Chapter II

Linear vibratory systems with a single degree of freedom

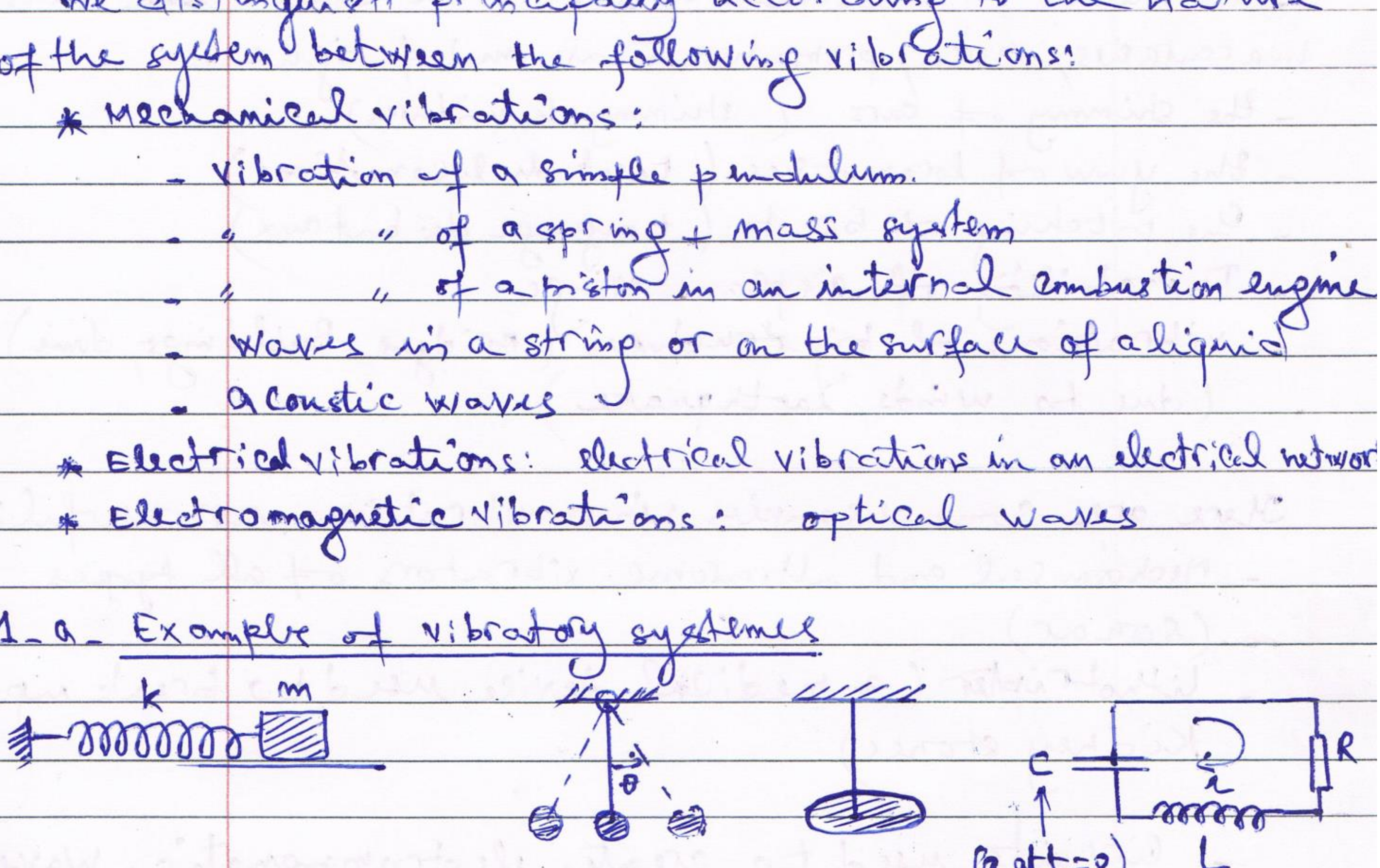
II_1_ vibratory phenomena

Vibration or vibratory motion refers to any movement (any change of the system state) that repeats itself periodically or nearly periodically over time. Thus these are back_and forth movements that occur around a stable equilibrium point (free vibrations).

