Chapter 2: Lagrangian Formalism

2-1. Generalized Coordinates:

Let us consider a mechanical system consisting of N particles moving in three dimensions. The positions of these particles will be denoted as $\vec{r}_I$ (for i=1,2,..., ,N). Each position vector has three components, so 3N coordinates are required to fully specify the configuration of the entire system. Furthermore, let us assume that these 3N coordinates are not independent, meaning they cannot evolve independently, but rather are linked by a certain number K of constraints, which can be expressed as a set of explicit mathematical relations:

$$C_\alpha(\vec{r}_1,...,\vec{r}_N,t)=0 \quad (\alpha = 1,2,\dots,K)$$

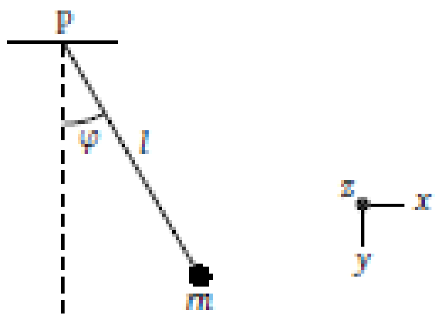
We define n=3N−K as the number of degrees of freedom of the mechanical system under study. The n variables q_α , α = 1, 2, ..., n, which are sufficient to describe the system, are called generalized coordinates.

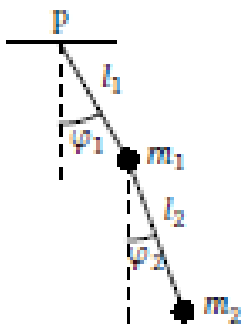
The variables q_α are, in principle, known functions of the particle coordinates and, possibly, of time:

$$q_\alpha=q_\alpha(\vec{r}_1,...,\vec{r}_N,t); \quad (\alpha=1,2,...,n)$$

These are the transformation relations between the generalized coordinates q_α and the $\vec{r}_i$.

Example 1: Simple Pendulum





Simple Pendulum **double Pendulum**

Consider a mass m suspended from a point P (taken as the origin) by a rigid rod of length l and negligible mass.

Writing the constraint equations:

From the setup, the two constraints are:

1. $z = 0$ (since the motion is restricted to a plane)
2. $x^2 + y^2 = l^2$ (the distance between the mass and the origin is constant)

Determining the number of degrees of freedom:

The number of degrees of freedom is: $n = 3 \times 1 - 2 = 1$, Thus, there is only **1 degree of freedom**.

Generalized Coordinates and Transformation Relations:

We can express the position coordinates x and y as:

$$x = l.sin\varphi$$
$$y = l.cos\varphi$$

Knowing the angle φ is enough to determine x and y.

Hence, the generalized coordinate is φ.

The transformation relation is written as: $q = \varphi = arctan(x/y)$

Example 2: Double Pendulum

Now, let's add a second mass m_2, suspended from the mass m_1 of the simple pendulum by another rigid rod of length l_2 and negligible mass, also constrained to move in the xy-plane. Let φ_2 represent the angle made by the second rod relative to the vertical (with φ_1 for the first rod of length l_1).

Let (x_1,y_1,z_1) and (x_2,y_2,z_2) be the Cartesian coordinates of the two masses.

Writing the constraint equations:

The four constraints are:

1. z1=0 (motion of mass 1 is in the plane)
2. z2=0 (motion of mass 2 is in the plane)
3. $x_1^2+y_1^2=l_1^2$ (mass 1 is at a constant distance from the origin)
4. $(x_2-x_1)^2+(y_2-y_1)^2=l_2^2$ (mass 2 is at a constant distance from mass 1)

Determining the number of degrees of freedom:

The number of degrees of freedom is: $n = 3 \times 2 - 4 = 2$ Thus, there are **2 degrees of freedom**.

