Chapter 6

Improper integrals

The Riemann integrals studied in the first year concern functions f defined and bounded on a closed bounded interval $[a, b]$. It is useful to generalise this notion to the case of functions f defined on any interval, except perhaps at a finite number of points and not necessarily bounded. In other words, to define the Riemann integral, we start with a function bounded on a compact interval. How to deal with the case of

1. of an unbounded function on a bounded interval ?
2. of any function on an unbounded interval ?

We will place ourselves in the following situation:

I is a non-compact interval and $f : I \longrightarrow \mathbb{R}$ is any function.

6.1. Generalities

Definition 1. We say that f is *locally integrable* (*loc. int.*) on I if f is Riemann integrable on any compact interval $[a, b]$ included in I.

Example 1. Any continuous or piecewise continuous function is locally Riemann-integrable. Throughout the following, the functions considered will be locally integrable, and it is assumed that I is of the form $[a, b[$ (resp. $]a, b]$) with a, b two real numbers such that $a < b$ (bounded interval), i.e., the function is not defined in b (resp. is not defined in a), i.e. where the curve C_f of the function f has a vertical asymptote of equation $x = b$ (resp. $x = a$), or of the form $[a,+\infty[$ (resp. $]-\infty,b]$) i.e., the domain of integration is unbounded.

Definition 2. Let $f:[a,b[\longrightarrow \mathbb{R}$, where $a<b\leqslant +\infty$, a locally integrable function.

i) The integral $\displaystyle\int_a^b f(t)\,dt$ is called **integral improper**, or **generalized integral** of f.

ii) We say that the integral $\displaystyle\int_a^b f(t)\,dt$ converges if $\displaystyle\lim_{x\to b^-}\int_a^x f(t)\,dt$ exists and is finite (i.e., this limit is a single real number). In this case, note:

$$\int_a^b f(t)\,dt=\lim_{x\to b^-}\int_a^x f(t)\,dt$$

and we call it the value of the generalized integral.

Otherwise, we say that the generalized integral $\displaystyle\int_a^b f(t)\,dt$ diverges.

Similarly if $f:]a,b]\longrightarrow \mathbb{R}$, where $-\infty\leqslant a<b$ a locally integrable function.

i) The integral $\displaystyle\int_a^b f(t)\,dt$ is called *generalized integral*, or *improper integral* of f.

ii) The integral $\displaystyle\int_a^b f(t)\,dt$ is said to converge if $\displaystyle\lim_{x\to a^+}\int_x^b f(t)\,dt$ exists and is finite (i.e., this limit is equal to a unique real number). In this case, we note:

$$\int_a^b f(t)\,dt=\lim_{x\to a^+}\int_x^b f(t)\,dt$$

and we call it the value of the generalized integral.

Otherwise, we say that the generalized integral $\displaystyle\int_a^b f(t)\,dt$ diverges.

Example 2.

1.

$$\int_0^{+\infty}\frac{1}{1+t^2}\,dt,\text{for all } x>0,\text{we have}$$

$$\int_0^x \frac{1}{1+t^2}\,dt=\arctan(x),$$

then,

$$\lim_{x\to+\infty}\int_0^x \frac{1}{1+t^2}\,dt=\lim_{x\to+\infty}\arctan(x)=\frac{\pi}{2},$$

thus,

$$\int_0^{+\infty} \frac{1}{1+t^2}\,dt \ \text{ converges.}$$

2. $\displaystyle\int_{\frac{\pi}{2}}^{+\infty} \sin(t)\,dt$ for all $x > \frac{\pi}{2}$ we have,

$$\int_{\frac{\pi}{2}}^{x} \sin(t)\,dt = -\cos(x),$$

then,

$$\lim_{x\to+\infty} \int_{\frac{\pi}{2}}^{x} \sin(t)\,dt = \lim_{x\to+\infty} (-\cos(x)) = -\lim_{x\to+\infty} \cos(x),$$

this limit can take an infinity of values, so does not exist. Consequently the integral $\displaystyle\int_{\frac{\pi}{2}}^{+\infty} \sin(t)\,dt$ diverges.

