Chapter I

Recall in Mathematics and classical mechanics and electricity

I-1- Taylor series

Consider a function $f(x)$ derivable up to $(n+1)$ order near x_0 then we can develope this function by Taylor law as below:

$$f(x) = f(x_0) + \frac{x-x_0}{1!} f'(x_0) + \frac{(x-x_0)^2}{2!} f''(x_0) + \dots + \frac{(x-x_0)^n}{n!} f^{(n)}(x_0) + R_n(x)$$

$R_n(x)$ is called the remainder of Taylor series and is given by:

$$R_n(x) = \frac{(x-x_0)^{n+1}}{(n+1)!} f^{(n+1)}\left(x_0 + \theta(x-x_0)\right) \quad 0<\theta<1$$

if $f(x)$ is infinitely derivable then:

$$f(x) = f(x_0) + \sum_{n=1}^{\infty} \frac{(x-x_0)^n}{n!} f^{(n)}(x_0)$$

Noting: if $x_0 = 0$ this series is called Mac Laurin series and we write:

$$f(x) = f(0) + \sum_{n=1}^{\infty} \frac{x^n}{n!} f^{(n)}(0)$$

Example:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + (-1)^n \frac{x^{2n}}{(2n)!} + \dots$$

$$e^x = 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots$$

$\varepsilon \ll 1 \Rightarrow (1+\varepsilon)^n \simeq 1 + n\varepsilon \quad \left[(1+\varepsilon)^{-n} \simeq 1 - n\varepsilon\right]$

$x \ll$ $\quad \sin x \simeq x, \quad \cos x \simeq 1 - \frac{x^2}{2}$

I - 2 - Periodic function

$f(x)$ is said to be periodic function if it satisfies:

$$f(t) = f(t+T) = f(t+2T) = \dots = f(t+nT)$$

where n is an integer number

T = is called the period of the function and it's the lowest value which verifies the above equality.

- some times t represents time, in which case we retain the same notation; if t expresses a distance then we use x in stead of t.

variable	period	frequency	angular frequency
time (t) unit	T (s)	$\nu = 1/T$ (f) Hz s^{-1}	$\omega = 2\pi\nu = 2\pi f$ $rad.s^{-1}$
distance (x) unit	λ (m)	$\sigma = 1/\lambda$ (m^{-1})	$k = 2\pi\sigma$ $rad.m^{-1}$

I - 3 - Development of a periodic function as fourier series

Consider a periodic function $f(t)$ [T period of $f(t)$] defined on the interval $]0, T[$ and satifies the following conditions:

1 - $f(t)$ is a bounded fct on $[0, T]$
2 - $f(t)$ accepts a limited number of extremums
3 - $f(t)$ " " " " " discontinuity points

the above function can be developed as a series of trigonometric functions called fourier series and denoted $p(t)$ as below:

$$f(t) = \rho(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega t + b_n \sin n\omega t$$

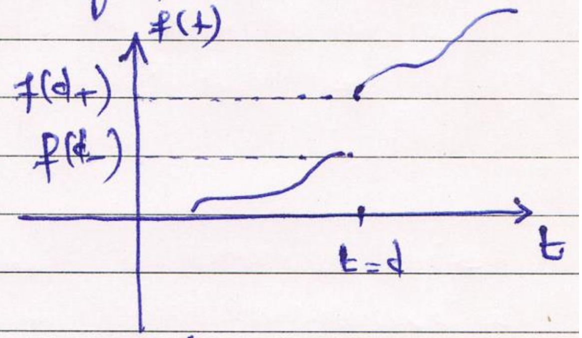
where: $\omega = \dfrac{2\pi}{T}$

n = a positive integer number

$n \in \mathbb{N}^{*+}$ $(n > 0)$

when $t = d$ is a discontinuity point then

$$\rho(d) = \frac{f(d_+) + f(d_-)}{2}$$



The parameters a_0, a_n, b_n are given by:

$$a_0 = \frac{2}{T}\int_0^T f(t)\,dt$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos(n\omega t)\,dt$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin(n\omega t)\,dt$$

Noting

- $\rho(t)$ can be written as

$$\rho(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} d_n \cos(n\omega t + \varphi_n)$$

where: $d_n = \dfrac{a_n}{\cos\varphi_n}$ and $tg\,\varphi_n = -\dfrac{b_n}{a_n}$

- the first terms in the series which period is T (the same period of $f(t)$) is called principal harmonic

- The term which order is n is called harmonic of order n

- if $f(t)$ is an even function (pair) then:

  $f(-t) = f(t)$ and consequently we obtain:

  $\forall n : \quad b_n = 0$

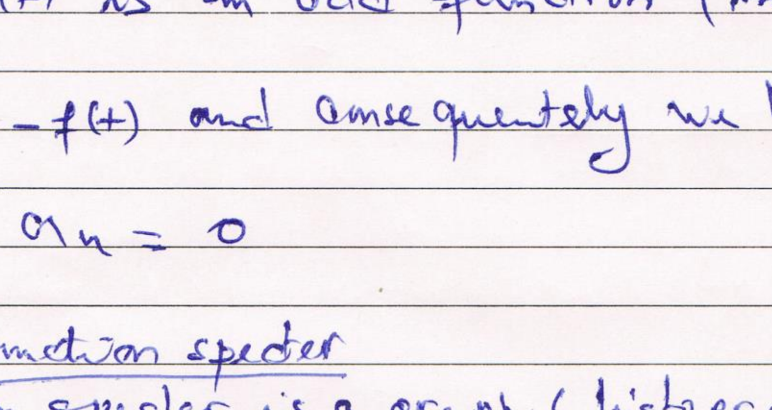
- if $f(t)$ is an odd function (impair) then

  $f(-t) = -f(t)$ and consequently we have:

  $\forall n, \quad a_n = 0$

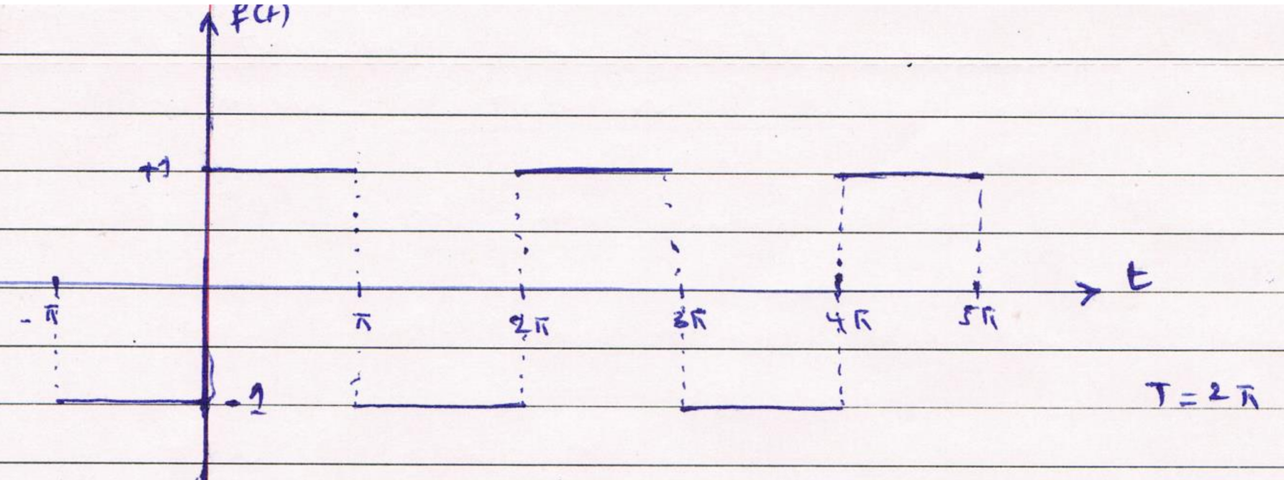
I-3-1. Function specter

the function specter is a graph (histogramme) which represents the amplitude of each term in the Fourier series of the function of the corresponding angular frequency.