Generalized Coordinates and Transformation Relations:

We can express the Cartesian coordinates of both masses as follows:

$$x_1 = l_1 sin\varphi_1$$
$$y = l_1 cos\varphi_1$$
$$x_2 = l_1 sin\, \varphi 1 + l_2 sin\varphi_2$$
$$y_2 = l_1 cos\varphi 1 + l_2 cos\varphi_2$$

Knowing φ1 and φ2 is enough to determine x_1, y_1, x_2, and y_2. The transformation relations are written as:

$$q_1 = \varphi_1 = arctan(x_1/y_1)$$
$$q_2 = \varphi_2 = arctan(\frac{x_2 - x_1}{y_2 - y_1})$$

2-2. Functional Variation:

Real Differentials:

Let $f(q_i, t)$ be a function where $i = 1,2, \dots, n$. Its real differential is:

$$df = \sum_i \frac{\partial f}{\partial q_i} dq_i + \frac{\partial f}{\partial t} dt$$

where dq_i and dt are the differentials of q_i and t.

Virtual Differentials (Variation):

The variations of the coordinatesq_i follow laws q_i =q_i (t), making f ultimately a function of time through the variablesq_i.

By definition, the virtual differential of f at time t is expressed as:

$$\delta f = \sum_i \frac{\partial f}{\partial q_i} dq_i$$

with time held constant, $dt = 0$.

Let dW e the work done by force $\vec{F}$ during an infinitesimal displacement $\overrightarrow{dr}$:

$$dW = \sum_{i=1}^{N} \vec{F_i}.d\vec{r_i} = \sum_{i=1}^{N} \left\{ \sum_{\alpha=1}^{n} \vec{F_i} . \frac{\partial \vec{r_i}}{\partial q_\alpha} \right\} dq_\alpha$$

$$dW = \sum_{i=1}^{N} \vec{F}_i . d\vec{r_i} = \sum_{\alpha=1}^{n} \Phi_\alpha . dq_\alpha$$

where Φ_α is called the generalized force. $\Phi_\alpha = \sum_{\alpha=1}^{n} \vec{F}_i \cdot \frac{\partial \vec{r_i}}{\partial q_\alpha}$

$$\Phi_\alpha = \frac{\partial W}{\partial q_\alpha}$$

Important Relations:

1. $\frac{\partial \vec{r_i}}{\partial q_\alpha} = \frac{\partial \dot{\vec{r_i}}}{\partial \dot{q_\alpha}}$

Proof:
Given $\vec{r_i}=\vec{r_i}\,(q_1,\ldots,q_\alpha,\ldots,q_n)$, we can write the differential of $\vec{r_i}$ as:

$$d\vec{r_i} = \frac{\partial \vec{r_i}}{\partial q_1} dq_1 + \cdots + \frac{\partial \vec{r_i}}{\partial q_\alpha} dq_\alpha + \cdots + \frac{\partial \vec{r_i}}{\partial q_n} dq_n$$

$$\frac{d\vec{r_i}}{dt} = \frac{\partial \vec{r_i}}{\partial q_1}\frac{dq_1}{dt} + \cdots + \frac{\partial \vec{r_i}}{\partial q_\alpha}\frac{dq_\alpha}{dt} + \cdots + \frac{\partial \vec{r_i}}{\partial q_n}\frac{dq_n}{dt}$$

Thus,

$$\frac{d\vec{r_i}}{dt} = \frac{\partial \vec{r_i}}{\partial q_1}\dot{q}_1 + \cdots + \frac{\partial \vec{r_i}}{\partial q_\alpha}\dot{q}_\alpha + \cdots + \frac{\partial \vec{r_i}}{\partial q_n}\dot{q}_n$$

Therefore, we conclude that:

$$\frac{\partial \vec{r_i}}{\partial q_\alpha} = \frac{\partial \dot{\vec{r_i}}}{\partial \dot{q_\alpha}}$$

2. $\frac{d}{dt}\left(\frac{\partial \vec{r_i}}{\partial q_\alpha}\right) = \frac{\partial \dot{\vec{r_i}}}{\partial q_\alpha}$