We distinguish principally according to the nature of the system between the following vibrations:

* Mechanical vibrations:
  - vibration of a simple pendulum.
  - " " of a spring + mass system
  - " " of a piston in an internal combustion engine
  - waves in a string or on the surface of a liquid
  - acoustic waves
* Electrical vibrations: electrical vibrations in an electrical network
* Electromagnetic vibrations: optical waves

1_a_ Example of vibratory systemes



1_b_ periodic vibrations

Each vibration whose physical quantities, (which caracterizing the system) take equal values at equal time intervals. the smallest interval is called the period of vibration and denoted by T. However the frequency express the number of periods per unit of time. It is denoted by f where:

$$f = \frac{1}{T}$$

vibrations are they disturbing or useful ?

for a long time, the study of machine and structure vibrations focused almost exclusively on attenuating and, if possible eliminating them. While this preoccupation remains essential, it is no longer the only one. Increasingly, we are now designing machines and devices that use mechanical vibrations to perform the desired function.

Here are some examples where vibrations are a disturbing element and must be eliminated.

- vibrations in machines where some components cause inaccuracies, noise, premature wear and fatigue.
- the shimmy of cars (shimmy ds voitures)
- the yaw of locomotives (Lacet ds locomotives)
- the pitching of boats (tangage ds bateaux)
- Instability of airplane wings
- vibrations of big structures (Bridges, Buildings, dams) (due to winds, Earthquake)

Here are some examples where vibrations are useful:

- Mechanical and ultrasonic vibrators of all types (sonar)
- lithotripter (a medical device used to break up Kidney stones)
- circuits used to create electromagnetic waves for telecommunication purposes.

Disturbing = مشوش

1_C_ free vibrations (3)

Any vibration that appears in the studied system after it has been displaced from its stable equilibrium position and then released with or without initial velocity.

Simple harmonic motion (simple harmonic oscillation (sho)) is the most simple free vibration

- Simple harmonic Oscillation (sho)

Any vibration that has a sinusoidal form i.e:

$$x(t) = x_0 \cos(\omega_0 t + \varphi) = x_0 \sin(\omega_0 t + \varphi_0)$$

where $\varphi_0 = \varphi + \pi/2$

- velocity and acceleration of sho

$$v(t) = \frac{dx}{dt} = \dot{x}(t) = -x_0 \omega_0 \sin(\omega_0 t + \varphi)$$
$$= x_0 \omega_0 \cos(\omega_0 t + \varphi + \pi/2)$$

which means that elongation and velocity are in phase quadrature, meaning the velocity leads the elongation by a phase of $\pi/2$.

$$\gamma(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2} = \ddot{x} = -\omega_0^2 x_0 \cos(\omega_0 t + \varphi) =$$
$$= x_0 \omega^2 \cos(\omega_0 t + \varphi + \pi) = -\omega_0^2 x(t)$$

$\Rightarrow$ acceleration and elongation are in phase opposition i.e $\gamma(t)$ leads elongation by a phase of π.

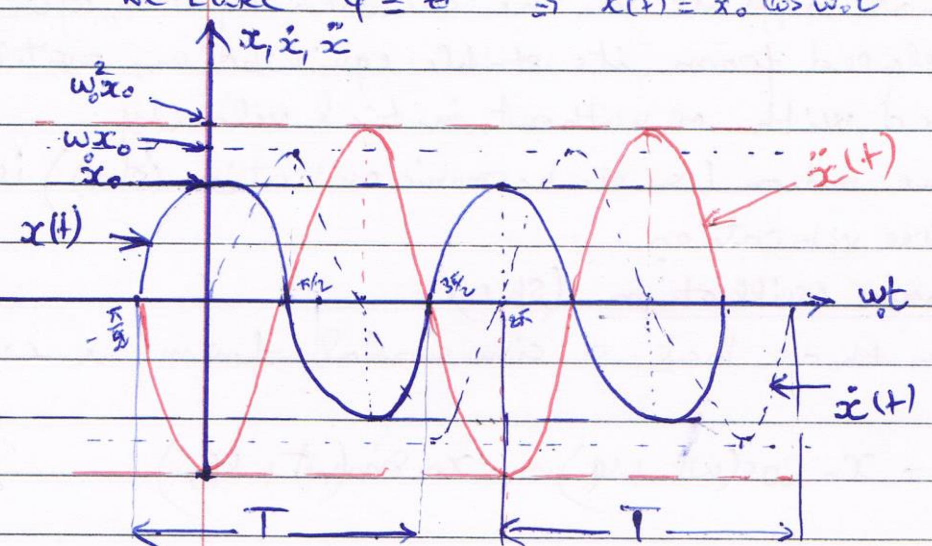
The relation between γ and x find above gives the following differential equation:

$$\boxed{\ddot{x} + \omega_0^2 x = 0} \quad \ldots\ldots (1)$$

It's the differential equation of a sho where ω_0 is the angular frequency of this sho i.e the square root of the constant near x express the angular frequency. So we say that any vibration verifying the equation (1) is a sho (non damping sho).

- Graphical representation of x, $\dot{x}$ and $\ddot{x}$ (4)

We take $\varphi = 0 \Rightarrow x(t) = x_0 \cos w_0 t$



Generalization

Any system with single degree of freedom described by the generalized coordinate q and satisfying a differential equation of the type:

$$\ddot{q} + w^2 q = 0$$

is a system performing simple harmonic oscillation (sho)

The coordinate q can be, in accordance with the nature of the system and its type of vibration, a spring deformation denoted x, or angular displacement α, current intensity i, potential difference v or gaz pressure p ... etc

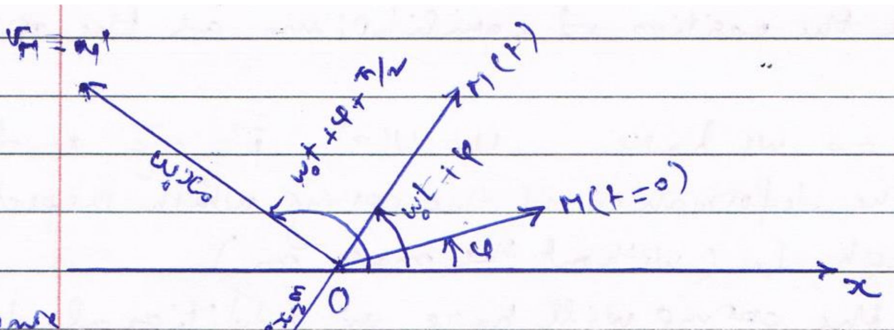
II-2- Fresnal representation of sho

Consider a vector $\vec{r} = \vec{OM}$ where

$|\vec{OM}| = x_0$; x_0 = amplitude of sho

$\vec{OM}$ = a vector which rotating with a constant angular velocity w around a certain center denoted O in counterclockwise direction

Consider an axis Ox, which indicate the phase origine so the vector $\vec{OM}$ make an angle $wt + \varphi$ at time t and consequently an angle φ at $t = 0$ (see the fig below)



The figure

The figure shows that at any time t we have:

$$x(t) = \text{proj}_{/ox}\left(\overrightarrow{OM}\right)$$: $x(t)$ instantaneous elongation of sho.

On the same diagramme we can represente v_M and γ_M where:

$$v_M \longrightarrow \overrightarrow{OM'} \quad \text{and} \quad \gamma_M \longrightarrow \overrightarrow{OM''}$$

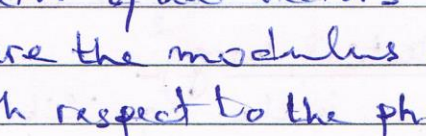
so that:

$$\overrightarrow{OM'} \perp \overrightarrow{OM} \quad \text{and } \overrightarrow{OM''} \text{ in opposite of } \overrightarrow{OM}$$

