mathbb{N} : \; a_n = \dfrac{f^{(n)}(0)}{n!}$.

Proof. f coincides with the sum of an integer series on $]-r, r[$. The result follows from theorem **??**. ∎

4.3.3. Sufficient condition for integer series development

Theorem 13.

Let $f :]-r, r[\to \mathbb{R}$ be a function of class C^∞ in a neighborhood of 0. We assume that there exists $M > 0$ such that

$$\text{for all} n \in \mathbb{N}, \text{ and for all } x \in]-r, r[, |f^{(n)}(x)| \leqslant M. \tag{4.3.1}$$

Then the series $\sum_{n=0}^{+\infty} \frac{f^{(n)}(0)}{n!} x^n$ is simply convergent in $]-r, r[$ and we have

$$f(x) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(0)}{n!} x^n, \forall x \in]-r, r[.$$

Proof. By hypothesis, there exists $M > 0$ such that for any $n \in \mathbb{N}$, and for any $x \in]-r, r[$, we have

$$|f^{(n)}(x)| \leqslant M.$$

The Taylor expansion of f in the neighbourhood of 0 to order n gives,

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k + \frac{f^{(n+1)}(\theta x)}{(n+1)!} x^{n+1}, \text{with} 0 < \theta < 1.$$

To prove the theorem, it suffices to prove that $\lim_{n \to +\infty} \frac{f^{(n+1)}(\theta x)}{(n+1)!} x^{n+1} = 0$. Indeed, $x \in]-r, r[$ $\Rightarrow |x| < r \Rightarrow |\theta x| < r \Rightarrow |f^{(n+1)}(\theta x)| < r$, and so,

$$\left| \frac{f^{(n+1)}(\theta x)}{(n+1)!} x^{n+1} \right| \leqslant \frac{M}{(n+1)!} r^{n+1}.$$

Now, the series with general term $u_n = \frac{M}{(n+1)!} r^{n+1}$ is convergent because,

$$\lim_{n \to +\infty} \frac{u_{n+1}}{u_n} = \lim_{n \to +\infty} \frac{r}{(n+1)} = 0 < 1,$$

and as a result

$$\lim_{n \to +\infty} \frac{f^{(n+1)}(\theta x)}{(n+1)!} x^{n+1} = 0,$$

which yields

$$f(x) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

∎

Theorem 14.

Let $f :]-r,r[\to \mathbb{R}$ be a function of class C^∞ in a neighborhood of 0. We assume that there exists $M > 0$ such that

$$\forall n \in \mathbb{N}, \text{and } \forall x \in]-r,r[, |f^{(n)}(x)| \leqslant M. \tag{4.3.2}$$

Then the series $\sum_{n=0}^{+\infty} \frac{f^{(n)}(0)}{n!} x^n$ is simply convergent in $]-r,r[$ and we have,

$$f(x) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(0)}{n!} x^n, \forall x \in]-r,r[.$$

Proof. By hypothesis, there exists $M > 0$ such that for all $n \in \mathbb{N}$, and for all $x \in]-r,r[$, we have

$$|f^{(n)}(x)| \leqslant M.$$

The Taylor development of f near 0 to order n gives

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k + \frac{f^{(n+1)}(\theta x)}{(n+1)!} x^{n+1}, \text{with } 0 < \theta < 1.$$

To prove the theorem, it suffices to prove that $\lim_{n \to +\infty} \frac{f^{(n+1)}(\theta x)}{(n+1)!} x^{n+1} = 0.$

In fact, $x \in]-r,r[\Rightarrow |x| < r \Rightarrow |\theta x| < r \Rightarrow |f^{(n+1)}(\theta x)| < r$, and so,

$$\left| \frac{f^{(n+1)}(\theta x)}{(n+1)!} x^{n+1} \right| \leqslant \frac{M}{(n+1)!} r^{n+1}.$$