Proof:
Since

$$\frac{d}{dt}\left(\frac{\partial \vec{r_i}}{\partial q_\alpha}\right) = \frac{\partial}{\partial q_\alpha}\left(\frac{d\vec{r_i}}{dt}\right) = \frac{\partial \dot{\vec{r_i}}}{\partial q_\alpha}$$

2-3. Lagrange's Equation:

Let $\vec{F_i}$ be the external force acting on the i-th particle in a system of N particles with n degrees of freedom:

We have:

$$m_i \ddot{\vec{r_i}} = \vec{F_i}$$

Multiplying both sides by $\frac{\partial \vec{r_i}}{\partial q_\alpha}$, we get:

$$m_i \ddot{\vec{r_i}} \frac{\partial \vec{r_i}}{\partial q_\alpha} = \vec{F_i} \frac{\partial \vec{r_i}}{\partial q_\alpha} \qquad (1)$$

On the other hand, we also have:

$$\frac{d}{dt}\left(\dot{\vec{r_i}} \frac{\partial \vec{r_i}}{\partial q_\alpha} \right) = \ddot{\vec{r_i}} \frac{\partial \vec{r_i}}{\partial q_\alpha} + \dot{\vec{r_i}} \frac{d}{dt}\left(\frac{\partial \vec{r_i}}{\partial q_\alpha} \right)$$

which implies:

$$\ddot{\vec{r_i}} \frac{\partial \vec{r_i}}{\partial q_\alpha} = \frac{d}{dt}\left(\dot{\vec{r_i}} \frac{\partial \vec{r_i}}{\partial q_\alpha} \right) - \dot{\vec{r_i}} \frac{d}{dt}\left(\frac{\partial \vec{r_i}}{\partial q_\alpha} \right) \qquad (2)$$

Multiplying Equation (2) by m_i

$$m_i \ddot{\vec{r_i}} \frac{\partial \vec{r_i}}{\partial q_\alpha} = m_i \frac{d}{dt}\left(\dot{\vec{r_i}} \frac{\partial \vec{r_i}}{\partial q_\alpha} \right) - m_i \dot{\vec{r_i}} \frac{d}{dt}\left(\frac{\partial \vec{r_i}}{\partial q_\alpha} \right) = \vec{F_i} \frac{\partial \vec{r_i}}{\partial q_\alpha} \qquad (3)$$

Let E_c be the kinetic energy of a system of N particles:

$$E_c = \frac{1}{2} \sum_{i=1}^{N} m_i \dot{\vec{r_i}}^2$$

For the partial derivative of E_c with respect to q_α:

$$\frac{\partial E_c}{\partial q_\alpha} = \sum_{i=1}^{N} m_i \dot{\vec{r_i}} . \frac{\partial \dot{\vec{r_i}}}{\partial q_\alpha} \qquad (4)$$

And for the partial derivative of E_c with respect to $\dot{q_\alpha}$:

$$\frac{\partial E_c}{\partial \dot{q}_\alpha} = \sum_{i=1}^{N} m_i \dot{\vec{r_i}} . \frac{\partial \dot{\vec{r_i}}}{\partial \dot{q}_\alpha} = \sum_{i=1}^{N} m_i \dot{\vec{r_i}} . \frac{\partial \vec{r_i}}{\partial q_\alpha} \qquad (5)$$

Substituting Equations (4) and (5) into Equation (3) gives:

$$\frac{d}{dt}\left(\frac{\partial E_c}{\partial \dot{q}_\alpha}\right) - \frac{\partial E_c}{\partial q_\alpha} = \Phi_\alpha \qquad (6)$$

Equation (6) is the Lagrange equation. There is one Lagrange equation for each generalized coordinate q_α . Thus, for a system with n degrees of freedom, we have a system of n Lagrange equations.

2-4. Generalized Momentum:

The quantity p_α defined as:

$$p_\alpha = \frac{\partial E_c}{\partial q_\alpha}$$

is called the generalized momentum associated with the generalized coordinate q_α.