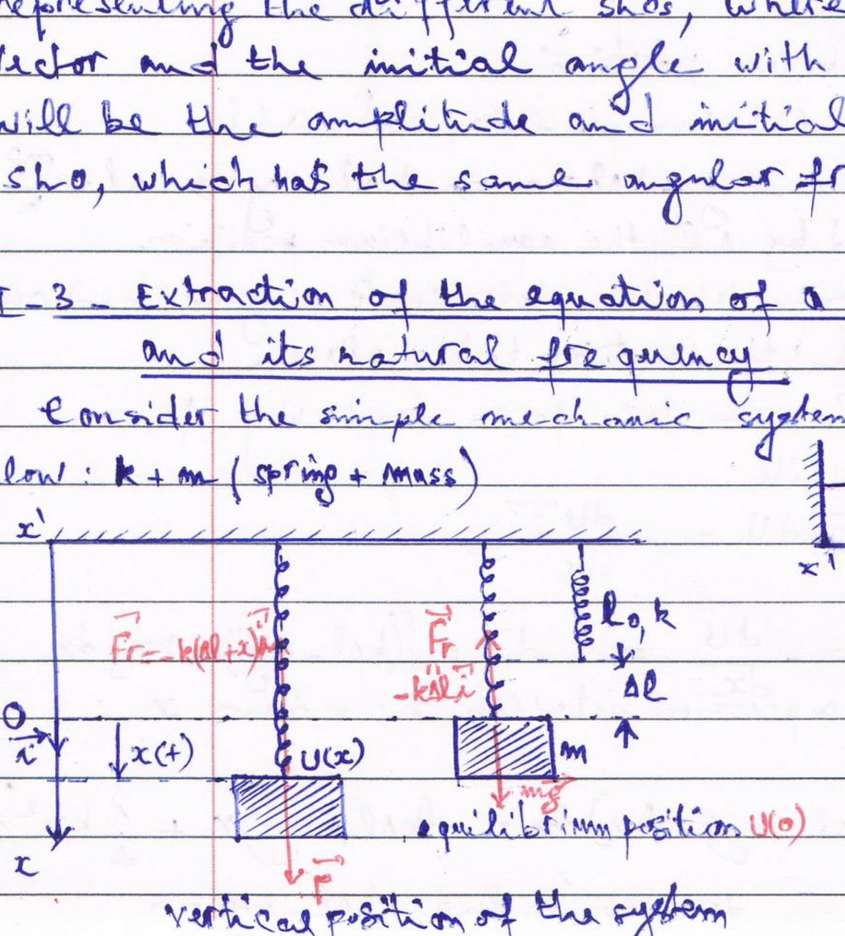
This representation can help us to determine a sum of a number of shos, which have the same angular frequency ω_0 by calculating the resulting vector of all vectors representing the different shos, where the modulus of this vector and the initial angle with respect to the phas axis will be the amplitude and initial phase of the resulting sho, which has the same angular frequency ω_0.

II-3 - Extraction of the equation of a simple harmonic oscillator and its natural frequency

Consider the simple mechanic system represented in the figure below: k + m (spring + mass)



Horizontal position of the system



vertical position of the system

let's suppose the position of equilibrium as the origine of coordinates

so at : $x=0$ we have $U=U(0)$, $\vec{p}=m\vec{g}$ and $\vec{F_r}=-k\Delta l\vec{i}$

where Δl is the deformation of the spring whith respect to its natural lengh l_0 (without the mass m).

in motion the spring will have an additional deformation $x(t)$ then:

$$\vec{F_r}=-k(\Delta l+x)\vec{i}$$

The potential energy in this position will be $U(x)$.

3_1_ SHO energy

a_ Kinetic energy

In the following paragraphe we demontrate that the solution of the differential equation governing the oscillations of the system has the form below:

$$x(t)=x_0\cos(\omega_0 t+\varphi) \quad \text{where} \quad \omega_0=\sqrt{\frac{k}{m}}$$

So the Kinetic energy of the system will be :

$$T=\frac{1}{2}m\dot{x}^2=\frac{1}{2}m\omega_0^2x_0^2\sin^2(\omega t+\varphi)$$

b_ Potential energy

$x=0$ is the equilibrium position

The condition of equilibrium is expressed as :

$$\vec{p}+\vec{F_r}=0 \Rightarrow mg-k\Delta l=0 \Rightarrow k\Delta l=mg \Rightarrow \Delta l=\frac{mg}{k}$$