Or the series with general term $u_n = \frac{M}{(n+1)!} r^{n+1}$ is convergent because

$$\lim_{n \to +\infty} \frac{u_{n+1}}{u_n} = \lim_{n \to +\infty} \frac{r}{(n+1)} = 0 < 1,$$

and consequently

$$\lim_{n \to +\infty} \frac{f^{(n+1)}(\theta x)}{(n+1)!} x^{n+1} = 0,$$

which gives

$$f(x) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

■

The condition (**??**) is not necessary, as shown in the following example

Example 5. Let $f(x) = \sum_{n=0}^{+\infty} \frac{2^n}{n!} x^n$. We put $a_n = \frac{2^n}{n!}$. We calculate R.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{2^{n+1}}{(n+1)!} \frac{n!}{2^n}\right| \underset{n\to+\infty}{\longrightarrow} 0 \Rightarrow R = +\infty.$$

Then f is of class C^∞ on $\mathbb{R}$, and we have

$$a_n = \frac{f^{(n)}(0)}{n!}$$

So, $f^{(n)}(0) = a_n n! = \frac{2^n}{n!} n! = 2^n \underset{n\to+\infty}{\longrightarrow} +\infty$. $f^{(n)}(0)$ is not bounded, so we conclude that the derivatives of f are not bounded.

4.3.4. Necessary and sufficient condition for integer series development

Theorem 15.

Let f be a function of class C^∞ on $]-r, r[$. f is developable as an integer series if and only if the Mac-Laurin remainder $\frac{f^{(n+1)}(\theta x)}{(n+1)!} x^{n+1}$, with $0 < \theta < 1$ holds,

$$\forall x \in]-r, r[: \lim_{n\to+\infty} \frac{f^{(n+1)}(\theta x)}{(n+1)!} x^{n+1} = 0$$

Example 6.

1. **The exponential function:** $x \mapsto f(x) = e^x$.

   This function is infinitely differentiable in $\mathbb{R}$, and we have

   $$\forall n \in \mathbb{N}: f^{(n)}(x) = e^x.$$

   The Mac-Laurin remainder is $\frac{e^{(0x)}}{(n+1)!} x^{n+1}$. We check as before (proof of Theorem **??**) that this limit tends to 0 when n tends to $+\infty$, and this whatever x in $\mathbb{R}$. Finally,

   $$\forall x \in \mathbb{R}: e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots = \sum_{n=0}^{+\infty} \frac{x^n}{n!}.$$

2. **The hyperbolic functions.**

   The cosine-hyperbolic and sine-hyperbolic functions have the same radius of convergence as the exponential function, in other words, $R = +\infty$.

$$\cosh x = \frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{+\infty} \frac{x^{2n}}{(2n)!},$$
$$\sinh x = \frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \sum_{n=0}^{+\infty} \frac{x^{2n+1}}{(2n+1)!}.$$

3. **The circular functions.**

   (a) **The sine function:** $x \mapsto f(x) = \sin x$. $f \in C^\infty(\mathbb{R})$, and we have,

   $$f(x) = \sin x, f'(x) = \cos x, f''(x) = -\sin x, f'''(x) = -\cos x,$$

   which allows us to deduce that for all $p \in \mathbb{N}$:

   $$f^{(4p)}(x) = \sin x \Rightarrow f^{(4p)}(0) = 0,$$
   $$f^{(4p+1)}(x) = \cos x \Rightarrow f^{(4p+1)}(0) = 1,$$
   $$f^{(4p+2)}(x) = -\sin x \Rightarrow f^{(4p+2)}(0) = 0,$$
   $$f^{(4p+3)}(x) = -\cos x \Rightarrow f^{(4p+3)}(0) = -1.$$

   Derivatives of any order are bounded above(upper bounded) by 1, and this whatever x in $\mathbb{R}$. We then have,

   $$\sin x = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \text{ and } R = +\infty.$$

   (b) **The cosine function:** $x \mapsto f(x) = \cos x$.

   $$f(x) = \cos x = (\sin x)' = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \text{ and } R = +\infty.$$

4. **The binomial series (Binomial family)**

   Considérons la fonction

   $$x \mapsto f(x) = y = (1+x)^\alpha, \ \alpha \in \mathbb{R}.$$

   Its domain of definition is $]-1, +\infty[$. We have a simple relation between the function and its derivative

   $$y = (1+x)^\alpha \Rightarrow y' = \alpha(1+x)^{\alpha-1},$$

   hence the differential equation,

   $$y'(1+x) = \alpha y. \tag{4.3.3}$$

All the solutions to this equation are of the form $y = c(1+x)^\alpha$, where c is an arbitrary constant. Now let's see if there is a solution f that can be developed into an integer series in the neighbourhood of 0.