Example:

$$f(t) = \begin{cases} -1 & \text{if} \quad -\pi < t < 0 \\ +1 & \text{if} \quad 0 < t < \pi \end{cases} \qquad : T = 2\pi \Rightarrow w = 1$$



f(t) is then square wave function الدالة مربع (fct créneau)

discontinuity points of f(t): $d = \mp n\pi$, $d = 0$

$\rho(d) = \frac{f(d_+) + f(d_-)}{2} = 0$: $\rho(0) = \rho(\mp n\pi) = 0$

impair

f(t) is an odd function therefore:

$\forall n \neq 0 \quad a_n = 0$

calculate of a_0 and b_n:

$$a_0 = \frac{2}{2\pi}\int_0^{2\pi} f(t)\,dt = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(t)\,dt = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-1)\,dt + \int_0^{+\pi} 1\,dt\right] = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(t)\sin n\omega t\,dt = \frac{1}{\pi}\left[-\int_{-\pi}^{0}\sin nt\,dt + \int_0^{\pi}\sin nt\,dt\right]$$

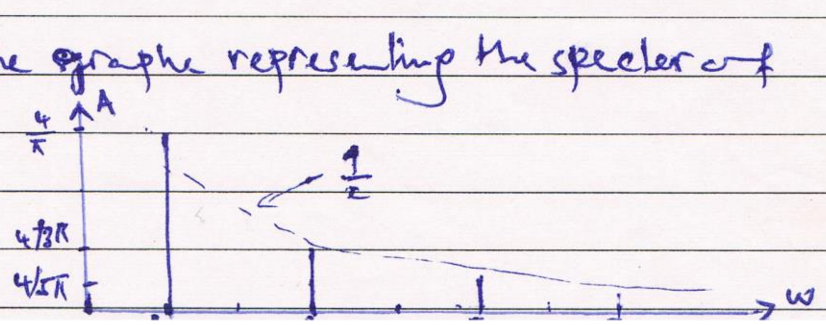
$$= \frac{2}{\pi}\int_0^{\pi}\sin nt\,dt = \boxed{\frac{2}{n\pi}(1 - \cos n\pi)}$$

if n is an even number: $b_{2p} = 0$ $\qquad n = 2p$

" " " " odd " " $b_{2p+1} = \frac{4}{(2p+1)\pi}$ $\qquad n = 2p+1$

$$\Rightarrow f(t) = \frac{4}{\pi}\sum_{p=0}^{\infty}\frac{1}{(2p+1)}\sin(2p+1)t = \frac{4}{\pi}\left(\sin t + \frac{\sin 3t}{3} + \frac{\sin 5t}{5} + \dots\right)$$

Then we faind finally the graphe representing the specter of the function as below:



I-4- sinusoidal function

a sinusoidal function is of the form:

$$x(t) = A\cos(wt+\varphi)$$

where:

- $x(t)$: instantaneous elongation
- A : amplitude (modulus)
- φ : initiale phase
- $wt+\varphi$: instantaneous phase
- w : angular frequency ($rad.s^{-1}$)
- $T = \frac{2\pi}{w}$ = period and $f = \frac{1}{T}$ frequency

$x(t)$ can be written as the sum of two sinusoidals functions in quadrature i.e:

$$x(t) = A\cos(wt+\varphi) = B\cos wt + C\sin wt$$
$$= B\cos wt + C\cos(wt - \tfrac{\pi}{2})$$

as we can see above the difference in phase between the two sinusoidale functions is $\frac{\pi}{2}$

I-4-1- Representation of sinusoidale function in the space of complex numbers

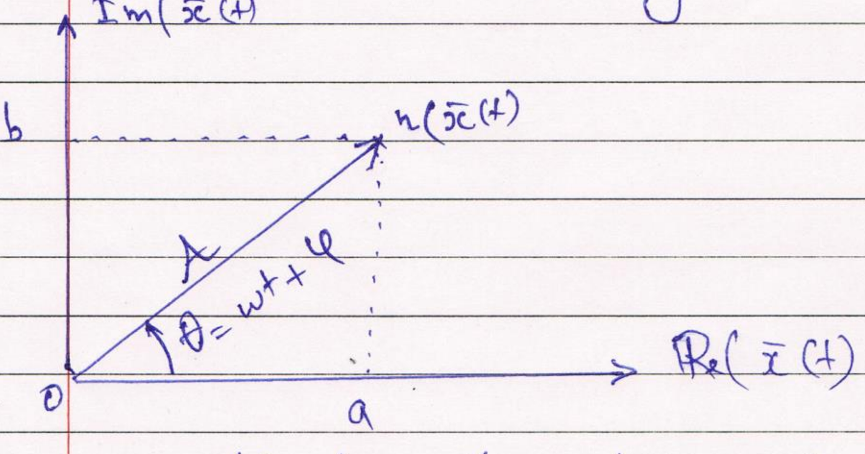
We can correspond for each sinusoidale function $x(t)$ ($x(t) = A\cos(wt+\varphi)$) a complex number denoted $\bar{x}(t)$

The question is which number $\bar{x}(t)$ can be associated to $x(t)$

Consider $\bar{x}(t) = a + jb$: $j = \sqrt{-1}$ imaginary number

and $a, b \in \mathbb{R}$ (a and b belonging to $\mathbb{R}$)

The space of complex numbers is represented by two axes (the horizontal one represente real numbers and the second one imaginary numbers) see the figure below and a complex number is represented by a point in this space by its two parts the real and the imaginary one



from the coordinate system above we can write:

$$\bar{x}(t) = a + jb = A\cos\theta + jA\sin\theta = A(\cos\theta + j\sin\theta)$$

A = modulus of $\bar{x}(t)$ and θ = argument of $\bar{x}(t)$

if we suppose that:

$$x(t) = \text{proj}_{\mathbb{R}e}(\bar{x}(t)) = A\cos(\omega t + \varphi) = \mathbb{R}e(\bar{x}(t))$$

then $\bar{x}(t)$ will be:

$$\bar{x}(t) = A[\cos(\omega t + \varphi) + j\sin(\omega t + \varphi)]$$

and consequentely:

$$\text{Im}(\bar{x}(t)) = \text{proj}_{\text{Im}}(\bar{x}(t)) = A\sin(\omega t + \varphi)$$

<u>conclusion</u>

$$x(t) = A\cos(\omega t + \varphi) \longrightarrow \bar{x}(t) = A[\cos(\omega t + \varphi) + j\sin(\omega t + \varphi)]$$

the vector $\overrightarrow{On}$ is then rotating vector, which rotate with a constant angular speed ω in clockwise direction and at t = 0 it makes an angle φ with respect to the real axes

Euler's identity is given by:

$$e^{jx} = \cos x + j \sin x$$

therefor:

$$\bar{x}(t) = A e^{j(\omega t + \varphi)} = A e^{j\varphi} e^{j\omega t} = \bar{A} e^{j\omega t}$$

where: $\bar{A} = A e^{j\varphi}$

$\bar{A}$ is called complex amplitude.

Noting:

- Modulus of $\bar{A}$: $|\bar{A}| = A$ amplitude of $x(t)$
- $\arg(\bar{A})$: $\arg(\bar{A}) = \varphi$ initiale phase of $x(t)$

- if $x(t) = A \sin(\omega t + \varphi)$ then

$$= A \cos(\omega t + \varphi - \tfrac{\pi}{2})$$

$$\bar{x}(t) = A e^{j(\omega t + \varphi - \frac{\pi}{2})} = A e^{j(\varphi - \frac{\pi}{2})} e^{j\omega t}$$

I-4-2- Common trigonometric identities.