3. $\displaystyle\int_0^5 \frac{1}{t}\,dt$, for all $0 < x < 5$ we have

$$\int_x^5 \frac{1}{t}\,dt = \ln 5 - \ln x,$$

then,

$$\lim_{x\to 0^+} \int_x^5 \frac{1}{t}\,dt = \lim_{x\to 0^+} (\ln 5 - \ln x) = +\infty,$$

thus the integral $\displaystyle\int_0^5 \frac{1}{t}\,dt$ diverges.

Theorem 1 (**Cauchy criterion for generalized integrals**).

i) Let f be a loc.int function on $[a, +\infty[$, then the generalized integral $\displaystyle\int_a^{+\infty} f(t)\,dt$ converges if and only if

$$\forall \epsilon > 0, \exists A > 0 : \ \forall x, x' \geqslant A \implies \left| \int_x^{x'} f(t)\,dt \right| < \epsilon.$$

ii) Let f be a loc.int function on $[a, b[$ ($a, b \in \mathbb{R}$ and $a < b$), then the generalized integral

$\displaystyle\int_a^b f(t)\,dt$ converges if and only if

$$\forall \varepsilon > 0, \exists \delta \in]0, b-a[:\ \forall x, x' \in]b-\delta, b[\Longrightarrow \left| \int_x^{x'} f(t)\,dt \right| < \varepsilon.$$

For the type interval $]a, b]$, we have a similar criterion.

Definition 3. We say that two integrals are of **the same nature** if they are either both convergent or both divergent.

Proposition 1. Let $f : I \longrightarrow \mathbb{R}$, ($I = [a, b[$ where $a < b \leqslant +\infty$)a locally integrable function on I. Then for any real number c such that $a \leqslant c < b$ we have

$$\int_a^b f(t)\,dt \ \text{ converges} \Longleftrightarrow \int_c^b f(t)\,dt \ \text{ converges,}$$

namely, the two integrals $\displaystyle\int_a^b f(t)\,dt$ and $\displaystyle\int_c^b f(t)\,dt$ have the same nature.

Doubly improper integrals

In this part we are interested in the case of the intervals $]a, b[$, $]-\infty, b[$, $]a, +\infty[$ and $]-\infty, +\infty[$, that is to say the case of doubly improper integrals.

Proposition 2. Let f be a loc. int function on $]a, b[$ and $c \in]a, b[$. Then, the integral $\displaystyle\int_a^b f(t)\,dt$ is convergent if and only if the two integrals $\displaystyle\int_a^c f(t)\,dt$ and $\displaystyle\int_c^b f(t)\,dt$ are convergent. In this case, we denote

$$\int_a^b f(t)\,dt = \int_a^c f(t)\,dt + \int_c^b f(t)\,dt.$$

Otherwise, we say that the improper integral $\displaystyle\int_a^b f(t)\,dt$ is divergent.

Remark 1. The previous proposition shows that, to study a doubly improper integral, it is necessary to cut it into two improper integrals.

Example 3. We know that $\int_{-x}^{x} \sin(t)\,dt = 0 \underset{x\to+\infty}{\longrightarrow} 0$ and yet the improper integral $\int_{-\infty}^{+\infty} \sin(t)\,dt$ makes no sense (diverges) because the improper integral $\int_{-\infty}^{+\infty} \sin(t)\,dt$ doesn't exist.

Falsely improper integrals

We are situated here in the interval $[a, b[$ where $-\infty < a < b < +\infty$.

Proposition 3. Let f be a continuous function on $[a, b[$. If f has a finite limit on the left of b then the improper integral $\int_{a}^{b} f(t)\,dt$ is convergent.

