Course of Analytical Mechanics
Code: F122
Evaluation Mode: 33% continuous assessment and 67% exam
Weekly Hours: 3h lecture and 1h30 TD

Chapter 1: Review of Classical Mechanics
Kinematics of a particle. Dynamics of a particle. Work and energy. Systems with N particles and external forces. Degrees of freedom.

Chapter 2: Lagrangian Formalism
Generalized coordinates. Functional variation. The Lagrangian. Curvilinear coordinates. Holonomic and non-holonomic constraints. Applications: Particle in a gravitational field, particle connected to a spring, two-body problem, central potential.

Chapter 3: Hamiltonian Formalism
Legendre transformation. The Hamiltonian. Canonical variables and Poisson brackets. Generalized moments. Canonical transformations. Hamilton-Jacobi method. Phase space. Angle-action variables and generating function. Integrable systems.

Chapter 4: Motion of a Rigid Body
Degrees of freedom of a rigid body. Kinetic energy. Principal axes and inertia tensor. Angular momentum of a rigid body. Vector approach and Euler's equations. Lagrangian approach and Euler angles. Symmetric top.

Chapter 5: Lagrangian Mechanics of Continuous Media
Transition to the continuous limit. Classical field theory. Euler-Lagrange field equations.

Chapter 6: Liouville's Theorem
Hamilton-Jacobi equation.

Chapter 1: Review of Classical Mechanics

Kinematics of a Particle

Kinematics is the study of the motion of a particle or systems of particles without investigating the causes of this motion. The causes of motion are found in forces, which will be the subject of dynamics.

- The description of a particle's motion involves three vectors:
  i) The position vector.
  ii) The velocity vector.
  iii) The acceleration vector.

Position Vector

Consider a given reference frame, R, with origin at point O. To locate a particle, we search for its relative position with respect to the origin:

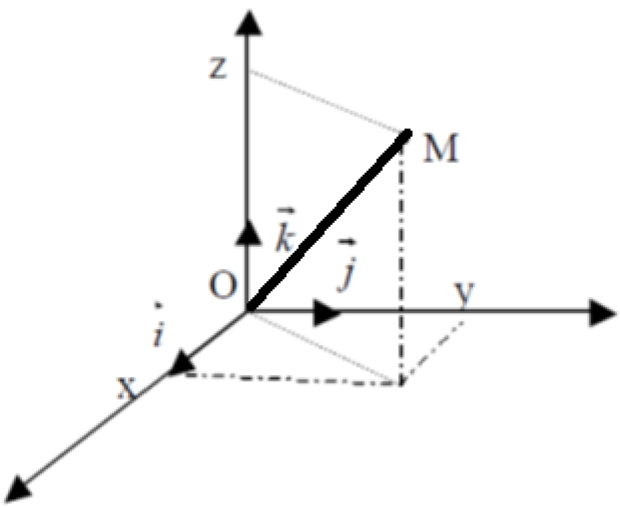
$\vec{r} = \overrightarrow{OM}$

The position vector **r⃗** varies during motion, and the set of successive positions over time forms a curve called the trajectory of position P.

Cartesian Coordinate System

Using the orthogonal Cartesian coordinate system with base vectors **i⃗**, **j⃗**, and **k⃗**, the position vector $\vec{r}$ can be expressed as:

$\vec{r} = \mathbf{x}\vec{\imath} + \mathbf{y}\vec{\jmath} + \mathbf{z}\vec{k}$.(Within R , $\frac{d}{dt}\vec{\imath} = \frac{d}{dt}\vec{\jmath} = \frac{d}{dt}\vec{k} = \vec{0}$)



The functions **x = f(t)**, **y = g(t)**, **z = h(t)** are the time equations of motion and can be obtained by integrating the equations of motion. The trajectory equation is obtained by eliminating time **t** between the different time equations, which may not always be practical.