$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$

$$\sin(a+b) = \sin a \cos b + \cos a \sin b$$

$$\cos a + \cos b = 2 \cos\left(\frac{a+b}{2}\right) \cos\left(\frac{a-b}{2}\right)$$

$$\cos a - \cos b = 2 \sin\left(\frac{a+b}{2}\right) \sin\left(\frac{a-b}{2}\right)$$

$$\sin a + \sin b = 2 \sin\left(\frac{a+b}{2}\right) \cos\left(\frac{a-b}{2}\right)$$

$$\sin a - \sin b = 2 \sin\left(\frac{a-b}{2}\right) \cos\left(\frac{a+b}{2}\right)$$

$$\cos x + j \sin x = e^{jx}$$

$$\cos x = \frac{e^{jx} + e^{-jx}}{2}, \quad \sin x = \frac{e^{jx} + e^{-jx}}{2j}$$

I- 5- scalar, vectorial and axial quantity. and notion of scalar and vectorial field

- scalar quantity

quantity (which is) defined by the number, which expresses it.

Example: mass, temprature, potentiel, distance, angle

- vectorial quantity

quantity defined by:

1- its modulus
2- its direction
3- its sense.

$\Big\}$ free vector

4- its origin if the vector is linked or bound vector (vecteur lié) i.e when this vector has a specific initial point or position in space.

. EX: velocity vector, force, acceleration vector

- Axiale vecteur

a vetor is said to be axiale if it requires knowing the axis of rotation or rotating direction.

EX: $\vec{M}$ - torque force (moment d'une force)

let G be a scalar quantity

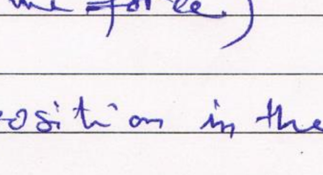
if G is defined in particular position in the space then

$$G \longrightarrow G(x, y, z, t)$$

The whole values of G in the considered space are called scalar field of G

if $G(x, y, z, t) = G(t)$ the value of G depend only on t then the field is called uniform field

by the same maner we call $\vec{A}(x, y, z, t)$ vectorial field and if $\vec{A}(x, y, z, t) = \vec{A}(t)$ the field is uniform.



I-5-1- scalar product

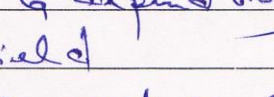
$\vec{A}$ and $\vec{B}$ two vectors

$\vec{A}.\vec{B}$ ($\vec{A}$ dot product $\vec{B}$) is a scalar defined by

$$\vec{A}.\vec{B} = \|\vec{A}\|.\|\vec{B}\|.\cos\varphi = \vec{B}.\vec{A}$$ a permutative operation

Analytically if $\vec{A}(a_1, a_2, a_3)$ and $\vec{B}(b_1, b_2, b_3)$ in an orthonormal cartesian coordinate system then

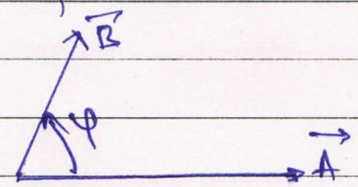
$$\vec{A}.\vec{B} = \begin{array}{ccc} (a_1 & a_2 & a_3) \\ \vec{i} & \vec{j} & \vec{k} \end{array} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \begin{array}{c} \vec{i} \\ \vec{j} \\ \vec{k} \end{array} = a_1b_1 + a_2b_2 + a_3b_3$$



the coordinate syst represented above is called a right-handed coordinate system.

I-5-2- Vectorial product

$\vec{A}$, $\vec{B}$ two vectors. vectorial product of $\vec{A}$ and $\vec{B}$ is a third vector $\vec{C}$ as following:



$$\vec{A} \wedge \vec{B} = \vec{C} \cdot \text{ where}$$

modulus of $\vec{C}$: $|\vec{C}| = |\vec{A} \wedge \vec{B}| = |\vec{A}| \cdot |\vec{B}| \sin\varphi$

the direction of $\vec{C}$ is determined by the method of right hand (or method of tir bouchon) and $\perp$ on plane $(\vec{A}, \vec{B})$
corkscrew method

Noting: vectorial product isn't permutative because:
$\vec{A} \wedge \vec{B} \neq \vec{B} \wedge \vec{A}$.

analytically we have:

$$\vec{A} \wedge \vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 b_3 - a_3 b_2 \\ -a_1 b_3 + a_3 b_1 \\ a_1 b_2 - a_2 b_1 \end{vmatrix} \begin{matrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{matrix}$$

I. 6- Second-order linear differential equation with constant and positive coefficients

Equation which form is:

$$\ddot{x} + p\dot{x} + qx = f(t) \qquad (p, q) = \text{constant and positive}$$

I-6-1- Homogeneous equation:

the differential equation is set to zero $\Rightarrow f(t) = 0$
therefore the equation will be as following:

$$\ddot{x} + p\dot{x} + qx = 0$$

the caracteristic equation of the above diff equation is

$$r^2 + pr + q = 0$$

The equation has two negative roots r_1 and r_2:

$$r_{1,2} = \frac{-p \mp \sqrt{\Delta}}{2} < 0$$

where $\Delta \geq 0$ is called the discriminant of the equation and is given by:

$$\Delta = p^2 - 4q$$

if $\Delta < 0$ then $r_{1,2}$ are two complex roots

then we can distinguish three cases:

* $\Delta > 0$

$$x_h(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} \quad : \quad r_1 \neq r_2 \text{ two real roots}$$

* $\Delta = 0$ $\quad x_h(t) = (C_1 t + C_2) e^{rt} \qquad r_1 = r_2 = r = -\frac{p}{2}$

r = one real double root

* $\Delta < 0 \Rightarrow r_{1,2}$ two complex conjugate roots:

$$r_{1,2} = -\frac{p}{2} \mp \sqrt{-|\Delta|} = -\delta \mp j\, w_a$$

where: $\quad \delta = -\frac{p}{2}$ and $w_a = \sqrt{|\Delta|}$

so the solution will be:

$$x_h(t) = e^{-\delta t}\left[C_1 e^{j w_a t} + C_2 e^{-j w_a t}\right]$$

because the solution $x_h(t)$ is a real one ($\in \mathbb{R}$) the C_1 and C_2 will be two ~~and~~ complex numbers but conjugate. The solution can be written finally as:

$$x_h(t) = C e^{-\delta t} \cos(w_a t + \varphi)$$

where C and φ can be fixed from the initial conditions of the system.

The solution above is called pseudo sinusoidal with pseudo angular frequency w_a and pseudo period $T_a = \frac{2\pi}{w_a}$

and with amplitude $C e^{-\delta t}$ which decreases exponentially over time.