2-5. The Lagrangian:

In conservative systems, the applied force on the system derives from a potential U, which gives:

$$\Phi_\alpha = -\frac{\partial U}{\partial q_\alpha}$$

Thus, the Lagrange equation becomes:

$$\frac{d}{dt}\left(\frac{\partial E_c}{\partial \dot{q}_\alpha}\right) - \frac{\partial E_c}{\partial q_\alpha} = -\frac{\partial U}{\partial q_\alpha}$$

or equivalently:

$$\frac{d}{dt}\left(\frac{\partial (E_c - U)}{\partial \dot{q}_\alpha}\right) - \frac{\partial (E_c - U)}{\partial q_\alpha} = 0$$

We define the Lagrangian function L as:

$$L = E_c - U$$

where L is known as the Lagrangian of the system.

Therefore, the Lagrange equation for a conservative system is:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) - \frac{\partial L}{\partial q_\alpha} = 0$$

2-6. Curvilinear Coordinate System:

Unlike the Cartesian coordinate system, the reference frame of a curvilinear coordinate system is not fixed but depends on the position of a point in space.

A curvilinear coordinate system is called an orthogonal system if the coordinate lines are orthogonal to each other at every point M in space. The three base vectors are tangent to the coordinate lines at M, resulting in these vectors being orthogonal to each other at every point in space.

Examples of curvilinear coordinate systems:

- Cylindrical coordinates
- Spherical coordinates

Kinetic energy of a particle of mass m in curvilinear coordinates:

$$E_c = \frac{1}{2} m \left(\frac{ds}{dt}\right)^2$$

Where s is the curvilinear abscissa, defined such that : $ds^2 = d\vec{r}.d\vec{r}$

In Cartesian coordinates: $d\vec{r} = dx\vec{\imath} + dy\vec{\jmath} + dz\,\overrightarrow{k}$

$$ds^2 = \left(dx\vec{\imath} + dy\vec{\jmath} + dz\,\overrightarrow{k}\right).\left(dx\vec{\imath} + dy\vec{\jmath} + dz\,\overrightarrow{k}\right) = (dx)^2 + (dy)^2 + (dz)^2$$

In cylindrical coordinates (see Chapter 1):

$$\vec{r} = \overrightarrow{OM} = \rho\overrightarrow{e_\rho} + z\vec{k} \Rightarrow d\vec{r} = d\rho\overrightarrow{e_\rho} + \rho d\overrightarrow{e_\rho} + dz\vec{k} = d\rho\overrightarrow{e_\rho} + \rho d\varphi\vec{e}_\varphi + dz\vec{k}$$

Thus:

$$d\vec{r} = d\rho\overrightarrow{e_\rho} + \rho d\varphi\vec{e}_\varphi + dz\vec{k}$$

$$ds^2 = d\vec{r}.d\vec{r} = \left(d\rho\overrightarrow{e_\rho} + \rho d\varphi\vec{e}_\varphi + dz\vec{k}\right).\left(d\rho\overrightarrow{e_\rho} + \rho d\varphi\vec{e}_\varphi + dz\vec{k}\right)$$

$$= (d\rho)^2 + (\rho d\varphi)^2 + (dz)^2$$

In spherical coordinates (see Chapter 1):

$\vec{r} = \overrightarrow{OM} = r\vec{e}_r \Rightarrow d\vec{r} = dr\vec{e}_r + rd\vec{e}_r$

$\vec{e}_r = sin\theta\cos\varphi\,\vec{\imath} + sin\theta\,\sin\varphi\,\vec{j} + cos\theta\vec{k} \Rightarrow d\vec{e}_r = \frac{\partial\vec{e}_r}{\partial\theta}d\theta + \frac{\partial\vec{e}_r}{\partial\varphi}d\varphi = d\theta\vec{e}_\theta + sin\theta.d\varphi\vec{e}_\varphi$

$$ds^2 = d\vec{r}.d\vec{r} = \left(dr\vec{e}_r + rd\theta\vec{e}_\theta + rsin\theta.d\varphi\vec{e}_\varphi\right).\left(dr\vec{e}_r + rd\theta\vec{e}_\theta + rsin\theta.d\varphi\vec{e}_\varphi\right)$$
$$= (dr)^2 + (rd\theta)^2 + (rsin\theta.d\varphi)^2$$

In general, we write: $ds^2 = \sum_{i,j} g_{ij} q_i q_j$

Where g_{ij} is called the metric (metric tensor).