Elementary Displacement

The elementary displacement vector $\overrightarrow{MM'}$ (with M' close to M) is written as:

$\overrightarrow{MM'} = d\overrightarrow{OM} = dx\vec{\imath} + dy\vec{\jmath} + dz\,\vec{k}$

Velocity Vector

Average Velocity

The average velocity vector of a particle M located at point M_1 at time t_1 and at point M_2 at time t_2, relative to a reference frame with origin O, is given by:

$$\vec{v}_{avg} = \frac{\overrightarrow{OM_2} - \overrightarrow{OM_1}}{t_2 - t_1}$$

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Instantaneous Velocity

The instantaneous velocity vector of a particle at point M, relative to an orthonormal reference frame **R(O,xyz)**, is:

$$\vec{v} = \lim_{\Delta t \to 0} \vec{v}_m = \frac{d}{dt}\overrightarrow{OM}$$

Acceleration Vector

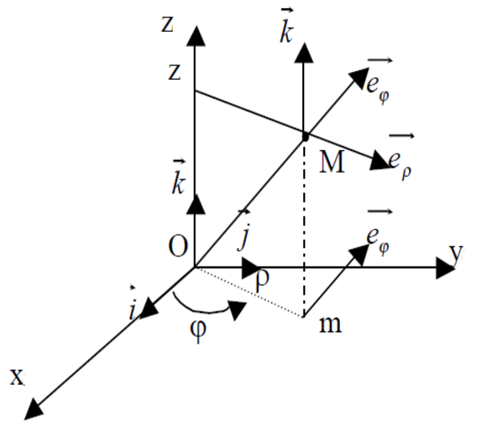
The derivative of the velocity vector with respect to time gives the acceleration vector $\vec{\mathbf{a}}$, which is written as:

$$\vec{a} = \frac{d}{dt}\vec{v} = \frac{d^2}{dt^2}\overrightarrow{OM}$$

Position, Velocity, and Acceleration in Cylindrical Coordinates

If the trajectory of point M exhibits axial symmetry, it is useful to use cylindrical coordinates **(ρ, φ, z)** defined as:

- **ρ = OM**, where **m** is the projection of point M onto the plane **(XOY)**,
- **φ** is the angle between **OX** and **OY**,
- **z** is the projection of the position vector $\overrightarrow{(\mathbf{OM})}$ onto the **OZ** axis.



A new orthonormal base $(\vec{e}_\rho, \vec{e}_\varphi, \vec{k})$ is associated with this coordinate system and relates to the vectors **i⃗**, **j⃗**, and **k⃗** as follows:

$\vec{e}_\rho = \cos\varphi\,\vec{i} + \sin\varphi\,\vec{j}$
$\vec{e}_\varphi = -\sin\varphi\,\vec{i} + \cos\varphi\,\vec{j}$

$\vec{k} = \vec{k}$

When point M moves through space, the ranges for **ρ**, **φ**, and **z** are:

- $0 \le \rho < +\infty$
- $0 \le \varphi \le 2\pi$
- $-\infty < z < +\infty$

In the base $(\vec{e}_\rho, \vec{e}_\varphi, \vec{k})$, the position vector $\overrightarrow{OM}$ is written as:
$\overrightarrow{OM} = \overrightarrow{Om} + \overrightarrow{mM} = \rho\vec{e}_\rho + z\vec{k}$.

$\overrightarrow{OM} = \rho\vec{e}_\rho + z\vec{k}$.

Velocity Vector in Cylindrical Coordinates

$$\vec{v} = \frac{d}{dt}\overrightarrow{OM} = \frac{d}{dt}\left(\rho\vec{e}_\rho + z\vec{k}\right) = \left(\frac{d}{dt}\rho\right)\vec{e}_\rho + \rho\left(\frac{d}{dt}\vec{e}_\rho\right) + \left(\frac{d}{dt}z\right)\vec{k} + z\frac{d}{dt}\vec{k}$$