I-6-2- Non-homogeneous equation

in this case the second part (right hand side) of the equation is different of zero i.e

$$f(t) \neq 0$$

the equation is then written as:

$$\ddot{x} + p\dot{x} + qx = f(t) \quad \ldots\ldots (1)$$

the general solution has the following form:

$$x(t) = x_h(t) + x_p(t)$$

where:

$x_h(t)$ is the previous solution

$x_p(t)$ is the particular solution which take into account the second term of the equation [the right hand side i.e: $f(t)$] whose form depends on that of $f(t)$

- 1st case: $f(t) = f_0 = cte$

easily we can be sure that: $x_p(t) = \frac{f_0}{q}$

so the general solution will be:

* $\Delta > 0$: $x(t) = \frac{f_0}{q} + C_1 e^{r_1 t} + C_2 e^{r_2 t}$ $\quad r_{1,2} \in \mathbb{R}^-$
* $\Delta = 0$ $\quad x(t) = \frac{f_0}{q} + (C_1 t + C_2) e^{rt}$ $\quad r \in \mathbb{R}^-$
* $\Delta < 0$ $\quad x(t) = \frac{f_0}{q} + C e^{-\delta t} \cos(\omega_a t + \varphi)$

- 2nd case: $f(t) = f_0 \cos \Omega t$ ($f(t)$ sinusoidal function)

Because $f(t)$ has a sinusoidal form, so the particular solution $x_p(t)$ will have the same form and the same angular frequency Ω, then we write:

$$x_p(t) = x_0 \cos(\Omega t + \theta)$$

in this solution the unknown quantities are x_0 and θ

We introduce this solution $x_p(t)$ in the above equation (1) with calculating $\dot{x}_p(t)$ and $\ddot{x}_p(t)$ we find:

$$(q-\Omega^2)x_0\cos(\Omega t+\theta) - p\Omega\, x_0\sin(\Omega t+\theta) = f_0\cos\Omega t$$

by identification between the two sides of the equation:

$\Rightarrow$

$$x_0 = \frac{f_0}{\sqrt{(q-\Omega^2)^2+(p\Omega)^2}} \quad , \quad \theta = -\operatorname{arctg}\frac{p\Omega}{q-\Omega^2}$$

This algebraic method is quite long. For this reason and because the diff equation is linear, we will use the complex method which involves associating a complex number to each sinusoidal function by referring to Euler's identity that we saw earlier: In a second step we will resolve the equation in the complex set by substituting each term in the equation by its associated complex number. The solution of (1) is then the real part of the complex solution.

<u>Noting</u>: the complex number is denoted: $\overline{f}(t)$, $\overline{x}_p$, $\dot{\overline{x}}_p$, $\ddot{\overline{x}}_p$

<u>sinusoidal function</u>	<u>Complex number</u>
$f(t) = f_0\cos\Omega t$	$\overline{f}(t) = \overline{F}_0 e^{j\Omega t}$: $\overline{f_0} = f_0 e^{j0}$
$x_p(t) = x_0\cos(\Omega t+\theta)$	$\overline{x}_p(t) = \overline{x}_0 e^{j\Omega t}$: $\overline{x}_0 = x_0 e^{j\theta}$
$\dot{x}_p(t) = -x_0\Omega\sin(\Omega t+\theta)$ $= \Omega x_0\cos(\Omega t+\theta+\frac{\pi}{2})$	$\dot{\overline{x}}_p(t) = \overline{x}_0 \cdot e^{j\frac{\pi}{2}} e^{j\Omega t} \cdot \Omega$
$\ddot{x}_p(t) = -x_0\Omega^2\cos(\Omega t+\theta)$ $= x_0\Omega^2\cos(\Omega t+\theta+\pi)$	$\ddot{\overline{x}}_p(t) = \Omega^2 e^{j\pi}\overline{x}_0 e^{j\Omega t}$

The equation in complex set is:

$$\ddot{\overline{x}}_p + p\dot{\overline{x}}_p + q\overline{x}_p = \overline{f}(t) \Rightarrow \left[\Omega^2 e^{j\pi} + p\Omega e^{j\frac{\pi}{2}} + q\right]\overline{x}_0 e^{j\Omega t} = \overline{f_0} e^{j\Omega t}$$

$$\Rightarrow \overline{x}_0 = \frac{\overline{f_0}}{(q-\Omega^2)+jp\Omega} \Rightarrow \begin{cases} x_0 = |\overline{x}_0| = \dfrac{f_0}{\sqrt{(q-\Omega^2)^2+(p\Omega)^2}} \\ \theta = \arg(\overline{x}_0) = -\operatorname{arctg}\dfrac{p\Omega}{q-\Omega^2} \end{cases}$$

The general solution will be then: (with respect to the signe of Δ):

* $\Delta > 0$: $x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} + x_0 \cos(\Omega t + \theta)$
* $\Delta = 0$ $x(t) = (c_1 t + c_2) e^{rt} + x_0 \cos(\Omega t + \theta)$
* $\Delta < 0$ $x(t) = c e^{-\delta t} \cos(w_a t + \varphi) + x_0 \cos(\Omega t + \theta)$

<u>Noting</u>:

In all thee cases the general solution is the sum of two terms: $x_h(t)$ and $x_p(t)$, where $x_h(t)$ vanishes over time. Then we can divide the solution into two phases with respect to time.

- the first phase when $x_h(t)$ is still importante compared to $x_p(t)$. This phase is called transitive phase
- the second one when $x_h(t)$ become negligeable ($t \gg$) compared to $x_p(t)$ and consequently: $x(t) \simeq x_p(t)$ is called permanent phase.

- <u>3rd case</u>

$f(t)$ is a periodic function but not sinusoidal

In this case and according to fourier's theorem:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\Omega t + b_n \sin n\Omega t = \frac{a_0}{2} + \sum_{n=1}^{\infty} d_n \cos(n\Omega t + \varphi_n)$$

The eqation(1) can be written in this case as:

$$\ddot{x} + p\dot{x} + qx = \frac{a_0}{2} + \sum_{n=1}^{\infty} d_n \cos(n\Omega t + \varphi_n)$$

Because of the linearity of the diff eq we search the solution for each term: $\frac{a_0}{2}$, $d_1 \cos(\Omega t + \varphi_1)$, $d_2 \cos(2\Omega t + \varphi_2)$ separately and then the solution will be the sum of all those solutions:

$$x(t) \simeq x_h(t) = \frac{a_0}{2q} + \sum_{n=1}^{\infty} \frac{a_n \cos(n\Omega t + \theta_n) + b_n \sin(n\Omega t + \theta_n)}{\sqrt{[(n\Omega)^2 - q]^2 + p^2(n\Omega)^2}}$$

I-7- Recall in classical mechanics

I-7-1- position vilocity of a material point

three types of coordinate systems can be used to fixe the position and velocity of a material point. the choise between them depend on the nature of the system which we want to study and the kind of its movement.

a- cartesian coordinates

the position of the material point M is determined by the position vector $\vec{r} = \overrightarrow{OM}$:

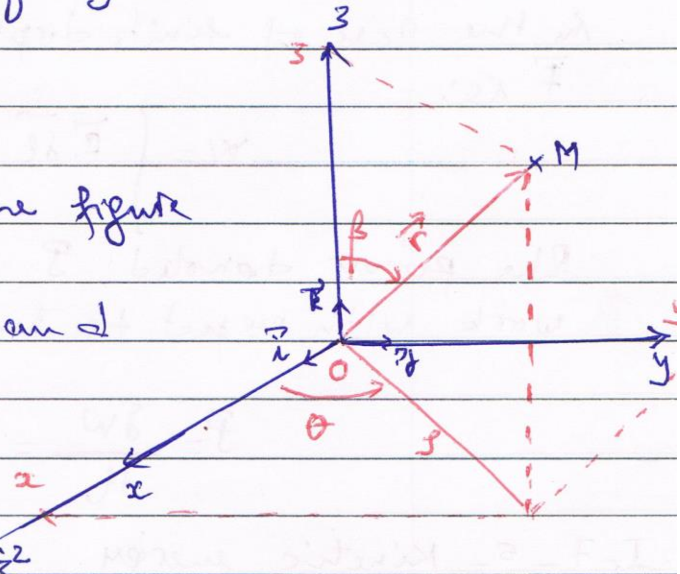
$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

where x, y, z are coordinates (cartesian) of M which are function of t

the vilocity vector $\vec{v}$ is derivative of $\overrightarrow{OM}$ with respect to t (time) then:

$$\vec{v} = \frac{d\overrightarrow{OM}}{dt} = \dot{x}\vec{i} + \dot{y}\vec{j} + \dot{z}\vec{k}$$

$$\Rightarrow \quad v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2.$$



b- cylindrical coordinates

ρ, θ and z as is shown in the figure

the relation between cylindrical and cartesian coordinates is:

$x = \rho\cos\theta$

$y = \rho\sin\theta$

$z = z \qquad \Rightarrow \; v^2 = \dot{\rho}^2 + \rho^2\dot{\theta}^2 + \dot{z}^2$

c- spherical coordinates

r, θ, β as shown in the figure above.

we can easily find the relation between spherical and cartesian coordinates:

$$x = r\sin\beta\cos\theta$$

$$y = r\sin\beta\sin\theta$$

$$z = r\cos\beta.$$

I-7-2- principal of action and reaction (3^{rd} Law of Newton)

if a body (1) exert a force F_{12} on an other body (2) then this one also exert a force F_{21} on the first one such as

$$\vec{F}_{21} = -\vec{F}_{12}.$$

I-7-3 - Fundamental law in dynamics (2nd Law of Newton)

in case of translational motion:

$$\vec{F} = \sum_i \vec{f_i} = m\vec{\gamma} = \frac{dp}{dt} = m\frac{d\vec{v}}{dt}$$

$\vec{P} = m\vec{v}$ is called momentum of a material point which mass is m

$\vec{F}$ = resultant of all forces acting on the material point of mass m.