The spring is elongated by Δl the equilibrium position.

this position is choosen as the coordinate origine ($x=0$) on the axis xx' on which the motion takes place.

knowing that $\vec{p}$ and $\vec{F_r}$ are two fores deriving from a potential then we write :

$$\vec{p}+\vec{F_r}=-\vec{grad}\,U=-\frac{dU}{dx}\vec{i}$$

$\Rightarrow$

$$mg-k(\Delta l+x)=-\frac{dU}{dx} \Rightarrow dU=[(k\Delta l-mg)+kx]dx$$

by integrating this expression between $x=0$ and x :

$$\int_{U(0)}^{U(x)}dU=\int_0^x[(k\Delta l-mg)+kx]dx=(k\Delta l-mg)x+\frac{1}{2}kx^2$$

for $x=0$ we find $k\Delta l-mg=0$

The equilibrium condition is:

$$\left.\frac{dU}{dx}\right|_{x=0} = 0 \Rightarrow (k\Delta l - mg) - kx\Big|_{x=0} = 0 \Rightarrow$$

$$k\Delta l - mg = 0$$

The same condition found previously.

Finally, we find that:

$$U(x) = \frac{1}{2}kx^2 + U(0)$$

The potential is always calculated up to a constant $U(0)$, so with a suitable choice of the origin of potentials, let put $U(0) = 0$ and then we obtain:

$$U(x) = \frac{1}{2}kx^2$$

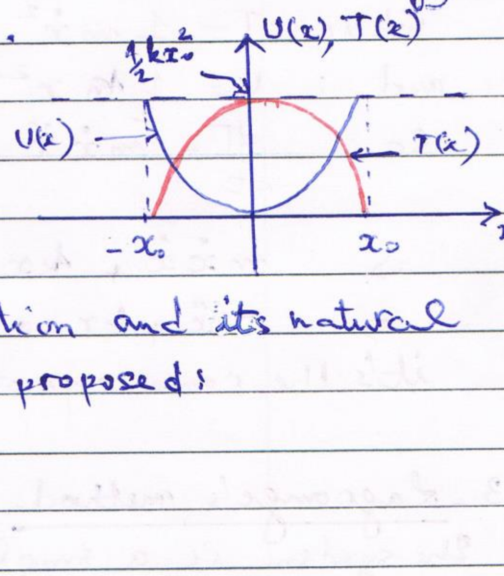
It's easy to show that at any time t we have:

$$E = T + U = \frac{1}{2}kx_0^2 = cte$$

Thus, the syst is conservative because its total energy is maintained constant all the time.

<u>Result</u>

There is a complete exchange of system's kinetic energy and potential energy.



In order to extract the motion equation and its natural angular frequency several methodes are proposed:

1- Newton's method
2- Conservation energy method
3- Lagrange's method
4- Rayleigh method

1- Newton's method

a- at the equilibrium position we have:

$$\sum \vec{F_i} = 0 \Rightarrow \vec{P} + \vec{F_r} = 0 \Rightarrow \boxed{mg = k\Delta l}$$

it's the equilibrium condition found previously

b- In motion:

$$\sum \vec{F_i} = m\vec{\gamma} = m\ddot{\vec{x}}(t) \Rightarrow m\vec{g} + \vec{F_r} = m\ddot{\vec{x}}$$

By projection of this equation on the axis xx' we obtain:

$$mg - k(\Delta l + x) = m\ddot{x}$$ (8)

taking into account the equilibrium condition that we have found previously and dividing the two sides of the above equation by m we get:

$$\boxed{\ddot{x} + \frac{k}{m}x = 0}$$

This equation has the form $\ddot{q} + \omega_0^2 q = 0$

So:

$$\omega_0 = \sqrt{\frac{k}{m}}$$

ω_0 is the natural angular frequency of the motion

2_ Conservation energy method

If the system is conservative (no friction forces) then the total energy will be constant i.e: $E = cte$

$$\Rightarrow \quad E = T + U = cte \Rightarrow \frac{dE}{dt} = 0 \Rightarrow \frac{dT}{dt} + \frac{dU}{dt} = 0$$

where $T = \frac{1}{2}m\dot{x}^2$

and : $U = \frac{1}{2}kx^2$

So: $\frac{dT}{dt} = m\dot{x}\ddot{x}$ and $\frac{dU}{dt} = kx\dot{x}$

$$\Rightarrow \quad m\dot{x}\ddot{x} + kx\dot{x} = 0 \quad \Rightarrow \dot{x}(m\ddot{x} + kx) = 0$$

$$\Rightarrow \quad m\ddot{x} + kx = 0 \quad \text{and finally} \quad \ddot{x} + \frac{k}{m}x = 0$$

it's the same equation previously obtained.