With : $\frac{d}{dt}\rho = \dot{\rho}$, $\frac{d}{dt}\vec{k} = \vec{0}$, $\frac{d}{dt}z = \dot{z}$

$$\vec{e}_\rho = \cos\varphi\,\vec{i} + \sin\varphi\,\vec{j} \rightarrow \overrightarrow{\frac{d}{dt}e_\rho} = \frac{d}{dt}(\cos\varphi\,\vec{i} + \sin\varphi\,\vec{j}) = \dot{\varphi}(-\sin\varphi\,\vec{i} + \cos\varphi\,\vec{j}) = \dot{\varphi}\,\vec{e}_\varphi$$

$$\vec{v} = \dot{\rho}\,\vec{e}_\rho + \rho\dot{\varphi}\,\vec{e}_\varphi + \dot{z}\vec{k}$$

Acceleration Vector in Cylindrical Coordinates

The acceleration vector **a⃗** is the time derivative of the velocity vector:

$$\vec{a} = \frac{d}{dt}\vec{v} = \frac{d}{dt}(\dot{\rho}\,\vec{e}_\rho + \rho\dot{\varphi}\,\vec{e}_\varphi + \dot{z}\vec{k}) = \left(\frac{d}{dt}\dot{\rho}\right)\vec{e}_\rho + \dot{\rho}\left(\frac{d}{dt}\vec{e}_\rho\right) + \left(\frac{d}{dt}\rho\right)\dot{\varphi}\,\vec{e}_\varphi + \rho\left(\frac{d}{dt}\dot{\varphi}\right)\vec{e}_\varphi + \rho\dot{\varphi}\left(\frac{d}{dt}\vec{e}_\varphi\right) + \left(\frac{d}{dt}\dot{z}\right)\vec{k} + \dot{z}\frac{d}{dt}\vec{k}$$

We have: $\left(\frac{d}{dt}\vec{e}_\varphi\right) = \frac{d}{dt}\left(-\sin\varphi\,\vec{i} + \cos\varphi\,\vec{j}\right) = -\dot{\varphi}(\cos\varphi\,\vec{i} + \sin\varphi\,\vec{j}) = -\dot{\varphi}\,\vec{e}_\rho$

$\frac{d}{dt}\dot{\rho} = \ddot{\rho}$, $\frac{d}{dt}\vec{k} = \vec{0}$, $\frac{d}{dt}\dot{z} = \ddot{z}$; $\frac{d}{dt}\rho = \dot{\rho}$; $\frac{d}{dt}\dot{\varphi} = \ddot{\varphi}$

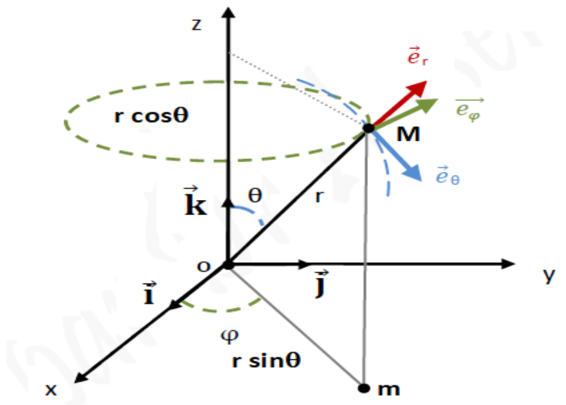
After calculating the derivatives, the acceleration vector becomes:

$$\vec{a} = (\ddot{\rho} - \rho\dot{\varphi}^2)\vec{e}_\rho + (\rho\ddot{\varphi} + 2\dot{\rho}\dot{\varphi})\vec{e}_\varphi + \ddot{z}\vec{k}$$

Position, Velocity, and Acceleration in Spherical Coordinates

In spherical coordinates **(r, φ, θ)**, the position vector is written as:

: $\overrightarrow{OM}=r\vec{e}_r$; Ou : $|\overrightarrow{OM}| = r$



The relationships between the spherical base vectors ($\vec{e}_r, \vec{e}_\theta, \vec{e}_\varphi$) and the Cartesian base vectors are $\vec{i}, \vec{j}, \vec{k}$:

$\vec{e}_r = \sin\theta\cos\varphi\,\vec{i} + \sin\theta\,\sin\varphi\,\vec{j} + \cos\theta\,\vec{k}$
$\vec{e}_\theta = \cos\theta\cos\varphi\,\vec{i} + \cos\theta\,\sin\varphi\,\vec{j} - \sin\theta\,\vec{k}$
$\vec{e}_\varphi = -\sin\varphi\,\vec{i} + \cos\varphi\,\vec{j}$

Velocity Vector in Spherical Coordinates

$$\vec{v} = \frac{d}{dt}\overrightarrow{OM} = \frac{d}{dt}(r\vec{e}_r) = \left(\frac{d}{dt}r\right)\vec{e}_r + r\left(\frac{d}{dt}\vec{e}_r\right)$$

We have : $\frac{d}{dt}r = \dot{r}$,

$$\left(\frac{d}{dt}\vec{e}_r\right) = \left(\frac{\partial}{\partial\theta}\vec{e}_r\right)\frac{d}{dt}\theta + \left(\frac{\partial}{\partial\varphi}\vec{e}_r\right)\frac{d}{dt}\varphi = \dot{\theta}\left(\frac{\partial}{\partial\theta}\vec{e}_r\right) + \dot{\varphi}\left(\frac{\partial}{\partial\varphi}\vec{e}_r\right)$$

$$\left(\frac{\partial}{\partial\theta}\vec{e}_r\right) = \cos\theta\cos\varphi\,\vec{i} + \cos\theta\,\sin\varphi\,\vec{j} - \sin\theta\,\vec{k} = \vec{e}_\theta$$

$$\left(\frac{\partial}{\partial\varphi}\vec{e}_r\right) = -\sin\theta\,\sin\varphi\,\vec{i} + \sin\theta\,\cos\varphi\,\vec{j} = \sin\theta\,(-\sin\varphi\,\vec{i} + \cos\varphi\,\vec{j}) = \sin\theta\,\vec{e}_\varphi$$

$$\rightarrow \left(\frac{d}{dt}\vec{e}_r\right) = \dot{\theta}\,\vec{e}_\theta + \dot{\varphi}\sin\theta\,\vec{e}_\varphi$$

$$\vec{v} = \dot{r}\,\vec{e}_r + r\dot{\theta}\,\vec{e}_\theta + r\dot{\varphi}\,\sin\theta\,\vec{e}_\varphi$$

Acceleration Vector in Spherical Coordinates

After calculating the time derivative of the velocity vector, the acceleration vector in spherical coordinates is:

$$\vec{a} = \frac{d}{dt}\vec{v} = \frac{d}{dt}(\dot{r}\,\vec{e}_r + r\dot{\theta}\,\vec{e}_\theta + r\dot{\varphi}\,\sin\theta\,\vec{e}_\varphi)$$

$\vec{a} = \left(\frac{d}{dt}\dot{r}\right)\vec{e}_r + \dot{r}\,\left(\frac{d}{dt}\vec{e}_r\right) + \left(\frac{d}{dt}r\right)\dot{\theta}\vec{e}_\theta + r\left(\frac{d}{dt}\dot{\theta}\right)\vec{e}_\theta + r\dot{\theta}\left(\frac{d}{dt}\vec{e}_\theta\right)+\left(\frac{d}{dt}r\right)\dot{\varphi}\ sin\theta\vec{e}_\varphi)+$
r$\left(\frac{d}{dt}\ \dot{\varphi}\right)\ sin\theta\vec{e}_\varphi$+ r$\dot{\varphi}\left(\frac{d}{dt}sin\theta\right)\vec{e}_\varphi$+ r$\dot{\varphi}\ sin\theta\left(\frac{d}{dt}\vec{e}_\varphi\right)$