I-7-4 - Work and power

if a force $\vec{F}$ causes a material point (or a body) to move by an infinitesimal displacement $\vec{\delta l}$, then the amount of work done by this force is:

$$\delta W = \vec{F}.\vec{\delta l} \qquad \delta W = \text{scalar quantity}.$$

in the case of finite displacement, the total work done by $\vec{F}$ is:

$$W = \int \vec{F}.\vec{\delta l} \quad \text{(curviligne integral)}$$

The power denoted P of $\vec{F}$ is derivative of the work with respect to time t:

$$P = \frac{\delta W}{dt} = \vec{F}.\vec{v}.$$

I-7-5 - Kinetic energy

we agree to denote Kinetic energy by T instead of E_c

- Kinetic energy of a material point of mass m

$$T = \frac{1}{2} m v^2 \quad : \quad v : \text{translation velocity}$$

- A solid body of mass M (planar motion)

$$T = T_{trans} + T_{rot}$$

T_{trans} = Kinetic energy (translation) of the gravity center G of the body $\Rightarrow T_{trans} = \frac{1}{2} m v_G^2$

T_{rot} = rotational Kinetic energy of rotational motion around the axis passing by the gravity center G and $\perp$ on plane of motion.

$$T_{rot} = \frac{1}{2} J_G \dot{\theta}^2$$

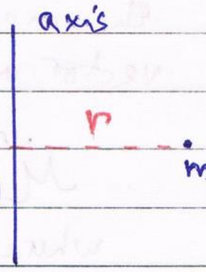
where :

J_G = Moment of inertia of the considered body with respect to the axis passing by G and ⊥ to the plane of motion

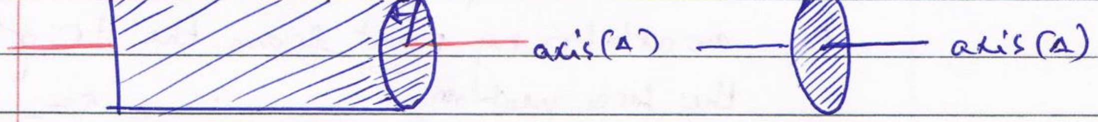
I_7_6 Moment of inertia

- Material point of mass m which rotate around an axis at a distance r (away from it)

$$J_m = m r^2$$

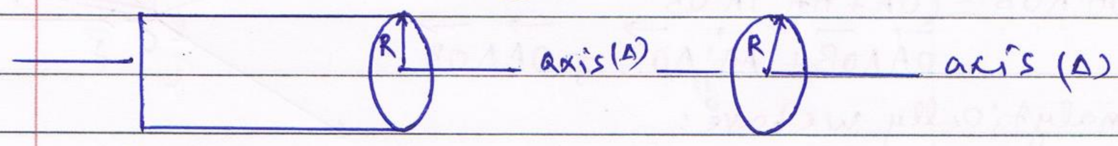


- solid cylinder or solid disk with respect to its axis of symmetry and of radius R and mass M



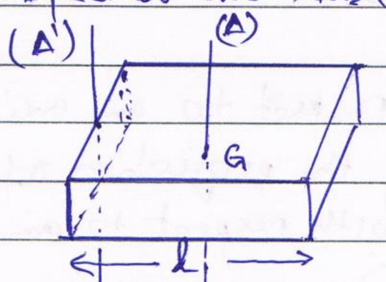
$$J_{/\Delta} = \frac{1}{2} M R^2$$

- hallow cylinder or a ring:



$$J_{/\Delta} = M R^2$$

- Solid body in the shape of a parallelepiped of mass m with respect to an axis (Δ) which passes by G



$$J_{/\Delta} = \frac{m l^2}{12}$$

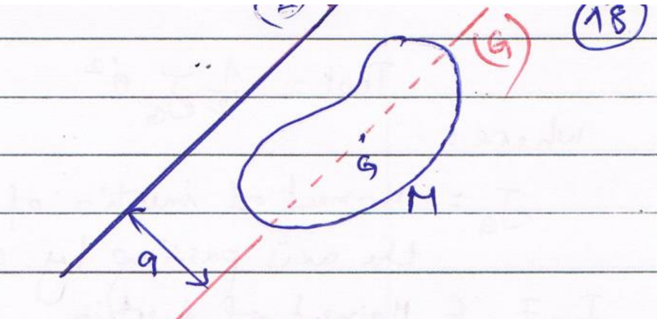
$$J_{/\Delta'} = \frac{m l^2}{3}$$

Δ' // Δ and passes by the bound of the solid.

Huygens theorem

The moment of inertia $I_{/\Delta}$ of a rigid body of mass M about an axis (Δ) is given by the moment of inertia about an axis (G) passing by the center of mass G parallel to (Δ) and adding the quantity Ma^2, where a is a distance between the two axis. Then we write:

$$J_{/\Delta} = J_{/G} + Ma^2$$



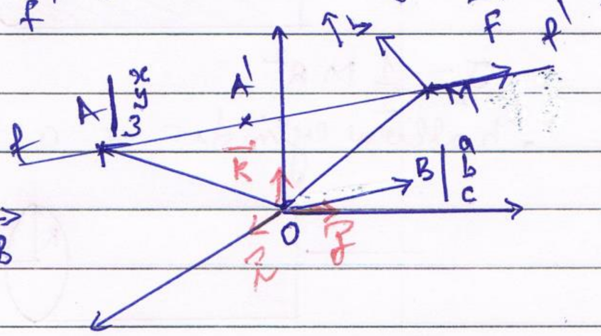
I-7-7- Moment of force (torque)

the moment of a force F with respect to a point O is an axial vector and denoted by $\overset{\frown}{\mathcal{M}}_{/O}(\vec{F})$ is given by:

$$\overset{\frown}{\mathcal{M}}_{/O}(\vec{F}) = \overrightarrow{OA} \wedge \overrightarrow{OB}$$

where: $\overrightarrow{OB} = \vec{F}$ equipollent vector

A: an arbitrary point from the stright holding the force vector $\vec{F}$



for an other point $A' \in ff'$

$$\overrightarrow{OA'} \wedge \overrightarrow{OB} = (\overrightarrow{OA} + \overrightarrow{AA'}) \wedge \overrightarrow{OB}$$
$$= \overrightarrow{OA} \wedge \overrightarrow{OB} + \underbrace{\overrightarrow{AA'} \wedge \overrightarrow{OB}}_{0} = \overrightarrow{OA} \wedge \overrightarrow{OB}$$

analytically we have:

$$\overset{\frown}{\mathcal{M}}_{/O}(\vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x & y & z \\ a & b & c \end{vmatrix} = (yc - zb)\vec{i} + (za - xc)\vec{j} + (xb - ya)\vec{k}$$