In this case, we say that the integral is falsely improper at b.

which proves that $\int_{a}^{b} f(t), dt$ converges and is

Remark 2. No falsely improper integral in $\pm\infty$, this is valid for a finite bound.

Example 4. The integral $\int_{0}^{\frac{\pi}{2}} \frac{sin(t)}{t}, dt$ is falsely improper at 0, so it is convergent, because

$$\frac{\sin(t)}{t} \underset{t\to 0}{\longrightarrow} 1.$$

6.2. Reference integrals

In this part, we give some fundamental results on improper integrals of special type, which can be used to study the nature of other improper integrals.

Proposition 4.

i) The improper integral $\displaystyle\int_1^{+\infty} \frac{1}{t^\alpha} dt$ is, $\begin{cases} \text{convergent if } \alpha > 1 \\ \text{divergent if } \alpha \leqslant 1 \end{cases}$

ii) The improper integral $\displaystyle\int_0^{1} \frac{1}{t^\alpha} dt$ is, $\begin{cases} \text{convergent if } \alpha < 1 \\ \text{divergent if } \alpha \geqslant 1 \end{cases}$

Example 5. The integrals $\displaystyle\int_1^{+\infty} \frac{1}{t^2} dt$ and $\displaystyle\int_0^{1} \frac{1}{\sqrt{t}} dt$ are convergent, and the integrals $\displaystyle\int_1^{+\infty} \frac{1}{\sqrt{t}} dt$ and $\displaystyle\int_0^{1} \frac{1}{t^2} dt$ are divergent.

Corollary 1. If $-\infty < a < b < +\infty$, and if α a real, then

i) $\displaystyle\int_a^{b} \frac{1}{(t-a)^\alpha} dt$ converges if and only if $\alpha < 1$.

ii) $\displaystyle\int_a^{b} \frac{1}{(b-t)^\alpha} dt$ converges if and only if $\alpha < 1$.

Proposition 5. The improper integral $\displaystyle\int_0^{+\infty} e^{-\alpha t} dt$ is $\begin{cases} \text{convergent if } \quad \alpha > 0 \\ \text{divergent if } \quad \alpha \leqslant 0 \end{cases}$

Example 6. The integral $\displaystyle\int_0^{+\infty} e^{-t} dt$ is convergent and the integral $\displaystyle\int_0^{+\infty} e^{t} dt$ is divergent.

6.3. Properties

6.3.1. Linearity

Theorem 2. Let f, g be two real loc. int. functions on $[a, b[$ and $\lambda \in \mathbb{R}$. Then,

i) if $\int_a^b f(t)dt$ converges and $\int_a^b g(t)dt$ converges then, $\int_a^b (f+g)(t)dt$ converges, and we have,

$$\int_a^b (f+g)(t)dt = \int_a^b f(t)dt + \int_a^{\vec{b}} g(t)dt$$

ii) If $\int_a^b f(t)dt$ converges then $\int_a^b (\lambda f)(t)dt$ converges and we have $\int_a^b (\lambda f)(t)dt = \lambda \int_a^b f(t)dt$.

Remark 3. If $\int_a^b f(t)\,dt$ and $\int_a^b g(t)\,dt$ are divergent then we cannot conclude anything about the nature of the integral $\int_a^b (f+g)(t)\,dt$, for example if we take the integral

$$\int_1^{+\infty} \frac{1}{t^2}\,dt.$$

On the one hand, we have,

$$1 = \int_1^{+\infty} \frac{1}{t^2}\,dt = \int_1^{+\infty} \left(\frac{1+t}{t^2} - \frac{1}{t}\right)dt$$

and on the other hand, the two improper integrals $\int_1^{+\infty} \frac{1+t}{t^2}\,dt$ and $\int_1^{+\infty} \frac{1}{t}\,dt$ are divergent, in other words

$$1 = \int_1^{+\infty} \frac{1}{t^2}\,dt = \int_1^{+\infty} \left(\frac{1+t}{t^2} - \frac{1}{t}\right)dt \neq \int_1^{+\infty} \frac{1+t}{t^2}\,dt - \int_1^{+\infty} \frac{1}{t}\,dt.$$