On a : $\left(\frac{d}{dt}\vec{e}_\theta\right)=\left(\frac{\partial}{\partial\theta}\vec{e}_\theta\right)\frac{d}{dt}\theta+\left(\frac{\partial}{\partial\varphi}\vec{e}_\theta\right)\frac{d}{dt}\varphi = \dot{\theta}\left(\frac{\partial}{\partial\theta}\vec{e}_\theta\right) + \dot{\varphi}\left(\frac{\partial}{\partial\varphi}\vec{e}_\theta\right)=-\dot{\theta}(sin\theta\cos\varphi\,\vec{i}+sin\theta\sin\varphi\,\vec{j}$
$+sin\theta\vec{k})+\dot{\varphi}cos\theta(-\sin\varphi\,\vec{i}+\cos\varphi\,\vec{j})=-\dot{\theta}\vec{e}_r + \dot{\varphi}\ cos\theta\ \vec{e}_\varphi$
And we have : $\left(\frac{d}{dt}\vec{e}_\varphi\right)=-\dot{\varphi}\ (\cos\varphi\,\vec{i}+\sin\varphi\,\vec{j})=-\dot{\varphi}\vec{e}_\rho$

We can write $\vec{e}_\rho$ *in* $(\vec{e}_r, \vec{e}_\theta, \vec{e}_\varphi)$ base as:

$\vec{e}_\rho = (\vec{e}_\rho.\vec{e}_r)\,\vec{e}_r+(\vec{e}_\rho.\vec{e}_\theta)\,\vec{e}_\theta+(\vec{e}_\rho.\vec{e}_\varphi)\,\vec{e}_\varphi$

Calculations of scalar products: $(\vec{e}_\rho.\vec{e}_r)$, $(\vec{e}_\rho.\vec{e}_\theta)$ *et* $(\vec{e}_\rho.\vec{e}_\varphi)$

$(\vec{e}_\rho.\vec{e}_r) = (\cos\varphi\,\vec{i}+\sin\varphi\,\vec{j}).(sin\theta\cos\varphi\,\vec{i}+sin\theta\sin\varphi\,\vec{j}+cos\theta\vec{k})=sin\theta$

$(\vec{e}_\rho.\vec{e}_\theta) = (\cos\varphi\,\vec{i}+\sin\varphi\,\vec{j}).(cos\theta\cos\varphi\,\vec{i}+cos\theta\sin\varphi\,\vec{j}-sin\theta\vec{k})=cos\theta$
$(\vec{e}_\rho.\vec{e}_\varphi) = (\cos\varphi\,\vec{i}+\sin\varphi\,\vec{j}).(-\sin\varphi\,\vec{i}+\cos\varphi\,\vec{j})=0$

$\left(\frac{d}{dt}\vec{e}_\varphi\right) = -\dot{\varphi}(sin\theta\vec{e}_r + cos\theta\vec{e}_\theta)$

So : $\vec{a} = \ddot{r}\vec{e}_r + \dot{r}(\dot{\theta}\ \vec{e}\theta + \dot{\varphi}sin\theta\vec{e}\varphi) + \dot{r}\ \dot{\theta}\vec{e}_\theta + r\ddot{\theta}\vec{e}_\theta + r\dot{\theta}(-\dot{\theta}\vec{e}_r + \dot{\varphi}\ cos\theta\ \vec{e}_\varphi)+\dot{r}\dot{\varphi}\ sin\theta\vec{e}_\varphi+$
r$\ddot{\varphi}\ sin\theta\vec{e}_\varphi$+ r$\dot{\varphi}\ \dot{\theta}cos\theta\vec{e}_\varphi$- r$\dot{\varphi}\ sin\theta\dot{\varphi}(sin\theta\vec{e}_r + cos\theta\vec{e}_\theta)$