Noting

the Moment of a force f with respect to an axis (Δ) is a scalar quantity given by the projection of the vector of the moment of this force with respect to an arbitrary point from the same axis (Δ).

$$\mathcal{M}_{/\Delta}(\vec{F}) = \underset{\Delta}{\text{proj}}\left(\overset{\frown}{\mathcal{M}}_{/O}(\vec{f})\right) \quad \text{where } O \in (\Delta)$$

I-7-8- Angular momentum vector of a material point

Angular momentum of a material point of mass m and position M with respect to O is a vector denoted by $\vec{L}_{/O}$ and given by:

$$\vec{L}_{/O}(M) = \overrightarrow{OM} \wedge \vec{P} \quad : \quad \vec{P} = m\vec{v}_M$$

$\vec{v}_M$: velocity vector of M

8-1 - Angular momentum theorem

The derivative of angular momentum vector of a material point M with respect to a fixed point O (in Galilean reference is given by the sum of the moments of all forces acting on this point M with respect to O. Then we write

$$\frac{d}{dt}\vec{L}_{/O}(M) = \sum_i \vec{\mathcal{M}}_{/O}(\vec{f_i}) = \vec{OM} \wedge \vec{f}$$

$\vec{f}$ = resultant force acting on M

We can demonstrate that for a material point M of mass m the angular momentum $\vec{L}_{/O}(M)$ is given by:

$$\vec{L}_{/O}(M) = m\,r^2(t)\,\vec{\omega}(t)$$

where: r : distance of M from O

ω = angular velocity of M around O.

8-2 - Angular momentum theorem for a solid body

a solid body (S) which is rotating about a point O means that this body is rotating about an axis (Δ) passing by O. So the angular momentum of this body with respect to (Δ) [scalar quantity) is the projection of the total angular momentum vector of this body with respect to a point O from (Δ)

we can show that:

$$L_{/\Delta}(S) = \text{proj}_{/\Delta}(\vec{L}_{/O}(S)) = J_{/\Delta}\cdot\omega(t) \quad \dots\dots(2)$$

$L_{/\Delta}(S)$ = angular momentum of the body / (Δ)

$\vec{L}_{/O}(S)$ = total angular momentum vector / O

$J_{/\Delta}$ = moment of inertia of the body / Δ

The angular momentum theorem of a solid body is then:

$$\frac{d}{dt}L_{/\Delta}(S) = \sum_i \mathcal{M}_{/\Delta}(\vec{f_i}) \quad \dots(3)$$

the equation (2) above gives:

$$\frac{d}{dt} L_{/\Delta}(S) = J_{/\Delta} \frac{d\omega}{dt} = J_{/\Delta} . \ddot{\theta} \ldots (4)$$

where $\ddot{\theta}$ is the angular acceleration of (S) about (Δ)

from (3) and (4) we obtain:

$$\boxed{J_{/\Delta} . \ddot{\theta} = \sum_i \mathcal{M}_{/\Delta}(\vec{f_i})}$$

I-7-9_ potential energy

It's a scalar physical quantity which depend on the position of the point, then on it's coordinates (sometime on it depend on time). the unit of this quantity is the same as the energy.

Example: - Gravitational potential energy
- Elastic potential energy of spring
- Electric " " " of an electric charge

the potential energy is generaly denoted by U

Force deriving from a potential

a conservative force (non_dissipative) is a force deriving from a potential. This means that a scalar field (potential) U excite such as this force can be written as a gradient of this potential i.e (latin phrase id est which translates to that is)

$$\vec{F} = -\overrightarrow{grad}\, U \iff \mathrm{Rot}\, \vec{F} = 0$$

In cartesian coordinates the force $\vec{F}$ has three components such as:

$$F_x = -\frac{\partial U}{\partial x}, \quad F_y = -\frac{\partial U}{\partial y} \quad \text{and} \quad F_z = -\frac{\partial U}{\partial z}$$

Example

- force of gravity: $U = -mgz \Rightarrow \vec{P} = mg\vec{k}$
- spring return force: $U = \frac{1}{2}kx^2 \Rightarrow \vec{F_r} = -kx\vec{i}$

k = stiffness constant (spring constant)
x = deformation of the spring

I-7-10 Stable equilibrium and unstable equilibrium

a solid body is in equilibrium state if the following conditions are presente in the system.

$$\sum_i \vec{F_i} = 0 \qquad \text{translation}$$

$$\sum \vec{\mathcal{M}_i} = 0 \qquad \text{rotation}$$

if the forces are diriving from a potential U, therfor the condition of equilibrium in the position $n(x_n, y_n, z_n)$ expressed by U will be written as followes:

$$\left.\frac{\partial U}{\partial x}\right|_{x_n} = \left.\frac{\partial U}{\partial y}\right|_{y_n} = \left.\frac{\partial U}{\partial z}\right|_{z_n} = 0$$

this condition means that the potential has an extremal value at this position.

this condition can be interpretated by two distinguesh maner

- 1st case: stable equilibrium (we take the simple exemple when U depend only on a single coordinate $\Rightarrow U(x)$)

$U(x)$ is minimal for $x = x_1$ $\Leftrightarrow$ $\left.\frac{dU}{dx}\right|_{x_1} = 0$ and $\left.\frac{d^2U}{dx^2}\right|_{x_1} > 0$

the position is a stable equilibrium (see fig below)

- 2nd case:

$U(x)$ is maximal for $x = x_2$ $\Leftrightarrow$ $\left.\frac{dU}{dx}\right|_{x_2} = 0$ and $\left.\frac{d^2U}{dx^2}\right|_{x_2} < 0$

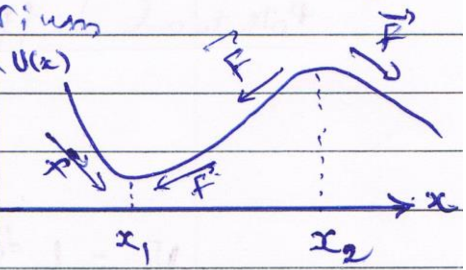
the position is an unstable equilibrium

U is min for $x < x_1$: $F = -\frac{dU}{dx} > 0$

$x > x_1$: $F = -\frac{dU}{dx} < 0$

$\vec{F}$ is always directed to x_1 : central force

U is max we find that the force is a repulsive one



Conclusion: A given system can perform oscillations around a certain point x if this one is a stable equilibrium point

I - 8 - Recall in electricity

8-1- Electric current

denoted by i and is defined as following:

$$i = \dot{q} = \frac{dq}{dt}$$

where

q = electric charge passing through a section of the considered circuit

8-2 - Capacity of parallel plan capacitor and the force applied between its two plates

the capacity is given by (denoted C):

$$C = \frac{\varepsilon S}{d}$$

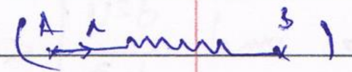
where:

C: capacity (capacitance)
ε: permitivity of the insolator (insulation permittivity)
S: surface of each plate
d: distance between the two plates

the force between the two plates is

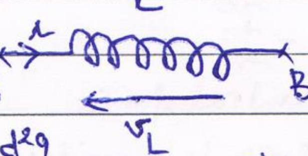
$$F = \frac{1}{2}\frac{q^2}{\varepsilon S} = \frac{1}{2}\frac{CV^2}{d}$$

8-3 - Potential difference across the terminals of an electric resistor



$$v_R = v_A - v_B = Ri$$

Potential difference across the terminals of self-inductance L



$$v_L = L\frac{di}{dt} = L\frac{d^2q}{dt^2} \Rightarrow i = \frac{1}{L}\int v(t)dt$$