Integration by part

Theorem 3. Let f, g be two functions of class C^1 on $[a, b[$ and with values in $\mathbb{R}$. If fg has a finite limit in b^-, then the improper integrals $\int_a^b (fg)(t)\,dt$ and $\int_a^b \left(fg'\right)(t)\,dt$ are of the same nature. If they are also convergent, then

$$\int_a^b \left(fg'\right)(t)\,dt = fg|_a^b - \int_a^b \left(f'g\right)(t)\,dt.$$

Change of variable

Theorem 4. Let a, b be two real numbers such that $-\infty \leqslant a < b \leqslant +\infty$, φ a class bijection C^1 of $]a, b[$ on its image and f a continuous function on $\varphi(]a, b[)$. Then, the integrals $\int_{\varphi(a)}^{\varphi(b)} f(t)\,dt$ and $\int_a^b f(\varphi(u))\varphi'(u)\,du$ are of the same nature and equal if they converge

6.4. Integral of positive functions

6.4.1. Convergence criteria

In this section we give the main comparison criteria adopted for studying the nature of a generalised integral.

In the following we will only use positive functions. For integrals of negative functions, it is sufficient to consider the function $-f$, which brings us to the case of integrals of positive functions

Proposition 6. If f is loc. int. and positive on $[a, b[$, then the generalized integral $\int_a^b f(t)\,dt$ converges if and only if there exists a real number $M > 0$ such that $0 \leqslant \int_a^x f(t)\,dt \leqslant M$ for all $x \in [a, b[$.

And so the integral $\int_a^b f(t)\,dt$ diverges if and only if $\int_a^b f(t)\,dt = +\infty$.

Proposition 7 (Comparison criteria)**.**

If $f, g : [a, b[\to \mathbb{R}$ are two positive loc. int. functions such that $f \leqslant g$, then,

i) $\displaystyle\int_a^b g(t)\,dt$ converges $\Rightarrow \displaystyle\int_a^b f(t)\,dt$ converges.

ii) $\displaystyle\int_a^b f(t)\,dt$ diverges $\Rightarrow \displaystyle\int_a^b g(t)\,dt$ diverges.

Corollary 2. Let $f, g : [a, b[\to \mathbb{R}$ be two positive loc. int. functions such that $f \overset{b}{\sim} g$ (the functions f and g are equivalent in the neighbourhood of b), then the two integrals $\displaystyle\int_a^b f(t)\,dt$ and $\displaystyle\int_a^b g(t)\,dt$ are of the same kind.

Generally we have the following proposition.

Proposition 8. Let $f, g : [a, b[\to \mathbb{R}$ be two positive loc. int. functions such that

$$\lim_{\substack{t \to b \\ <}} \frac{f(t)}{g(t)} = \ell \ (\ell \in \mathbb{R}).$$

i) If $\ell \neq 0$, then the two integrals $\displaystyle\int_a^b f(t)\,dt$ and $\displaystyle\int_a^b g(t)\,dt$ are of the same nature.

ii) If $\ell = 0$ ($f = o(g)$ in the neighborhood of b), then,

($\diamond$) $\displaystyle\int_a^b g(t)\,dt$ converges $\Rightarrow \displaystyle\int_a^b f(t)\,dt$ converges.

($\star$) $\displaystyle\int_a^b f(t)\,dt$ diverges $\Rightarrow \displaystyle\int_a^b g(t)\,dt$ diverges.

