Chapter 5

Fourier series

5.1. Trigonometric series

Definition 1. A series of functions of variable x with general term $u_n : \mathbb{R} \to \mathbb{C}$ is called a trigonometric series if of the form

$$u_0(x) = f raca_0 2, u_n(x) = a_n \cos n\omega x + b_n \sin n\omega x, \quad \text{for all} \quad n \geqslant 1,$$

where $(a_n)_n, (b_n)_n$ are two sequences of complex numbers, $\omega > 0$.

Remark 1. Let us suppose that the series $\frac{a_0}{2} + \sum_{n \geqslant 1} [a_n \cos n\omega x + b_n \sin n\omega x]$ converges for a certain $x \in \mathbb{R}$ and assume that

$$f(x) = \frac{a_0}{2} + \sum_{n \geqslant 1} [a_n \cos n\omega x + b_n \sin n\omega x] \tag{5.1.1}$$

We have for all $n \in \mathbb{N}$ and $k \in \mathbb{Z}$:

$$\cos\left(n\omega\left(x + \frac{2\pi k}{\omega}\right)\right) = \cos(n\omega x + 2\pi k n) = \cos n\omega x,$$
$$\sin\left(n\omega\left(x + \frac{2\pi k}{\omega}\right)\right) = \sin(n\omega x + 2\pi k n) = \sin n\omega x.$$

Then the series converges at any point of the form $x + \frac{2\pi k}{\omega}$, $k \in \mathbb{Z}$.

- If the series (5.1.1) converges in $\mathbb{R}$, we will have $f(x) = f\left(x + \frac{2\pi k}{\omega}\right)$ and consequently the function f is periodic with period $T = \frac{2\pi}{\omega}$. In conclusion, the following properties are equivalent:

1. The trigonometric series (5.1.1) converges in $\mathbb{R}$.

2. The trigonometric series (5.1.1) converges in $\left[0, \frac{2\pi}{\omega}\right]$.

3. The trigonometric series (5.1.1) converges in $\left[\alpha, \alpha + \frac{2\pi}{\omega}\right]$, $\forall \alpha \in \mathbb{R}$.

The results obtained for series of functions obviously apply to trigonometric series, and in particular we have

Proposition 1. If the series $\sum_{n \geqslant 0} |a_n|$, $\sum_{n \geqslant 0} |b_n|$ are convergent, then the trigonometric series (5.1.1) is normally convergent on $\mathbb{R}$, so it is absolutely convergent on $\mathbb{R}$.

Proposition 2. If the numerical series $(a_n)_n$, $(b_n)_n$ are decreasing and tend to 0, then the trigonometric series (5.1.1) is convergent for $x \neq \frac{2\pi k}{\omega}$ where $k \in \mathbb{Z}$.

5.1.1. Complex representation of a trigonometric series

Proposition 3. Une série de fonctions de A series of functions of variable x is a trigonometric series if and only if its general term $u_n : \mathbb{R} \to \mathbb{C}$ is of the form,

$$u_0(x) = c_0, u_n(x) = c_n e^{in\omega x} + c_{-n} e^{-in\omega x}, \text{ pour tout } n \geqslant 1,$$

where $(c_n)_n, (c_{-n})_n$ are two sequences of complex numbers.

Lemma 5.1.1. Let $f : \mathbb{R} \to \mathbb{R}$, a periodic function of period $T > 0$ and integrable in the interval $[0, T]$. Then,

$$\forall \alpha \in \mathbb{R} : \int_{\alpha}^{\alpha+T} f(t)dt = \int_0^T f(t)dt.$$

5.1.2. Calculation of trigonometric series coefficients

Real case

Let us put in the conditions of uniform convergence of the triginometric series (5.1.1), the following

$$f(x) = \frac{a_0}{2} + \sum_{k \geqslant 1} [a_k \cos k\omega x + b_k \sin k\omega x].$$