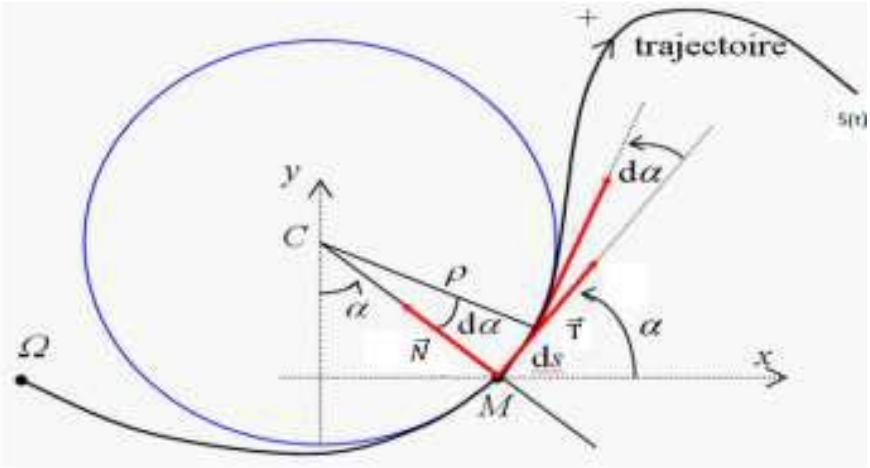
Curvilinear Coordinate Systems

In these systems, we generalize the relationships between position, velocity, and acceleration in curved spaces. The transformations and calculus follow similar principles but require adapting to the specific geometry of the curvilinear coordinates.

Frenet Frame (Frenet Trihedron)

It is useful to introduce a specific frame called the Serret-Frenet trihedron (Frenet frame). It allows the velocity and acceleration to be expressed intrinsically (i.e., expressing these kinematic quantities independently of any particular coordinate system). At point M, we have:

$$\vec{v} = \frac{ds}{dt}\vec{T} = v\vec{T}$$



The Frenet frame is defined by the vectors $\vec{T}, \vec{N}, \vec{B}$, originating from point M. This frame is a tool for studying the local behavior of curves. It is a local frame associated with a point MMM, describing a curve (S).

- $\vec{T}$: Tangent at M(t), oriented in the direction of motion.
- $\vec{N}$: Normal to the trajectory at M(t), perpendicular to $\vec{T}$, oriented toward the direction of curvature.
- $\vec{B}$: Binormal, defined as $\vec{B} = \vec{T} \wedge \vec{N}$

The acceleration vector is given by:

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(v\vec{T}\right) = \frac{dv}{dt}\vec{T} + v\frac{d\vec{T}}{dt}$$

Where $v = \frac{dS}{dt}$ and $\vec{v} = \frac{ds}{dt}\vec{T}$

At time t, at point M on the trajectory, the tangent vector $\vec{T}$ makes an angle α with the direction of the x-axis. At time t+dt, this vector rotates by an angle d α. Therefore, the time derivative of this unit vector is given by:

$$\frac{d\vec{T}}{dt} = \dot{\alpha}\,\vec{N}$$

We have: $ds = \rho d\alpha \Rightarrow d\alpha/dt = \dot{\alpha} = \rho$ dS/dt

Thus, the acceleration vector becomes:

$$\vec{a} = \frac{dv}{dt}\vec{T} + v\frac{d\alpha}{dt}\vec{N}$$

Finally, the expression for the acceleration vector in the Frenet frame is:

$$\vec{a} = \frac{dv}{dt}\vec{T} + \frac{v^2}{\rho}\vec{N}$$

Dynamics of a Particle: Dynamics of a particle is a branch of classical mechanics that studies bodies in motion under the influence of mechanical forces applied to them. It combines statics, which studies the equilibrium of bodies at rest, and kinematics, which studies motion.

1-2.1 Newton's Laws:

1-2.2. Newton's First Law (Principle of Inertia):

A body remains in a state of rest or in uniform motion in a straight line unless acted upon by external forces that compel it to change its state.

1-2.3. Newton's Second Law (Fundamental Principle of Dynamics):

The concept of momentum naturally arises in dynamics: the fundamental relationship in dynamics expresses that the action of an external force on a system leads to a change in its momentum:

$$\vec{F}_{ext} = \frac{d\vec{p}}{dt}$$

1-2.4. Newton's Third Law (Principle of Reciprocal Actions):

If a body A exerts a force $\vec{F}_{A/B}$, then B exerts a force $\vec{F}_{B/A}$ on A, such that $\vec{F}_{A/B} = -\vec{F}_{B/A}$.