Suppose that $f(x) = \sum_{n\geqslant 0} a_n x^n$ is a solution of (**??**). For such a function to exist it is necessary to have the relations,

$$(1+x)f'(x) - \alpha f(x) = 0 \Leftrightarrow (1+x)\sum_{n\geqslant 1} n a_n x^{n-1} - \alpha \sum_{n\geqslant 0} a_n x^n = \sum_{n\geqslant 0} [(n+1)a_{n+1} - (\alpha - n)a_n] x^n$$
$$= 0,$$

then we deduce that, for all $n \in \mathbb{N}$,

$$(n+1)a_{n+1} - (\alpha - n)a_n = 0,$$

and therefore, for all $n \in \mathbb{N}$,

$$a_{n+1} = \frac{\alpha - n}{n+1} a_n,$$

because an integer series is a zero series if and only if all its coefficients are zero,

$$\begin{cases} a_1 = \alpha a_0, \\ a_2 = \dfrac{\alpha - 1}{2} a_1, \\ \vdots \\ a_{n-1} = \dfrac{\alpha - n + 2}{n-1} a_{n-2}, \\ a_n = \dfrac{\alpha - n + 1}{n} a_{n-1}, \end{cases}$$

which finally gives

$$a_n = \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{n!} a_0.$$

Let the series $\sum_{n\geqslant 0} \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{n!} a_0 x^n$, the radius of convergence R is given by the relation

$$\frac{1}{R} = \lim_{n\to+\infty} \left| \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n)}{(n+1)!} \frac{n!}{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)} \right| = \lim_{n\to+\infty} \left| \frac{\alpha-n}{n+1} \right| = 1,$$

by construction, the series $f(x) = \sum_{n\geqslant 0} \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{n!} a_0 x^n$ is a solution of the differential equation (**??**), so it is of the form $f(x) = c(1+x)^\alpha$.

Since $c = a_0 = f(0)$, we can deduce that for all $x \in]-1,1[$:

$$(1+x)^\alpha = 1 + \sum_{n\geqslant 1} \frac{\alpha(\alpha-1)(\alpha-2)\dots(\alpha-n+1)}{n!} x^n, \; R = 1.$$

This series is known as the binomial series. For example

$$\sqrt{1+x} = 1 + \sum_{n\geqslant 1} (-1)^{n-1} \frac{1\cdot 1.3\cdot 5\dots(2n-3)}{2^n n!} x^n = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \cdots$$

$$\frac{1}{\sqrt{1+x}} = 1 + \sum_{n>1} (-1)^n \frac{1.3.5\dots(2n-1)}{2^n n!} x^n = 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \cdots$$

5. **The function:** $x \mapsto \dfrac{1}{1-x}$.

   We note on the one hand that for all $|x| < 1$, $\lim\limits_{n\to+\infty} |x|^n = 0$ and on the other hand

   $$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \frac{x^{n+1}}{1-x} \quad \text{for all } |x| < 1.$$

   hence,

   $$\frac{1}{1-x} = \sum_{n\geqslant 0} x^n \quad \text{with } R = 1 \quad \text{and} \quad \frac{1}{1+x} = \sum_{n\geqslant 0} (-1)^n x^n, \; R = 1.$$

6. **La fonction:** $x \mapsto \ln(1+x)$.