Then,

$$f(x)\cos n\omega x = \frac{a_0}{2}\cos n\omega x + \sum_{k\geqslant 1}[a_k \cos k\omega x \cos n\omega x + b_k \sin k\omega x \cos n\omega x],$$

$$f(x)\sin n\omega x = \frac{a_0}{2}\sin n\omega x + \sum_{k\geqslant 1}[a_k \cos k\omega x \sin n\omega x + b_k \sin k\omega x \sin n\omega x],$$

so,

- $\int_0^{\frac{2\pi}{\omega}} f(x)\cos n\omega x dx = \frac{a_0}{2}\int_0^{\frac{2\pi}{\omega}} \cos n\omega x dx + \int_0^{\frac{2\pi}{\omega}} \sum_{k\geqslant 1}(a_k \cos k\omega x \cos n\omega x) dx + \int_0^{\frac{2\pi}{\omega}} \sum_{k\geqslant 1}(b_k \sin k\omega x \cos n\omega$
  The trigonometric series (5.1.1) is uniformly convergent, so we get
- $\int_0^{\frac{2\pi}{\omega}} f(x)\cos n\omega x dx = \frac{a_0}{2}\int_0^{\frac{2\pi}{\omega}} \cos n\omega x dx + \sum_{k\geqslant 1} a_k \int_0^{\frac{2\pi}{\omega}} \cos k\omega x \cos n\omega x dx + \sum_{k\geqslant 1} b_k \int_0^{\frac{2\pi}{\omega}} \sin k\omega x \cos n\omega x d$

$$\cos k\omega x \cos n\omega x dx = \frac{1}{2}[\cos(k+n)\omega x + \cos(k-n)\omega x]$$

$$\cos n\omega x \sin k\omega x = \frac{1}{2}[\sin(k+n)\omega x + \sin(k-n)\omega x]$$

$$\sin k\omega x \sin n\omega x dx = \frac{1}{2}[\cos(k+n)\omega x - \cos(k-n)\omega x]$$

and

$$\int_0^{\frac{2\pi}{\omega}} f(x)\sin n\omega x\, dx = \frac{a_0}{2}\int_0^{\frac{2\pi}{\omega}} \sin n\omega x\, dx + \sum_{k\geqslant 1} a_k \int_0^{\frac{2\pi}{\omega}} \cos k\omega x \sin n\omega x\, dx + \sum_{k\geqslant 1} b_k \int_0^{\frac{2\pi}{\omega}} \sin k\omega x \sin n\omega x\, dx,$$

or we have,

$$\int_0^{\frac{2\pi}{\omega}} \cos k\omega x \cos n\omega x\, dx = \int_0^{\frac{2\pi}{\omega}} \sin k\omega x \sin n\omega x\, dx = \begin{cases} 0 & \text{si } n \neq k, \\ \frac{\pi}{\omega} & \text{si } n = k, \end{cases}$$

$$\int_0^{\frac{2\pi}{\omega}} \cos k\omega x \sin n\omega x\, dx = 0.$$

Then we deduce the coefficients of the series by the following relations

$$\begin{cases} a_n = \frac{\omega}{\pi}\int_0^{\frac{2\pi}{\omega}} f(x)\cos n\omega x\, dx, \\ b_n = \frac{\omega}{\pi}\int_0^{\frac{2\pi}{\omega}} f(x)\sin n\omega x\, dx. \end{cases}$$

By lemma 5.1.1, the coefficients can be written

$$\begin{cases} a_n = \frac{\omega}{\pi}\int_0^{\frac{2\pi}{\omega}} f(x)\cos n\omega x\, dx = \frac{\omega}{\pi}\int_\alpha^{\alpha+\frac{2\pi}{\omega}} f(x)\cos n\omega x\, dx, \ \forall \alpha \in \mathbb{R}, \\ b_n = \frac{\omega}{\pi}\int_0^{\frac{2\pi}{\omega}} f(x)\sin n\omega x\, dx = \frac{\omega}{\alpha}\int_\alpha^{\alpha+\frac{2\pi}{\omega}} f(x)\sin n\omega x\, dx, \ \forall \alpha \in \mathbb{R}. \end{cases}$$

In particular, in the case of 2π-periodic functions (if $\omega = 1$:)

$$\begin{cases} a_n = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} f(x)\cos nx\,dx = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \\ b_n = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} f(x)\sin nx\,dx = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\sin nx\,dx, \end{cases}$$

these expressions are valid even for $n = 0$.

Complex case

In this case we have $f(x) = \displaystyle\sum_{k=-\infty}^{+\infty} C_k e^{ik\omega x}$.

$$\int_0^{\frac{2\pi}{\omega}} f(x)e^{-in\omega x}\,dx = \sum_{k=-\infty}^{+\infty} C_k \int_0^{\frac{2\pi}{\omega}} e^{i(k-n)\omega x}\,dx,$$

or

$$\int_0^{\frac{2\pi}{\omega}} e^{i\omega(k-n)x}dx = \begin{cases} 0 & \text{si } n \neq k, \\ \dfrac{2\pi}{\omega} & \text{si } n = k. \end{cases}$$