1-2.5. Momentum:

The momentum $\vec{p}$ is the product of mass m and the velocity vector $\vec{v}$ of a material point-like body:

$$\vec{p} = m\vec{v}$$

This depends on the reference frame used.

1-2.6. Angular Momentum:The angular momentum $\vec{L}_{/o}$ of point M with respect to a fixed point O is given by:

$$\vec{L}_{/o} = \vec{r} \times \vec{p} = \overrightarrow{OM} \times \vec{p}$$

1-2.7. Torque (Moment of Force):

The torque $\vec{\Gamma}_{/o}$ of point M with respect to a fixed point O, in a reference frame R, is defined as:

$\vec{\Gamma}/o = \vec{r} \times \vec{F} = \overrightarrow{OM} \times \vec{F}$

1-2.8. Theorem of Angular Momentum:

The theorem states that the time derivative of the angular momentum with respect to a point (O) equals the torque applied to this point (O).

Proof:

$$\vec{L}/o = \vec{r} \times \vec{p} = m\,\overrightarrow{OM} \times \vec{p}$$

$$d\frac{\vec{L}/o}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p}) = \frac{d}{dt}\left(\overrightarrow{OM} \times m\vec{v}\right) = m\frac{d}{dt}\left(\overrightarrow{OM} \times \vec{v}\right) = m\frac{d\,\overrightarrow{OM}}{dt} \times \vec{v} + m\,\overrightarrow{OM} \times \frac{d\vec{v}}{dt}$$

$$= m\,\vec{v} \times \vec{v} + m\,\overrightarrow{OM} \times \vec{a} = m\,\overrightarrow{OM} \times \frac{\vec{F}}{m} = \overrightarrow{OM} \times \vec{F} = \vec{\Gamma}/o$$

1-3. Work and Energy:

1-3.1. Power and Work of a Force:

The instantaneous power PPP of a material point M is the scalar product of the resultant forces F\mathbf{F}F acting on M and the velocity v\mathbf{v}v of this point:

$$P = \sum \vec{F} \cdot \vec{v}$$

The unit of power is "Watt" (1 Watt = 1 kg m^2 s^{-3}).

The elementary work done by a material point moving a finite distance $d\overrightarrow{OM}$ is given by:

$$dW = \sum \vec{F} \cdot d\overrightarrow{OM}$$

Since $\vec{v} = \frac{d\overrightarrow{OM}}{dt}$, we can write:

$$dW = \sum \vec{F} \cdot \vec{v}dt = Pdt$$

Kinetic Energy:

The kinetic energy of a particle M of mass m and velocity vector $\vec{v}$ is the scalar E_C, defined as:

$$E_c = \frac{1}{2}mv^2$$

Theorem:

The work of the resultant $\sum F$ of all forces (conservative and non-conservative) applied to a material point M, in any reference frame R, between initial position A and final position B, equals the change in its kinetic energy between A and B.

Proof:

$W = \int \vec{F} \cdot d\vec{r}$

Given: $\vec{F} = m\vec{a} = m\ (d\vec{v}\ /dt)$ and $d\vec{r} = \vec{v}\ dt$, we get:

$W = \int m\ (d\vec{v}\ /dt) \cdot \vec{v}\ dt = (1/2)m \int (d/dt)(\vec{v}^2)\ dt$

As $d(\vec{v}^2)/dt = 2(d\vec{v}/dt) \cdot \vec{v}$,

$W = (1/2)m\vec{v}^2(t_2) - (1/2)m\vec{v}^2(t_1) = \Delta E_C$

Where $\vec{v}(t_2)$ and $\vec{v}(t_1)$ represent the velocities of the particle at times t_2 and t_1, respectively.

1-4. System of N Particles:

- **Internal and External Forces**: Internal forces are exerted between objects within the system, while external forces are applied by objects outside the system.