Corollary 3.

i) Let f be an int. loc. function on $[a, +\infty[$, and positive. If there exists $\alpha > 1$ such that $\displaystyle\lim_{t \to +\infty} t^{\alpha} f(t) = 0$, then the improper integral $\displaystyle\int_a^{+\infty} f(t)\,dt$ is convergent.

ii) Let f be an int. loc. function on $[a, b[$ ($a < b$), and positive. If there exists $\alpha < 1$ such that $\displaystyle\lim_{t \to b^-} (b-t)^{\alpha} f(t) = 0$, then the improper integral $\displaystyle\int_a^b f(t)\,dt$ is convergent.

6.4.2. Integral of functions with any sign

Absolute convergence

Definition 4. Let $f : [a,b[\longrightarrow \mathbb{R}$ be an int. loc. function. The improper integral $\int_a^b f(t)\,dt$ is said to be ***absolutely convergent*** if the improper integral $\int_a^b |f(t)|\,dt$ converges.

Proposition 9. Let $f : [a,b[\longrightarrow \mathbb{R}$ be an int. loc. function. Then

$$\int_a^b f(t)\,dt \text{ is absolutely convergent } \Rightarrow \int_a^b f(t)\,dt \text{ is convergent}$$

Example 7. The integral $\int_0^{+\infty} e^{-t}\sin\left(\ln\left(t^2+1\right)\right)\,dt$ is absolutely convergent.

Indeed,

$$\left|e^{-t}\sin\left(\ln\left(t^2+1\right)\right)\right| \leqslant e^{-t},$$

or the integral $\int_0^{+\infty} e^{-t}\,dt$ is convergent. Thus the integral $\int_0^{+\infty} e^{-t}\sin\left(\ln\left(t^2+1\right)\right)\,dt$ is absolutely convergent.

6.4.3. Abel's criterion for integrals of the form $\int f\,g$

Theorem 5 (Abel's Theorem)**.**

Let $f,g : [a,b[\longrightarrow \mathbb{R}$ be two loc. int functions.

i) If f is monotone and $\lim_{t\to b^-} f(t) = 0$, and

ii) if there exists $M > 0$ such that for all$x \in [a,b[$: $\left|\int_a^x f(t)g(t)\,dt\right| \leqslant M$,

Then the improper integral $\int_a^b f(t)g(t)\,dt$ converges.

Example 8. The generalized integral $\int_1^{+\infty} \frac{\sin t}{t}\,dt$ is convergent. Indeed . $f(x) = \frac{1}{t}$ is a decreasing function on $[1, +\infty[$ and $\lim_{t \to +\infty} \frac{1}{t} = 0$. On the other hand, for all $x \in [1, +\infty[$,

$$\left| \int_1^x \sin(t)\,dt \right| = |\cos 1 - \cos x| \leqslant 2.$$

Thus according to Abel's Theorem, the improper integral $\int_1^{+\infty} \frac{\sin t}{t}\,dt$ converges.

Semi-convergent integral

Definition 5. The improper integral $\int_a^b f(t)\,dt$ is said to be *semi-convergent* if it is convergent but not absolutely convergent.

Example 9.
The integral $\int_1^{+\infty} \frac{\sin t}{t}\,dt$ converges but it does not converge absolutely. So it is semi-convergent. Indeed, using Abel's theorem (theorem 5), it is clear that the integral $\int_1^{+\infty} \frac{\sin t}{t}\,dt$ is convergent.

Study of absolute convergence.

We have for all $t \geqslant 1$:

$$\frac{\sin^2 t}{t} \leqslant \frac{|\sin t|}{t} \quad \text{and} \quad \frac{\sin^2 t}{t} = \frac{1}{2t} - \frac{\cos(2t)}{2t}.$$

Or the integral $\int_1^{+\infty} \frac{1}{t}\,dt$ diverges, so the integral $\int_1^{+\infty} \frac{|\sin t|}{t}\,dt$ diverges.