Then the coefficients are given by,

$$c_n = \frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} f(x)e^{-in\omega x}dx = \frac{\omega}{2\pi}\int_\alpha^{\alpha+\frac{2\pi}{\omega}} f(x)e^{-in\omega x}dx, \quad \text{for all } \alpha \in \mathbb{R} \text{ and } n \in \mathbb{Z}.$$

5.1.3. Development in trigonometric series

So far we have started with a trigonometric series and studied the function defined by the sum of this series. In this part we start with a function $f : \mathbb{R} \to \mathbb{C}$ and we have two questions

1. Is there a trigonometric series that converges everywhere on $\mathbb{R}$ and whose sum is equal to f ?

2. If the answer to the question is yes, is this series unique?

Definition 2 (Fourier series of a periodic function)**.**

Let $f : \mathbb{R} \to \mathbb{C}$ be a 2π-periodic and absolutely integrable function on $[0, 2\pi]$. We call the Fourier series of f the trigonometric series $\dfrac{a_0}{2} + \displaystyle\sum_{n\geqslant 1}[a_n \cos nx + b_n \sin nx]$ whose coefficients are given by the formulas

$$a_n(f) = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx, \quad b_n(f) = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx, \quad a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx,$$

or else $\sum_{n=-\infty}^{+\infty} c_n e^{inx}$ where,

$$c_n(f) = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}dx, \quad \text{for all } n \in \mathbb{Z}.$$

The $c_n(f)$ are called the Fourier coefficients of f. We will denote $S_\infty(f)$ the Fourier series of f.

Remark 2. Let $f : \mathbb{R} \to \mathbb{C}$ be a 2π-periodic and absolutely integrable function on $[0, 2\pi]$. The sequence $(c_n(f))_{n\in\mathbb{Z}}$ is bounded. Indeed, for all $n \in \mathbb{Z}$, we have

$$|c_n(f)| \leqslant \frac{1}{2\pi}\int_0^{2\pi}\left|\int_0^{2\pi} f(x)e^{-inx}dx\right| \leqslant \frac{1}{2\pi}\int_0^{2\pi} |f(x)|dx.$$

The same result is valid for the sequences (a_n) and (b_n).

Remark 3. Let $f : \mathbb{R} \to \mathbb{C}$ be a 2π-periodic and absolutely integrable function on any bounded interval $[a, b]$ of $\mathbb{R}$. If f is developable in Fourier series, then

1. If f is even,

$$a_n(f) = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)dx = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)dx,$$

because the function $x \mapsto f(x)\cos(nx)$ is even, for all $n \in \mathbb{N}$.

$$b_n(f) = 0,$$

because the function $x \mapsto f(x)\sin(nx)$ is odd, for all $n \in \mathbb{N}$.

2. If f is odd

$$a_n(f) = 0,$$

because the function $x \mapsto f(x)\cos(nx)$ is odd, for all $n \in \mathbb{N}$.

$$b_n(f) = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)dx = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)dx,$$

because the function $x \mapsto f(x)\sin(nx)$ is even, for all $n \in \mathbb{N}$.

Theorem 1 (Dirichlet). (Necessary condition)

Let $f : \mathbb{R} \to \mathbb{C}$ be a periodic 2π-function satisfying the following Dirichlet conditions

1. The discontinuities of f (if they exist) are of the first kind and are of finite number in any finite interval,
2. f has a right derivative and a left derivative at every point.

Then the Fourier series associated with f is convergent and we have,

$$\frac{a_0}{2} + \sum_{n\geqslant 1} [a_n \cos nx + b_n \sin nx] = \begin{cases} f(x) & \text{if } f \text{ is continuous at } x, \\ \dfrac{f(x+0)+f(x-0)}{2} & \text{if } f \text{ is discontinuous at } x. \end{cases}$$

In addition, convergence is uniform on any interval where the function f is continuous.

The notations $f(x+0)$, $f(x-0)$ represent respectively the right and left limits of f at the point x.

Theorem 2 (Jordan).

Let $f : \mathbb{R} \to \mathbb{C}$ be a 2π-periodic function satisfying the following conditions

1. There exists $M > 0$ such that $|f(x)| \leqslant M$.
2. We can divide the interval $[\alpha, \alpha + 2\pi]$ into subintervals $[\alpha_1, \alpha_2], [\alpha_2, \alpha_3], \ldots [\alpha_{n-1}, \alpha_n]$, with $\alpha = \alpha_1$ and $\alpha_n = \alpha + 2\pi$ such that the restriction $f|_{\alpha_j, \alpha_{j+1}}$ is monotone and continuous.