   Some integer series developments can be obtained using the theorems on the integration and derivation of integer series, so from the series development of the function $\dfrac{1}{1+x}$ we deduce by integration that

   $$\ln(1+x) = \sum_{n\geqslant 0} \frac{(-1)^n}{n+1} x^{n+1}, \quad R = 1.$$

   Similarly, we have

   $$\ln(1-x) = -\sum_{n\geqslant 0} \frac{1}{n+1} x^{n+1}, \quad R = 1.$$

   So,

   $$\forall x \in [-1,1]: \; \ln(1-x) = -\sum_{n\geqslant 1} \frac{x^n}{n},$$

   $$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}} = 1 + \sum_{n\geqslant 1} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n n!} x^{2n} = 1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \cdots$$

   Knowing that $\arcsin 0 = 0$, we obtain

   $$\arcsin x = x + \sum_{n\geqslant 1} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot\dots\cdot(2n)(2n+1)} x^{2n+1} = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \cdots$$

   We use the same process to develop new functions $x \mapsto \arccos x$, $x \mapsto \operatorname{arg\,sinh} x$, $x \mapsto \arctan x$, $x \mapsto \operatorname{arg\,th} x$.

Example 7. Determine the integer series expansions in the neighbourhood of 0 of the functions defined by,

$$f(x) = \frac{e^x}{1-x}, \quad g(x) = \frac{5}{x^4 - 13x^2 + 36}.$$

a) We have $f(x) = f_1(x) \times f_2(x)$ with $f_1 : x \mapsto e^x$, $f_2 : x \mapsto \dfrac{1}{1-x}$.

The function f_1 is developable as an integer series on $\mathbb{R}$, f_2 is developable as an integer series on $]-1,1[$. We deduce from this,

$$\forall x \in]-1,1[: f(x) = \left(\sum_{n=0}^{+\infty} x^n\right) \times \left(\sum_{n=0}^{+\infty} \frac{x^n}{n!}\right) = \sum_{n=0}^{+\infty} a_n x^n \quad \text{with} \quad a_n = \sum_{p=0}^{n} \frac{1}{p!}.$$

hence,

$$\begin{aligned} f(x) &= a_0 + a_1 x + a_2 x^2 + \cdots \\ &= 1 + 2x + \left(1 + \frac{1}{1!} + \frac{1}{2!}\right)x^2 + \left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!}\right)x^3 + \cdots + \sum_{p=0}^{n} \frac{1}{p!} x^n + \cdots \end{aligned}$$

b. We gave $x^4 - 13x^2 + 36 = (x^2-9)(x^2-4)$.

The function g is therefore not defined at the points $-3, -2, 2, 3$. It follows that the largest interval with centre 0 on which g can be developed as an integer series is the interval $]-2,2[$. Let x be a point in this interval. We have

$$\frac{5}{(x^2-9)(x^2-4)} = \left(\frac{1}{(x^2-9)} - \frac{1}{(x^2-4)}\right) = -\frac{1}{9 \cdot 1 - \frac{x^2}{9} + \frac{1}{4} - \frac{x^2}{4}}.$$

For all y such that $|y| < 1$, we have $\dfrac{1}{1-y} = \displaystyle\sum_{n=0}^{+\infty} y^n$. We have $\left|\dfrac{x^2}{9}\right| < 1$ and $\left|\dfrac{x^2}{4}\right| < 1$. As a result

$$\forall x \in]-2,2[, g(x) = \sum_{n=0}^{+\infty} x\left(2n\frac{1}{4n+1} - \frac{1}{9n+1}\right).$$

4.3.5. Solving differential equations in the form of an integer series

Example 8. In each of the following cases, find the functions that are solutions of the differential equations and that can be developed into an integer series

1. Consider the differential equation

$$x^2(1-x)y'' - x(1+x)y' + y = 0 \tag{4.3.4}$$

If $\sum_{n\geqslant 0} a_n x^n$ is the integer series development of a function f which is a solution to the differential equation, we have

$$\begin{cases} y'(x) = \sum_{n\geqslant 1} n a_n x^{n-1}, \\ y''(x) = \sum_{n\geqslant 2} n(n-1) a_n x^{n-2}. \end{cases}$$