Coordonnées cartésiennes	Coordonnées cylindriques	Coordonnées sphériques
$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 sin^2\theta \end{pmatrix}$

g_{ij} is a symmetric tensor that allows us to define the kinetic energy E_c by:

$$E_c = \frac{m}{2}\sum_{i,j} g_{ij}\dot{q}_i\dot{q}_j$$

2-7. Holonomic and Non-Holonomic Constraints:

A holonomic constraint is a constraint that can be expressed in the form:

$$f(\overrightarrow{r_1}, \dots, \overrightarrow{r_N}, t) = 0$$

If it cannot, it is classified as non-holonomic. If the equation of the holonomic constraint depends on time, such that

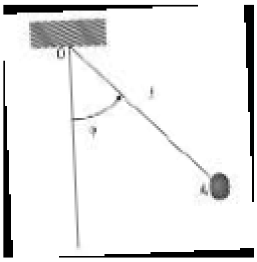
$$\frac{\partial f}{\partial t} \neq 0,$$

it is called rheonomic. If it does not depend on time, such that

$$\frac{df}{dt} = 0$$

it is called scleronomic.

Example:



For a simple pendulum, the constraint equations are:

1. $z = 0$
2. $x^2 + y^2 = l^2$

Both constraints are holonomic and scleronomic.

Example 1: In spherical coordinates, the Lagrangian of a particle subjected to a potential $U(r,\theta,\varphi)$ is written as:

$$L = \frac{m}{2}\left(\dot{r}^2 + \left(r\dot{\theta}\right)^2 + (rsin\theta.\dot{\varphi})^2\right) - U(r,\theta,\varphi)$$

Lagrange equations: 3 equations correspond to the variables r, θ, φ.

$$\begin{cases} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0 & (1) \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 & (2) \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) - \frac{\partial L}{\partial \varphi} = 0 & (3) \end{cases}$$

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r} \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = m\ddot{r} \qquad and \quad \frac{\partial L}{\partial r} = mr\dot{\theta}^2 + mr\dot{\varphi}^2 sin^2\theta - \frac{\partial U}{\partial r}$$

$$\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 2mr\dot{\theta} + mr^2\ddot{\theta} \qquad and \quad \frac{\partial L}{\partial \theta} = 2mr\dot{\varphi}^2 sin\theta cos\theta - \frac{\partial U}{\partial \theta}$$

$$\frac{\partial L}{\partial \dot{\varphi}} = mr^2\dot{\varphi}\, sin^2\theta \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) = 2mr\dot{r}\dot{\varphi}\, sin^2\theta + mr^2\ddot{\varphi}\, sin^2\theta + 2mr^2\dot{\varphi} sin\theta cos\theta$$

$$and \quad \frac{\partial L}{\partial \varphi} = -\frac{\partial U}{\partial \varphi}$$

$$\Rightarrow \begin{cases} m\ddot{r} - mr\dot{\theta}^2 - mr\dot{\varphi}^2 sin^2\theta + \frac{\partial U}{\partial r} = 0 & (1) \\ 2mr\dot{\theta} + mr^2\ddot{\theta} - 2mr\dot{\varphi}^2 sin\theta cos\theta + \frac{\partial U}{\partial \theta} = 0 & (2) \\ 2mr\dot{r}\dot{\varphi}\, sin^2\theta + mr^2\ddot{\varphi}\, sin^2\theta + 2mr^2\dot{\varphi} sin\theta cos\theta \ - \frac{\partial L}{\partial \varphi} = 0 & (3) \end{cases}$$

Applications:

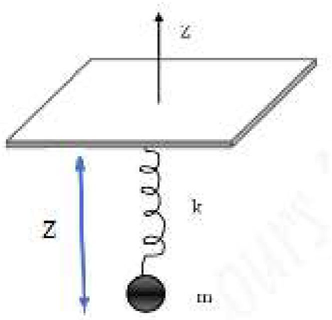
1 - Particle in a Gravitational Field:

A particle of mass mmm in a gravitational field has a potential energy U=mgz, where z measures its height, and g is the acceleration due to gravity. We set z=0; U=0 as the origin of gravitational potential energy. (x,y,z) are the coordinates of the material point in space.