6.5. Generalized integrals dependent on a parameter

Let I be an open interval of $\mathbb{R}$. Consider a semi-open interval $[a,b[$ and a function of two variables $f(t,x)$ where $t\in[a,b[$ and $x\in I$, with values in $\mathbb{R}$. It is assumed either that $b=+\infty$ or that $b<+\infty$ and that for some $x\in I$, the function $t\longmapsto f(t,x)$ is not defined in b. It is assumed that for all $x\in I$, the function $t\longmapsto f(t,x)$ is integrable on the semi-open interval $[a,b[$ in the sense of generalized integrals and we are interested in the properties of the function defined on I by the generalized integral

$$F(x)=\int_a^b f(t,x)\,dt.$$

We study the continuity and derivability of F on I.

6.5.1. Dominant convergence theorem

Theorem 6. Let $(f_n(t))_{n\in\mathbb{N}}$ be a sequence of integrable functions on a semi-open interval $[a,b[$ such that,

i) The sequence $(f_n(t))_{n\in\mathbb{N}}$ simply converges on $[a,b[$ to a locally integrable function f.

ii) There exists a function φ integrable (in the sense of generalized integrals) on the semi-open interval $[a,b[$ such that,

$$\forall n\in\mathbb{N},\forall t\in[a,b[,|f_n(t)|\leqslant\varphi(t).$$

Then f is integrable (in the sense of generalized integrals) on the semi-open interval $[a,b[$ and,

$$\int_a^b f(t)\,dt=\lim_{n\to+\infty}\int_a^b f_n(t)\,dt.$$

6.5.2. Continuité

Theorem 7. Let $f : [a, b[\times I \longrightarrow \mathbb{R}$, be a continuous function with respect to each of the two variables on $[a, b[\times I$. We assume that there exists a function $\varphi : [a, b[\longrightarrow \mathbb{R}_+$, integrable (in the sense of generalized integrals) on the semi-open interval $[a, b[$ such that

$$\forall (t, x) \in [a, b[\times I,\ |f(t, x)| \leqslant \varphi(t).$$

Then the function $t \longmapsto f(t, x)$ is integrable (in the sense of generalized integrals) on the semi-open interval $[a, b[$ and the function F, defined for $x \in I$ by

$$F(x) = \int_a^b f(t, x)\,dt,$$

is continuous on I. In particular, for all $x_0 \in I$, we have

$$F(x_0) = \int_a^b \lim_{x \to x_0} f(t, x)\,dt = \lim_{x \to x_0} \int_a^b f(t, x)\,dt,$$

which is a case of inversion of limit and generalized integral.

Example 10. Let $f(t, x) = \dfrac{e^{-xt}}{1+t^2}$ be defined for $(t, x) \in [0, +\infty[\times]0, +\infty[$. The function f is continuous with respect to each of the two variables on $[0, +\infty[\times]0, +\infty[$ and

$$\forall (t, x) \in [0, +\infty[\times]0, +\infty[,\ |f(t, x)| \leqslant \frac{1}{1+t^2},$$

which is an integrable function (in the sense of generalized integrals) on $[0, +\infty[$. The function $\displaystyle\int_0^{+\infty} \frac{e^{-xt}}{1+t^2}\,dt$ is therefore continuous on $]0, +\infty[$.

For the continuity of generalized integrals depending on a parameter, in general, one cannot reason on the entire interval I. One seeks dominations on subintervals of I and uses a saturation argument to obtain the result on the entire I. See the example 11, of the Gamma function below.

6.5.3. Derivation

Theorem 8. Let $f : [a, b[\times I \longrightarrow \mathbb{R}$, be a continuous function with respect to each of the two variables on $[a, b[\times I$. It is assumed that f has a partial derivative with respect to x, $\dfrac{\partial f}{\partial x}$, continuous with respect to each of the two variables on $[a, b[\times I$. It is assumed that there exist two functions φ and $\psi : [a, b[\longrightarrow \mathbb{R}_+$, which are integrable (in the sense of generalized integrals) on the semi-open interval $[a, b[$ such that

$$\forall (t, x) \in [a, b[\times I,\ |f(t, x)| \leqslant \varphi(t) \quad \text{and} \quad \left|\frac{\partial f}{\partial x}(t, x)\right| \leqslant \psi(t).$$