  Each internal force $\vec{F}_{int}$ (exerted on object A by object B within the system) is balanced by an equal and opposite internal force (exerted by object A on object B).

  Hence, the resultant internal forces sum to zero: $\sum \vec{F}_{int} = \vec{0}$.
- **Total Momentum of N Particles**:

  The total momentum of N particles is given by $\vec{p} = \sum m_i\vec{v}_i$ .The total mass of the system is $M = \sum_i m_i$.

  The center of mass position $\vec{r}_c$ of the system with masses m_1, m_2, ..., m_n and position vectors $\vec{r}_1$, $\vec{r}_2$, ..., $\vec{r}_n$ is defined as:

  $\vec{r}_c = (m_1\vec{r}_1 + m_2\vec{r}_2 + ... + m_n\vec{r}_n) / (m_1 + m_2 + ... + m_n) = (\sum_i m_i\vec{r}_i) / M$

The total momentum of the system is $\vec{P} = M\ \vec{v}_c$, where $\vec{v}_c$ is the velocity of the center of mass. Thus, the total momentum of the system is equal to the momentum of its center of mass.

- **System of N Particles and External Forces**:
  Theorem: A system of N particles behaves like a material point with total mass M located at the center of mass, subject to an external force equal to the sum of the external forces on each particle.

Proof:

The total momentum is $\vec{p}i = mi\ (d^2\vec{r}_i / dt^2) = \sum\vec{F}_{ij} + \vec{F}_{iext}$. Summing over all particles gives: $\sum\vec{p}_i = \sum m_i\ (d^2\vec{r}_i / dt^2) = \sum\sum\vec{F}_{ij} + \sum\vec{F}_{iext}$.

Since the internal forces sum to zero, $\sum\sum\vec{F}_{ij} = 0$, we have:
$\sum\vec{F}_{iext} = \vec{F}_{ext}$ and $M(d^2\vec{r}_c / dt^2) = M\vec{a}_c$, where $\vec{a}_c$ is the acceleration of the center of mass. This result is known as the theorem of the kinetic resultant or the center of inertia theorem.

Angular Momentum of a System of N Particles about Point O:

Defined as $\vec{L}/O = \sum\vec{r}i \times \vec{p}i = \sum mi\vec{r}i \times \vec{v}i$. Using barycentric coordinates with respect to the center of mass:

$\vec{r}_i = \vec{OC} + \vec{CM}$, where $\vec{r}_c$ is the vector to the center of mass, and $\vec{r}'_i$ is the vector to point M in the center of mass reference frame. Thus, $\vec{r}_i = \vec{r}_c + \vec{r}'_i$ and $\vec{v}_i = \vec{v}_c + \vec{v}'_i$.

The angular momentum about point O is:
$\vec{L}/O = M(\vec{r}_c \times \vec{v}_c) + \sum m_i\vec{r}'_i \times \vec{v}'_i$.

Therefore, the angular momentum about point O is the sum of the angular momentum of the center of mass and the angular momentum about the center of mass (intrinsic angular momentum of the system).

Kinetic Energy of a System of N Particles:

The total kinetic energy is the sum of the kinetic energies of the N particles:
$E_C = \sum(1/2)m_iv_i^2$.
Using $\vec{v}i = \vec{v}c + \vec{v}'i$,
$E_C = (1/2)M\vec{v}_c^2 + \sum(1/2)mi\vec{v}'_i{}^2$, where the cross term vanishes.

Thus, the total kinetic energy is the sum of the kinetic energy of the center of mass and the kinetic energy of the particles relative to the center of mass.

1-5. Degrees of Freedom:

To define the position of N particles in space, N position vectors are required, which involves specifying 3N coordinates. The number of independent coordinates needed to uniquely define the system's position is called the system's degrees of freedom, which in this case is 3N.

Examples:

- A particle that can move in one direction has 1 degree of freedom.
- A particle free to move in two dimensions has 2 degrees of freedom.
- A particle free to move in three dimensions has 3 degrees of freedom