Then the Fourier series associated with f is convergent and we have,

$$\frac{a_0}{2} + \sum_{n\geqslant 1} [a_n \cos nx + b_n \sin nx] = \begin{cases} f(x) & \text{if } f \text{ is continuous at } x, \\ \dfrac{f(x+0)+f(x-0)}{2} & \text{if } f \text{ is discontinuous at } x. \end{cases}$$

Moreover, convergence is uniform on any interval where f is continuous.

Example 1.

1. Let $f : \mathbb{R} \to \mathbb{R}$ be the 2π-periodic function defined by

$$f(x) = |x|, \quad \text{for all } x \in [-\pi, \pi].$$

Xe have,

(a) $|f(x)| \leqslant \pi,\ \forall x \in [-\pi, \pi]$.

(b) $f|_{[-\pi,0]}$ is decreasing, continuous and $f|_{[0,\pi]}$ is increasing, continuous.

f satisfies Jordan's conditions, and can therefore be developed into a Fourier series.

Since f is even

- $b_n(f) = 0$,
- $a_0(f) = \frac{1}{\pi}\int_{-\pi}^{\pi} x \cos 0 dx = \frac{2}{\pi}\int_0^{\pi} dx = \pi$,
- $$\begin{aligned} a_n(f) &= \frac{2}{\pi}\int_0^{\pi} x \cos nx dx = \frac{2}{\pi}\left[\frac{x}{n}\sin nx\right]_0^{\pi} - \frac{2}{\pi} \times \frac{1}{n}\int_0^{\pi} \sin nx dx \\ &= 0 - \frac{2}{n\pi}\left[-\frac{\cos nx}{n}\right]_0^{\pi} = \frac{2}{n^2\pi}(1-(-1)^n) \\ &= \begin{cases} 0 & \text{if } n \text{ is even,} \\ -\frac{4}{n^2\pi} & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

The Fourier series of f is therefore

$$S_\infty(f) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n \geqslant 1} \frac{1}{(2n+1)^2}\cos(2n+1)x.$$

We have uniform convergence since f is continuous.

Finally, note that $f(0) = 0$ translates as

$$\frac{\pi}{2} - \frac{4}{\pi}\sum_{n \geqslant 1} \frac{1}{(2n+1)^2} = 0 \Leftrightarrow \sum_{n \geqslant 1} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.$$

On the other hand, since $\sum_{n \geqslant 1} \frac{1}{(2n)^2} = \frac{1}{4}\sum_{n \geqslant 1} \frac{1}{n^2}$, we have,

$$\sum_{n \geqslant 0} \frac{1}{(2n+1)^2} = \sum_{n \geqslant 1} \frac{1}{n^2} - \sum_{n \geqslant 1} \frac{1}{(2n)^2} = \frac{3}{4}\sum_{n \geqslant 1} \frac{1}{n^2},$$

it follows that

$$\sum_{n \geqslant 1} \frac{1}{n^2} = \frac{4}{3}\sum_{n \geqslant 0} \frac{1}{(2n+1)^2} = \frac{\pi^2}{6}.$$

2. Let $f : \mathbb{R} \to \mathbb{R}$ be the 2π-periodic function defined by $f(k\pi) = 0$, for all $k \in \mathbb{Z}$ and

$$f(x) = \operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x \in]0,\pi], \\ -1 & \text{if } x \in]-\pi, 0[. \end{cases}$$

Since f is odd, we will have

- $a_n(f) = 0$, and for $n \geqslant 1$,
- $b_n(f) = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} \sin nx\,dx = \begin{cases} 0 & \text{if } n \text{ is even}, \\ \dfrac{4}{n\pi} & \text{if } n \text{ is odd}. \end{cases}$

this function verifies the hypotheses of Dirichlet's theorem for all $x \in \mathbb{R}$). The Fourier series of f is therefore,

$$S_\infty(f) = \frac{4}{\pi}\sum_{n\geqslant 0}\frac{1}{(2n+1)}\sin((2n+1)x).$$

In particular, for $x = \dfrac{\pi}{2}$, we obtain,

$$S_\infty(f) = \frac{4}{\pi}\sum_{n\geqslant 0}\frac{(-1)^n}{(2n+1)} = 1 \Leftrightarrow \sum_{n\geqslant 0}\frac{(-1)^n}{(2n+1)} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}.$$

For $x = \pm\pi$, we have,

$$S_\infty(f) = 0 = \frac{1}{2}(f(x+0) - f(x-0)).$$