Then for all $x \in I$, the functions $t \longmapsto f(t, x)$ and $t \longmapsto \dfrac{\partial f}{\partial x}(t, x)$ are integrable (in the sense of the generalised integrals) on the semi-open interval $[a, b[$ and the function F, defined for $x \in I$ by

$$F(x) = \int_a^b f(t, x)\, dt,$$

is derivable on I and

$$F'(x) = \left(\int_a^b f(t, x)\, dt\right)' = \int_a^b \frac{\partial f}{\partial x}(t, x)\, dt,$$

which is a case of derivative inversion and a generalized integral.

Remark 4. For derivation under the sum sign, we have the same remark as for continuity, in general, we cannot reason on the whole interval I. We look for dominations on sub-intervals of I and use a saturation argument to obtain the result on I as a whole. See the following example.

Example 11 (The Gamma function.).

For $x \in \mathbb{R}$, we pose

$$\Gamma(x) = \int_0^{+\infty} e^{-t} t^{x-1}\, dt.$$

Since there are two integration problems, in 0 and in $+\infty$, we separate this generalised integral into two generalized integrals

$$\Gamma_1(x) = \int_0^1 e^{-t} t^{x-1}\, dt \quad \text{et} \quad \Gamma_2(x) = \int_1^{+\infty} e^{-t} t^{x-1}\, dt.$$

The function to be integrated $f(t,x) = e^{-t}t^{x-1}$ is positive and locally integrable. For a fixed $x > 0$, we can therefore apply the Comparison Criteria for generalized integrals.

Study of Γ_1 : Let $x \in \mathbb{R}$ be fixed. When $t \longrightarrow 0$, $f(t,x)$ is equivalent to t^{x-1} and as $\int_0^1 t^{x-1}\,dt$ converges if and only if $x > 0$, then $\Gamma_1(x)$ is well defined for $x \in]0,+\infty[$.

Study of Γ_2 : For any fixed x, when $t \longrightarrow +\infty$, we have

$$\lim_{t\to+\infty} e^{t/2} f(t,x) = \lim_{t\to+\infty} e^{-t/2}t^{x-1} = 0.$$

So $f(t,x)$ is dominated in the neighbourhood of $+\infty$ by $e^{-t/2}$ which is integrable in $+\infty$. So $\Gamma_2(x)$ is also well defined for $x \in \mathbb{R}$.

Thus the function Γ is well defined for $x > 0$.

We can easily establish that $\Gamma_1(1) = 1$ and that $\forall x > 0$, $\Gamma(x+1) = x\Gamma(x)$. In particular, we can deduce that $\forall n \in \mathrm{N}^*$, $\Gamma(n) = (n-1)! The \Gamma$ function therefore appears to be an extension to $\mathbb{R}_+$ of the "factorial" function.

Continuity: The function $f(t,x) = e^{-t}t^{x-1}$ is continuous on $]0,+\infty[\times]0,+\infty[$.

Let $a > 0$, we have

$$\forall x \in [a,+\infty[, \forall t \in]0,1], e^{-t}t^{x-1} \leqslant e^{-t}t^{a-1}.$$

The function $\psi_1(t) = e^{-t}t^{a-1}$ is integrable in the neighbourhood of 0. According to theorem 7, the function $\Gamma_1(x)$ is continuous on $[a,+\infty[$. Since this is true for all $a > 0$, we deduce that $\Gamma_1(x)$ is continuous on $]0,+\infty[$.

Let $b > 0$. We have in the same way

$$\forall x \in]0,b], \forall t \in [1,+\infty[, e^{-t}t^{x-1} \leqslant e^{-t}t^{b-1}.$$

The function $\psi_2(t) = e^{-t}t^{b-1}$ is integrable in $+\infty$. According to theorem 7, the function $\Gamma_2(x)$ is continuous on $]0,b]$. This being true for all $b > 0$, we deduce that $\Gamma_2(x)$ is continuous on $]0,+\infty[$.