Potential difference across the terminals of a capacitance C

$$v_c = \frac{1}{C}\int i(t)\,dt \Rightarrow i = C\frac{dv_c}{dt}..$$

8-4 Instantaneous electrical power

$$P = v i$$

8-5 Electrical energy stored in a capacitance

denoted by W_e and given by:

$$W_e = \frac{1}{2} C v_c^2 = \frac{1}{2}\frac{q^2}{C}$$

8-6 Magnetic energy stored in a self-inductance L

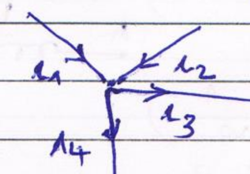
denoted by W_m and given by:

$$W_m = \frac{1}{2} L i^2$$

8-7 Kirchhoff's Laws (Lois de Kirchhoff)

- Kirchhoff's current law (Loi des noeuds)

The sum of the currents entering a node is equal to the sum of the currents leaving the node



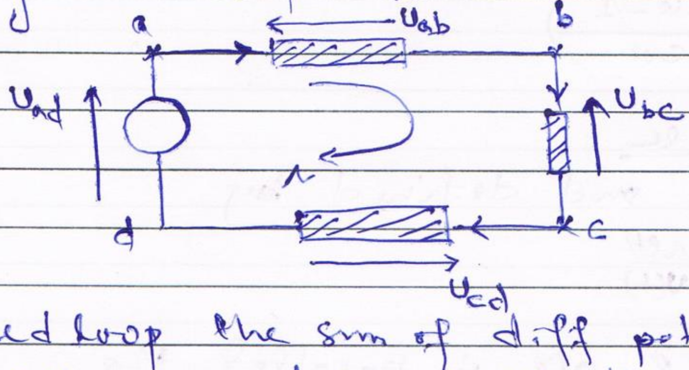
$$i_1 + i_2 = i_3 + i_4$$

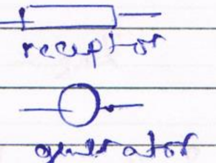
or

$$i_1 + i_2 - i_3 - i_4 = 0$$

8-8 Kirchhoff's voltage law (Loi des mailles)

The algebraic sum of the potential difference around any closed loop is zero.





in a closed loop the sum of diff potentials in the same direction of the current is equal to the sum of those in the opposite direction of the current.

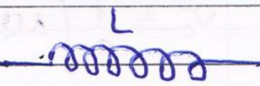
$$\Rightarrow U_{ab} + U_{bc} + U_{cd} = U_{ad}$$

8-8 Impedance in RLC circuit

- Impedance of a pure resistor: —▭— (—∿∿∿—)

$$Z_R = R \quad \text{real quantity.}$$

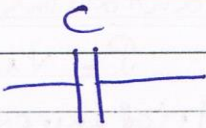
- pure inductance of self inductance L:



$Z_L = jL\omega$

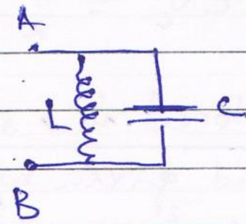
complex quantity

- pure capacitor of capacitance C



$Z_C = \frac{1}{jC\omega}$ ω = angular frequency of the current.

- L // C

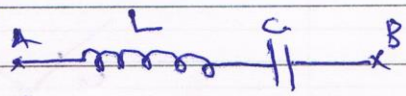


$$\frac{1}{Z_{AB}} = \frac{1}{Z_L} + \frac{1}{Z_C} \Rightarrow Z_{AB} = j\frac{L\omega}{1-LC\omega^2}$$

if : $\omega^2 = \frac{1}{LC} \Rightarrow Z_{AB} \longrightarrow \infty$ (an open circuit)

we say in this case that L and C form together a trap circuit (circuit bouchon)

- L in series with C

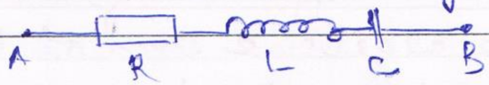


$$Z_{AB} = Z_L + Z_C = j\left(\frac{LC\omega^2 - 1}{C\omega}\right)$$

if $\omega^2 = \frac{1}{LC}$ then $Z_{AB} = 0$

we say in this case that L and C form together

- L, C and R in series:



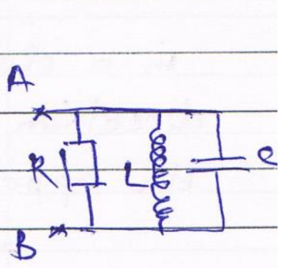
$$Z_{AB} = R + j\left(\frac{LC\omega^2 - 1}{C\omega}\right)$$

8-9 Admittance

denoted Y and defined by

$$Y = \frac{i(t)}{v(t)}$$

if L, C and R are in parallel the



$$Y_{AB} = \frac{1}{Z_R} + \frac{1}{Z_L} + \frac{1}{Z_C} = \frac{1}{R} + j\left(C\omega - \frac{1}{L\omega}\right)$$

I_9_Lagrangian and lagrange equations

I_9_1_Lagrange equation for a material point

_Number of degrees of freedom

let's have a material point which move (without friction) in the space. the point of a mass m can be localised by three coordinates x, y, z (general case) i.e we say that this point have three degrees of freedom, which mean that it can perform three different motions in the space (3 independent motions) $\Rightarrow N = 3$

assume now the point is moving according to a trajectory given by the following equations:

$$\begin{cases} z = 0 & \text{planar motion} \\ f(x,y) = 0 & \text{trajectory equation} \end{cases}$$

those two equations are called constraints applied to the point. In this case the number of degrees of freedom will be reduced and we write:

$$S = N - l = 3 - 2 = 1 \text{ degree of freedom}$$

S = number of degrees of freedom

l = number of constraints عدد القيود

N = number of degrees of freedom without constraints.

then the material point in the example below has one degree of freedom. this means that the position of the point can be determined by one coordinate. the choise of this coordonate is arbitrary, it depends only on the nature of motion (for example it can be, x, y, z, θ or ρ). the chosen coordinate is called generalized coordinate and is denoted q.

q = generalized coordinate

Assume that the material point is subjected to a force $\vec{F}(x,y,z)$

The generalized force associated to q is given by: (denoted f_q): (القوة المعممة)

$$f_q = \frac{\delta W}{\delta q} = \vec{F} \cdot \frac{\partial \vec{r}}{\partial q}$$

where : δW = is the work of f

We can demonstrat for a system with single degree of freedom that :

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q} = f_q \qquad T = \text{Kenetic energy of the system.}$$

$\dot{q}$ = generalized velocity (السرعة المعممة)

the above equation is called lagrange equation of a single degree of freedom system.

9-1-1- Case of a Conservative system

a Conservative system is a system whose total energy is conserved (i.e syst without dissipation) then we have:

$$E = T + U = cte$$

E = total energy

T = Kinetic energy

U = potential energy

In this case the force acting on the system is deriving from a potential and so we can write:

$$f_q = -\frac{\partial U}{\partial q}$$

then we have:

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{q}}\right] - \frac{\partial T}{\partial q} = -\frac{\partial U}{\partial q}$$

In general the potential energy U is not depending on $\dot{q}$ (generalized velocity) (noting $\dot{q} = \frac{dq}{dt}$)

$\Rightarrow \frac{\partial U}{\partial \dot{q}} = 0$

then we find easily :

$$\frac{d}{dt}\left[\frac{\partial (T-U)}{\partial \dot{q}}\right] - \frac{\partial (T-U)}{\partial q} = 0$$

we put that:

$$\mathcal{L}(q,\dot{q}) = T - U$$

$\mathcal{L}$ is called the lagrangian of the system (Kinetic energy minus potential energy)

finally the lagrange equation can be written as:

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{q}}\right] - \frac{\partial \mathcal{L}}{\partial q} = 0$$

9_1_2_ Non conservative system

It's a system with energy is not conserved i.e

$$E = T + U \neq cte$$

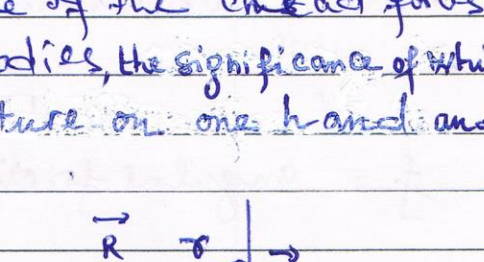
The system is then subject of friction forces which dissipate the energy (total energy) and transform it in thermal one (heat energy)

We distinguish between two principale kinds of friction forces

a. first kind: solid friction

This kind of friction appears when a solid body is moving over a solid surface because of the contact forces that are created between the two bodies, the significance of which depends on surface molecular structure on one hand and surface roughness on the other.