Substituting y' and y'' into (**??**), we obtain,

$$(\mathbf{??}) \Leftrightarrow x^2(1-x)\sum_{n\geqslant 2} n(n-1)a_n x^{n-2} - x(1+x)\sum_{n\geqslant 1} n a_n x^{n-1} + \sum_{n\geqslant 0} a_n x^n = 0$$

$$\Leftrightarrow 2a_2x^2 + \sum_{n\geqslant 3}(n-1)\left[na_n - (n-2)a_{n-1}\right]x^n - a_1 x - \sum_{n\geqslant 2}\left[na_n + (n+1)a_{n-1}\right]x^n + \sum_{n\geqslant 0} a_n x^n$$

$$= 0$$

$$\Leftrightarrow a_0 + a_2x^2 - a_1x^2 + \sum_{n\geqslant 3}\left[(n(n-1) - n + 1)a_n - ((n-1)(n-2) + (n+1))a_{n-1}\right]x^n = 0$$

$$\Leftrightarrow \begin{cases} a_0 = 0 \\ a_2 - a_1 = 0 \\ (n-1)^2 a_n - (n-1)^2 a_{n-1} = 0, \forall n \geqslant 3. \end{cases}$$

$$\Leftrightarrow \begin{cases} a_0 = 0, \\ a_2 = a_1, \\ a_n = a_{n-1}, \forall n \geqslant 3. \end{cases}$$

On the other hand, by making x zero in the expression of f, we have $f(0) = a_0 = 0$. So the series $\sum_{n\geqslant 1} a_n x^n$ has radius of convergence $R = 1$. Consequently, any solution of (**??**) that can be developed as an integer series is proportional to the particular solution,

$$f_1(x) = \sum_{n\geqslant 1} x^n = \frac{x}{1-x}.$$

2. Consider the equation

$$x^2y'' + x(1+x)y' - y = 0 \tag{4.3.5}$$

If $\sum_{n\geqslant 0} a_n x^n$ is the integer series expansion of a function f which is the solution to the differential equation (**??**), we have

$$\begin{aligned}
(\textbf{??}) \Leftrightarrow\ & x^2\sum_{n\geqslant 2} n(n-1)a_n x^{n-2} + x(1+x)\sum_{n\geqslant 1} na_n x^{n-1} - \sum_{n\geqslant 0} a_n x^n = 0\\
\Leftrightarrow\ & \sum_{n\geqslant 2} n(n-1)a_n x^n + \sum_{n\geqslant 1} na_n x^n + \sum_{n\geqslant 1} na_n x^{n+1} - \sum_{n\geqslant 0} a_n x^n = 0\\
\Leftrightarrow\ & \sum_{n\geqslant 2}\left[n(n-1)a_n + na_n + (n-1)a_{n-1} - a_n\right]x^n + a_1 x - a_0 - a_1 x = 0\\
\Leftrightarrow\ & -a_0 + \sum_{n\geqslant 2}(n-1)\left[(n+1)a_n + a_{n-1}\right]x^n = 0\\
\Leftrightarrow\ & \begin{cases} a_0 = 0\\ (n-1)\left[(n+1)a_n + a_{n-1}\right], \forall n\geqslant 2.\end{cases}\\
\Leftrightarrow\ & \begin{cases} a_0 = 0\\ a_n = -\dfrac{1}{(n+1)}a_{n-1}, \forall n\geqslant 2\end{cases}
\end{aligned}$$

or else, $a_0 = 0$ and for all $n\in\mathbb{N}^*$

$$\begin{aligned}
a_{n+1} &= -\frac{1}{(n+2)}a_n\\
&= \frac{-1}{(n+2)}\times\frac{-1}{(n+1)}a_{n-1}\\
&= \frac{-1}{(n+2)}\times\frac{-1}{(n+1)}\times\frac{-1}{n}a_{n-2}\\
&= 2\frac{(-1)^n}{(n+2)!}a_1.
\end{aligned}$$

In addition

$$\lim_{n\to+\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0,$$

hence,

$$R = +\infty,$$

so the solutions of (**??**) that can be developed as an integer series are the functions defined on $\mathbb{R}$ by

$$f(x) = a_0 + a_1 x + 2a_1\sum_{n\geqslant 2}\frac{(-1)^n}{(n+2)!}x^n = 2a_1\sum_{n\geqslant 0}\frac{(-1)^n}{(n+2)!}x^{n+1},$$

and as a result we have

$$xf(x) = 2a_1 \sum_{n\geqslant 0} \frac{(-1)^{n+2}}{(n+2)!} x^{n+2} = 2a_1 \left(e^{-x} + x - 1\right),$$

then any function which can be developed into an integer series and is a solution of (**??**) is therefore proportional to the particular solution f_0 defined by,

$$f(0) = 0 \text{ and } \forall x \in \mathbb{R}^* : \ f(x) = \frac{e^{-x} + x - 1}{x}.$$