The function Γ is therefore continuous on $]0,+\infty[$.

Derivation: The function $f(t,x)=e^{-t}t^{x-1}$ has a partial derivative with respect to x which is continuous on $]0,+\infty[\times]0,+\infty[$. In fact, for any $(t,x)\in]0,+\infty[\times]0,+\infty[$,

$$\frac{\partial f}{\partial x}(t,x)=e^{-t}t^{x-1}\ln(t).$$

Let $\alpha,\beta>0$. As for continuity, we have

$$\forall x\in[\alpha,+\infty[,\forall t\in]0,1],\left|e^{-t}t^{x-1}\ln(t)\right|\leqslant e^{-t}t^{\alpha-1}\ln(t),$$

the function $e^{-t}t^{\alpha-1}\ln(t)$ is integrable in 0, because,

- For $\alpha>1$, $\lim_{t\to0}\frac{e^{-t}t^{\alpha-1}\ln(t)}{\ln(t)}=0$ and $\int_0^1\ln(t)\,dt=-1$, so $\int_0^1 e^{-t}t^{\alpha-1}\ln(t)\,dt$ converges,
- For $\alpha=1$, $\lim_{t\to0}\frac{e^{-t}\ln(t)}{\ln(t)}=1$ and $\int_0^1\ln(t)\,dt=-1$, so $\int_0^1 e^{-t}\ln(t)\,dt$ converges,
- And for $0<\alpha<1$, let $\gamma>0$ such that $1-\alpha<\gamma<1$, then, we have $\lim_{t\to0}\frac{e^{-t}t^{\alpha-1}\ln(t)}{\frac{1}{t^\gamma}}=0$ et $\int_0^1\frac{1}{t^\gamma}\,dt$ converges, so $\int_0^1 e^{-t}t^{\alpha-1}\ln(t)\,dt$ converges.

Similarly, we have,

$$\forall x\in[0,\beta],\forall t\in[1,+\infty[,\left|e^{-t}t^{\alpha-1}\ln(t)\right|\leqslant e^{-t}t^{\beta-1}$$

and

$$\lim_{t\to+\infty}\frac{e^{-t}t^{\beta-1}\ln(t)}{1/t^2}=0,$$

then the function $e^{-t}t^{beta-1}\ln(t)$ is integrable in $+\infty$.

We deduce from theorem 8 that the functions Γ_1 and Γ_1 are differentiable on $[\alpha,+\infty[$ and $[0,\beta]$ respectively. As this is true for all $\alpha,\beta>0$, these two functions are differentiable on $]0,+\infty[$. The same is true for Γ and we have

$$\forall x\in[0,+\infty[\Gamma'(x)=\int_1^{+\infty}e^{-t}t^{\alpha-1}\ln(t)\,dt.$$

Remark 5. The assumption of domination of the function $f(t,x)$ by an integrable function φ or of the partial derivative $\frac{\partial f}{\partial x}$ by an integrable function ψ on the semi-open interval $[a,b[$ in the two theorems (theorem 7 and theorem 8) is very strong. We can consider a less strong assumption, uniform convergence.

Definition 6. We say that the generalized integral depends on a parameter

$$F(x) = \int_a^b f(t,x)\,dt,$$

is **uniformly convergent** on I if:

$$\forall \epsilon > 0, \exists c \in [a,b] tel que \forall x \in I, \left| \int_b^c f(t,x)\,dt \right| \leqslant \epsilon.$$

It can be shown that the conclusion of the Theorems (Theorem 7 and Theorem 8) remains valid if we replace the assumption of domination (on f or on $\frac{\partial f}{\partial x}$) by this assumption of uniform convergence.