The adjacent figure shows a solid body moving over a horizontal plane and in the right direction



the forces acting on the body are:

$\vec{P}$ = force of gravity = $m\vec{g}$

$\vec{R_y}$ = reaction force due to $\vec{P}$

$\vec{F_s}$ = force of friction (solid friction)

$\vec{R_y}$ and $\vec{F_s}$ togther ($\vec{R} = \vec{R_y} + \vec{F_s}$) reprents the reaction force of the plane on the body.

from the figure above we have:

R_y = proj $\vec{R}$/vertical axis $= -mg$

R_x = proj $\vec{R}$/ horizontal axis = $- mg\, tg\gamma$

the angle γ is depending only of the surface of the plane. If this one is homogeneous then γ will be constant and therefor:

$$f_s = |R_x| = mg\, tg\gamma = cte$$

$\vec{f_s}$ has a constant modulus but it changes its direction every time when the body changes its direction of motion.

b_ Second Kind: fluid friction (viscous friction)

This kind of friction appears when the system moves in a viscous medium (fluid medium) such as liquid medium or a gaz medium. The the force of friction denoted $\vec{f_f}$ is proportional to the velocity of the mobile and we write:

$$\vec{f_f} = -\alpha \vec{v} \quad : \quad \alpha > 0$$

the minus signe means that the force $\vec{f_f}$ and the velocity are always in opposite directions.

α = linear friction coefficient (always positive)

If the mobile is in a rotational motion the a torque (الـعـزم) of friction appears which is denoted Γ and is proportional to the angular velocity $\dot{\theta}$ and we write:

$$\Gamma = -f\dot{\theta}$$

f = angular friction coefficient

b_1 Lagrange equation in the case of fluid friction

the force of fluid friction is given by:

$$\vec{f_f} = -\alpha \vec{v} \qquad \text{in cartesian coordinates}$$

if the system is described by a certain generalized coordinate denoted q then we must find the generalized force denoted f_q associated to $\vec{f_f}$.

we know that:

$$f_q = \frac{\delta W}{\delta q} = \vec{f_f}\cdot\frac{\partial \vec{r}}{\partial q} = -\alpha \frac{\partial \vec{r}}{\partial t}\cdot\frac{\partial \vec{r}}{\partial q} = -\alpha\frac{\partial \vec{r}}{\partial t}\cdot\frac{\partial \vec{r}}{\partial q}\cdot\frac{\partial q}{\partial q} = -\alpha\left(\frac{\partial \vec{r}}{\partial q}\right)^2\cdot\frac{\partial q}{\partial t}$$

$$\Rightarrow \quad f_q = \underbrace{-\alpha\left(\frac{\partial r}{\partial q}\right)^2}_{\beta} \cdot \dot{q} = -\beta\dot{q}$$

So the equation of Lagrange will be:

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{q}}\right] - \frac{\partial \mathcal{L}}{\partial q} = -\beta\dot{q}$$

b-2 - Dissipation function   دالة التبدد

let's calculate the work of the friction force $\vec{f}_f$ during a period of time δt

during this time δt the displacement is $\delta\vec{r}$ so:

$$\delta w_f = \vec{f}_f \cdot \delta\vec{r} = (-\alpha\vec{v})\frac{\delta\vec{r}}{dt}\,dt = -\alpha v^2 dt$$

the quantity of heat absorbed by the medium in contact with the system will be:

$$\delta Q = \alpha v^2 dt$$

then the dissipated power in the form of heat is:

$$P_d = \frac{\delta Q}{dt} = \alpha v^2 = \alpha\frac{\delta \vec{r}}{dt}\cdot\frac{\delta \vec{r}}{dt}\frac{(\partial q)^2}{(\partial q)^2}$$

$$= \underbrace{\alpha\left(\frac{\delta \vec{r}}{\partial q}\right)^2}_{\beta}\cdot\left(\frac{\partial q}{dt}\right)^2 = \beta\dot{q}^2$$

by definition the dissipation function denoted D is the half of the dissipated power so:

$$D = \frac{1}{2}P_d = \frac{1}{2}\beta\dot{q}^2$$

thus we find that the generalized force f_q will be:

$$f_q = -\frac{\partial D}{\partial \dot{q}}$$

and finally the Lagrange equation in presence of fluid friction force is:

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}}\right] - \frac{\partial L}{\partial q} + \frac{\partial D}{\partial \dot{q}} = 0$$

b.3 . Multi - degree - of - freedom system

let's have a system consisting of N material points. Each point needs three coordinates then it has three degrees of freedom. so to discribe the configuration of the whole system (N point) we have to determine 3N coordinates. consequently the system is 3N degrees of freedom. A solid body has 6 coordinates (3 cartesian coordinates + 3 angles of rotation). for M solid bodies we have 6M coordinates, so 6M degrees of freedom. the most general case is when the system is consisting of N material points + M solid bodies. the number of degrees of freedom will be 3N + 6M. If the system is subjekt of l constraints then the number of degrees of freedom will be:

$$S = 3N + 6M - l$$

S = number of degrees of freedom of the system and it's called s degrees of freedom system.

the system is then discribed by S generalized coordinates denoted $q_1, q_2, \ldots q_s$

the system admits in this case S lagrange equations which form is:

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right] - \frac{\partial \mathcal{L}}{\partial q_i} + \frac{\partial D}{\partial \dot{q}_i} = 0 \qquad i = 1 \ldots, S$$

$\mathcal{L}(q_1, q_2, \ldots q_s, \dot{q}_1, \dot{q}_2, \ldots \dot{q}_s)$

b-4 . General Case : External force depending on time

consider a system with S degrees of freedom subjected of an externale force $F(t)$ depending on time and not deriving from a potential energy, this force will have S components according to S degrees of freedom which are denoted as: $Q_1, Q_2, \ldots Q_s$ and each component is calculated by:

$$Q_i = \frac{\delta W}{\partial q_i} = \vec{F}(t)\frac{\delta \vec{r}}{\partial q_i} \qquad i = 1, \ldots\ldots S$$

$Q_i(t)$ = the component i of the generalized force associeted to $\vec{F}(t)$ corresponding to the generalized coordinati q_i

Therefor the system admits s lagrange equations which have the following form:

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_i}\right] - \frac{\partial L}{\partial \dot{q}} + \frac{\partial D}{\partial \dot{q}_i} = Q_i(t) \quad i = 1, \ldots\ldots s$$

In the field of vibrations and waves and in the limit of vibrations with small amplitudes the above s equations will be reduced to s linear differential equations of second order with constant and positive coefficients. With these criterias the system of s equations admits a solution with s angular frequencies denoted ω_j where $1 \leq j \leq s$ and $2s$ integration constantes whose values are determined by $2s$ initial conditions of the motion.

So the system, under a particular conditions, can undergo sinusoidal oscillations of angular frequency ω_j, called natural mode of vibration of which there is a number S of vibration modes. For each mode all parts of the system acquire a sinusoidal oscillation with the same angular frequency ω_j. But if the initial conditions are arbitrary, the vibrations in the system will be a linear combination of these natural modes, and therefor not sinusoidal.