4.3.6. Definitions of functions of complex variable in integer series

The complex exponential

We have seen that the only function which is equal to its derivative (over an interval) is the exponential function, and this is why it is used to solve differential equations of order 2. We have seen that the radius of convergence of the real integer series $\sum_{n\geqslant 0} \frac{x^n}{n!}$ is $+\infty$ and that for all $x \in \mathbb{R}$, the radius of convergence of the real integer series $\sum_{n\geqslant 0} \frac{x^n}{n!}$ is $+\infty$

$$e^x = \sum_{n\geqslant 0} \frac{x^n}{n!}.$$

We generalize this expression to all $z \in \mathbb{C}$ and we set $e^z = \sum_{n\geqslant 0} \frac{z^n}{n!}$.

Propriété.

1. *For all* $z_1, z_2 \in \mathbb{C} : e^{z_1} e^{z_2} = e^{z_1+z_2}$.
2. *For all* $x \in \mathbb{R} : \ \cos x = \sum_{n\geqslant 0} \frac{(-1)^n}{(2n)!} x^{2n}, \ \sin x = \sum_{n\geqslant 0} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$.
3. *For all* $x \in \mathbb{R} : \ e^{ix} = \cos x + i \sin x$.
4. *For all* $z \in \mathbb{C}$*, For all* $n \in \mathbb{N}^* : \ (e^z)^n = e^{nz}$.

Proof.

1. For all $z_1, z_2 \in \mathbb{C}$ we have
$$e^{z_1}e^{z_2} = \left(\sum_{n\geqslant 0}\frac{z_1^n}{n!}\right)\times\left(\sum_{n\geqslant 0}\frac{z_2^n}{n!}\right)\sum_{n\geqslant 0}c_n = \sum_{n\geqslant 0}c_n \text{ where } c_n = \sum_{k=0}^{n}\frac{z_1^k z_2^{n-k}}{k!(n-k)!}.$$
($\sum\limits_{n\geqslant 0}c_n$ is the product series of $\sum\limits_{n\geqslant 0}\frac{z_1^n}{n!}\times\sum\limits_{n\geqslant 0}\frac{z_2^n}{n!}$ which are absolutely convergent). Therefore
$$e^{z_1}e^{z_2} = \sum_{n\geqslant 0}\left(\frac{1}{n!}\sum_{k=0}^{n}\frac{z_1^k z_2^{n-k}n!}{k!(n-k)!}\right) = \sum_{n\geqslant 0}\sum_{k=0}^{n}\frac{z_1^k z_2^{n-k}\,n!}{n!k!(n-k)!}$$
$$= \sum_{n\geqslant 0}\frac{1}{n!}(z_1+z_2)^n = e^{z_1+z_2}.$$

2. According to the Taylor-Lagrange formula for the neighbourhood of 0 of order $2n+1$:
$$\cos x = \sum_{k=0}^{n}\left[\cos^{(2k)}(0)\frac{x^{2k}}{(2k)!} + \cos^{(2k+1)}(0)\frac{x^{2k+1}}{(2k+1)!}\right] + \cos^{(2n+2)}(c_k)\frac{x^{2k+2}}{(2n+2)!}.$$
or for all $n \in \mathbb{N}$:
$$\cos^{(2k)}(0) = (-1)^n\cos x0 = (-1)^n \text{ et } \cos^{(2k+1)}(0) = (-1)^{n+1}\sin 0 = 0,$$
so,
$$\lim_{n\to+\infty}\left|\cos x - \sum_{k=0}^{n}(-1)^k\frac{x^{2k}}{(2k)!}\right| \leqslant \lim_{n\to+\infty}\frac{|x|^{2n+2}}{(2n+2)!} = 0.$$
The same procedure for $\sin x$.