Describe the motion of this material point using the Lagrangian formalism.

2 - Particle Suspended by a Spring:

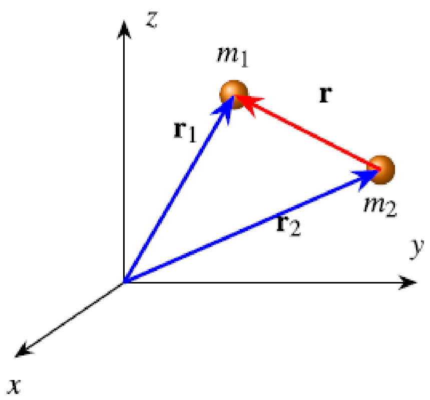
A particle of mass m is suspended by a spring with constant k in a gravitational field. Near the Earth's surface, it is assumed that only vertical motion is allowed, with no movement in the x and y directions, resulting in only one degree of freedom. The best choice of generalized coordinates is the Cartesian coordinate z (with U=mgzU). The motion is purely vertical.



Describe the motion of m using the Lagrangian formalism.

3 - Two-Body Problem:

This is the simplest closed physical system that exists. Two particles, with masses $m_1$and m_2, have instantaneous positions $\overrightarrow{r_1}$ and $\overrightarrow{r_2}$ and interact through a potential: U($\overrightarrow{r_1}$, $\overrightarrow{r_1}$)=U($\overrightarrow{r_1}$−$\overrightarrow{r_1}$).



The relative coordinate is:

$$\vec{r} = \overrightarrow{r_1} - \overrightarrow{r_2} ,$$

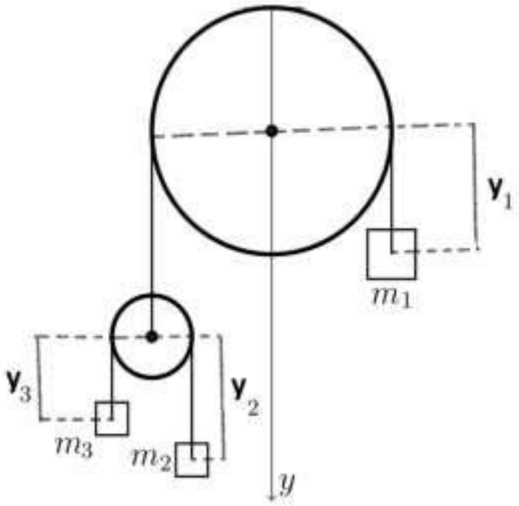
and the center of mass coordinate is:

$$\vec{r_c} = \frac{m_1\vec{r_1}+m_2\vec{r_2}}{m_1+m_2}$$

Write $\vec{r_1}$ and $\vec{r_2}$ as functions of $\vec{r}$ and $\vec{r}_c$

Write the Lagrangian in terms

Generalized Atwood Machine: Consider the system shown in which a mass m_1 is connected, via pulley A, to a second pulley B with mass M. Pulley B, in turn, connects two other masses m_2 and m_3. The masses of the ropes and pulleys are negligible, and gravity g acts downward. The vertical positions of the three masses are y_1, y_2, and y_3, and movement in any direction other than vertical is ignored. Pulley A is fixed, and the ropes have constant lengths.



1. Show that this problem has two degrees of freedom.
2. Write the Lagrangian of this system, using are $y_1$and y_2 as generalized coordinates.
3. Find an explicit expression for the accelerations $\ddot{y}_1$ and $\ddot{y}_2$ in terms of g and the three masses. Under what condition is $\ddot{y}_1$ zero?