3. $$e^{ix} = \sum_{n\geqslant 0}\frac{(ix)^n}{n!} = \sum_{p\geqslant 0}\frac{(ix)^{2p}}{(2p)!} + \sum_{p\geqslant 0}\frac{(ix)^{2p+1}}{(2p+1)!} = \sum_{p\geqslant 0}\frac{(-1)^p x^{2p}}{(2p)!} + i\sum_{p\geqslant 0}\frac{(-1)^p x^{2p+1}}{(2p+1)!}$$
$$= \cos x + i\sin x,$$
because $(i)^2 p = (-1)^n$, $(i)^{2p+1} = i(-1)^n$.

■

Remark 2. The function $z \mapsto e^z$ is periodic with period $2\pi i$ i.e.,
$$e^{z+2\pi i} = e^z e^{2\pi i} = e^z(\cos 2\pi i + i\sin 2\pi i) = e^z.$$

Hyperbolic functions with complex values

For all $z \in \mathbb{C}$:

$$\cosh z = \frac{e^z + e^{-z}}{2} = \sum_{n\geqslant 0} \frac{z^{2n}}{(2n)!}, \quad \sinh z = \sum_{n\geqslant 0} \frac{z^{2n+1}}{(2n+1)!}, \quad \operatorname{th} z = \frac{e^z - e^{-z}}{e^z + e^{-z}}.$$

All these integer series therefore have an infinite radius of convergence.

The functions $z \mapsto \cos z$, $z \mapsto \sin z$, $z \in \mathbb{C}$

$$e^{iz} = \cos z + i \sin z \mapsto \cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}, \quad \tan z = i\frac{1-e^{2iz}}{1+e^{2iz}},$$

$$\sin iz = i \sinh z, \quad \cos iz = \cosh z, \quad \tan iz = i \operatorname{th} z.$$

Definition of the function $z \mapsto \ln z$

Theorem 16. Let Z be a non-zero complex number, for any complex number$z$:

$$e^z = Z \Leftrightarrow \exists k \in \mathbb{Z}/z = \ln(|Z|) + i(\arg(Z) + 2\pi k).$$

In particular, the function $\exp : \mathbb{C} \to \mathbb{C}^*$, $z \mapsto e^z$ is surjective.

Proof. Let $Z \in \mathbb{C}, z \in \mathbb{C}$, put $z = x + iy$ where x and y are two real numbers

$$\begin{aligned} e^z = Z &\Leftrightarrow e^x e^{iy} = Z = |Z| e^{i \arg(Z)} \\ &\Leftrightarrow \begin{cases} e^x = |Z|, \\ \exists k \in \mathbb{Z} : \ y = \arg(Z) + 2\pi k, \end{cases} \\ &\Leftrightarrow \exists k \in \mathbb{Z} : z = \ln(|Z|) + i(\arg(Z) + 2\pi k). \end{aligned}$$

∎

Example 9. Find $\ln(-3)$ and solve the equation $\cos z = 2.$ in $\mathbb{C}$.

$$-3 = 3e^{i\pi} = 3(\cos \pi + i \sin \pi) \Rightarrow \ln(-3) = \ln 3 + i(\pi + 2\pi k)$$

$$\begin{aligned} &\cos z = 2 \\ &\Leftrightarrow \frac{e^{iz} + e^{-iz}}{2} = 2 \Leftrightarrow e^{2iz} + 1 - 4e^{iz} = 0 \\ &\Leftrightarrow e^{iz_1} = 2 - \sqrt{3} \vee e^{iz_2} = 2 + \sqrt{3} \\ &\Leftrightarrow iz_1 = \ln(2 - \sqrt{3}) + i2\pi k \vee iz_2 = \ln(2 + \sqrt{3}) + i2\pi k \\ &\Leftrightarrow z_1 = -i \ln(2 - \sqrt{3}) + 2\pi k \vee z_2 = -i \ln(2 + \sqrt{3}) + 2\pi k, k \in \mathbb{Z}. \end{aligned}$$