3_ Lagrange's method

The system is a single degree of freedom. Then it's described by a single generalized coordinate ($q = x$)

The Lagrange's equation of a single degree of freedom system is of the following form:

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{x}}\right] - \frac{\partial \mathcal{L}}{\partial x} = 0 \quad \text{(conservative system)}$$

(حيث أن)

whereas: $\mathcal{L}(x, \dot{x}) = T - U = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$

by derivating $\mathcal{L}$ corresponding to the Lagrange's equation above we find again:

$$\ddot{x} + \frac{k}{m}x = 0$$

4 - Rayleight's method

This method involves calculating the natural angular frequency of the system's vabritions without solving the differential equation. It is based on the fact that, at the system's equilibrium position, the kinetic energy is maximale while the potential energy is zero. Conversely, the extremal position is caracterized by zero kinetic energy and maximal potential energy. And since the system is assumed to be conservative, we can write:

$$E = T + U = T_{max} = U_{max} = cte$$

the motion is supposed to a sho then:

$$x(t) = x_0 \cos(\omega_0 t + \varphi) \Rightarrow \dot{x}(t) = -x_0 \omega \sin(\omega_0 t + \varphi)$$

$$\Rightarrow T = \frac{1}{2} m x_0^2 \omega_0^2 \sin^2(\omega_0 t + \varphi)$$

so:

$$T_{max} = \frac{1}{2} m x_0^2 \omega_0^2$$

$$U(x) = \frac{1}{2} k x^2 = \frac{1}{2} k x_0^2 \cos^2(\omega_0 t + \varphi)$$

$$\Rightarrow U_{max} = \frac{1}{2} k x_0^2$$

Therefore:

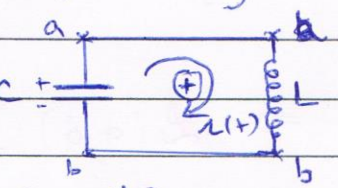
$$T_{max} = U_{max} \Rightarrow \frac{1}{2} m x_0^2 \omega_0^2 = \frac{1}{2} k x_0^2$$

and we get:

$$\omega_0 = \sqrt{\frac{k}{m}}$$

II - 4 - Electrical circuit LC

let's have the following LC circuit:



The differential equation governing the vibrations in the circuit can be established by the following methods:

a) Kirchhoff's method

The voltage Kirchhoff's law gives:

$$\sum V_i = 0 \Rightarrow \underbrace{(V_a - V_b)}_{V_L(t)} + \underbrace{(V_b - V_a)}_{V_C(t)} = 0$$

where:

$$\left.\begin{aligned} V_L(t) &= V_a - V_b = L\frac{di}{dt} \\ V_C(t) &= V_b - V_a = \frac{1}{C}\int i\,dt \end{aligned}\right\} \Rightarrow L\frac{di}{dt} + \frac{1}{C}\int i\,dt = 0 \;\ldots (2)$$

By differentiation of (2) with respect to time we find:

(10)

$$\boxed{\ddot{i} + \frac{1}{LC} i = 0}$$

the same equation above can be expressed through the quantity of electrical charge in the capacitor C (denoted q)

the relation between $q(t)$ and $i(t)$ is:

$$q(t) = \int i\,dt \Rightarrow \dot{q}(t) = i(t) \Rightarrow \ddot{q} = \dot{i}(t)$$

then we get:

$$\boxed{\ddot{q} + \frac{1}{LC} q = 0}$$

b- Energy method

Two types of energy are exchanged in the circuit:

- electrostatic energy in C : $W_e = \frac{1}{2}\frac{1}{C} q^2$
- magnetic energy in L : $W_m = \frac{1}{2} L i^2 = \frac{1}{2} L \dot{q}^2$

then (supposing that the totale energy is conserved):

$$\frac{dE}{dt} = \frac{d(W_e + W_m)}{dt} = 0 \Rightarrow \boxed{\ddot{q} + \frac{1}{LC} q = 0}$$

It's the same diff equation established earlier

c- Larange's method

$$T = W_m = \frac{1}{2} L \dot{q}^2$$
$$U = W_e = \frac{1}{2}\frac{1}{C} q^2 \Rightarrow \mathcal{L}(q, \dot{q}) = T - U$$

the Lagrange's equation is written as:

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{q}}\right] - \frac{\partial \mathcal{L}}{\partial q} = 0 \quad \text{(single degree of freedom)}$$

by derivating the above equation we find again:

$$\boxed{\ddot{q} + \frac{1}{LC} q = 0}$$

This is a typically equation of a sho. then the natural angular frequency is:

$$\omega_0 = \sqrt{\frac{1}{LC}} = \frac{1}{\sqrt{LC}}$$

II-5- solution of the differential equation (11)

In the two systems (mech or ele) the differential equation has the same form:

$$\ddot{q} + \omega_0^2 q = 0 \quad \text{where} \quad \begin{cases} \text{mech } q(t) = x(t) \text{ and } \omega_0 = \sqrt{\frac{k}{m}} \\ \text{ele: } q(t) = q(t) \text{ or } i(t) \text{ and } \omega_0 = \sqrt{\frac{1}{LC}} \end{cases}$$

this is a linear diff equation (homogeneous and of 2^{nd} ordre)

$\Rightarrow$

$$q(t) = A_1 e^{r_1 t} + A_2 e^{r_2 t}$$

where r_1 and r_2 the roots of the caracteristic equation:

$$r^2 + \omega_0^2 = 0 \Rightarrow r_{1,2} = \mp j\omega_0 \quad (j = \sqrt{-1})$$

$$\Rightarrow \quad q(t) = A_1 e^{j\omega_0 t} + A_2 e^{-j\omega_0 t} \in \mathbb{R}$$

A_1 and A_2 two conjugal complex numbers

It is easy to verifie that the solution above can be ~~puta~~ in the form:

$$q(t) = A\cos(\omega_0 t + \varphi)$$

We see in this solution that ω_0 is effectivilly the angular frequency. It is called the natural angular frequency of free oscillations in the system (النبض الذاتي أو الطبيعي)

A, φ two constant of integration which can be obtained from the initial conditions in the syst.

for example if the system is the mechanical one then

$$q(t) = x(t), \quad \omega_0 = \sqrt{\frac{k}{m}}$$

if the I.C are: $x(0) = x_0$

$\dot{x}(0) = 0$

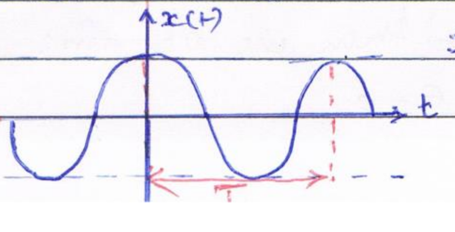
So: $x(t) = A\cos(\omega_0 t + \varphi) \Rightarrow x(0) = A\cos\varphi = x_0$

$\dot{x}(t) = -A\omega_0 \sin(\omega_0 t + \varphi) \Rightarrow \dot{x}(0) = -A\omega_0 \sin\varphi = 0$

$\Rightarrow \sin\varphi = 0 \Rightarrow \varphi = 0 \Rightarrow A = x_0 \Rightarrow x(t) = x_0 \cos(\sqrt{\frac{k}{m}}\, t)$

the solution is then:

$$x(t) = x_0 \cos(\sqrt{\frac{k}{m}}\, t) \Rightarrow T_0 = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m}{k}}$$



Condensider the electrical circuit LC (12)

$q(t) = q(t) \Rightarrow q(t) = Q\cos(\omega_0 t + \varphi)$ where $\omega_0 = \frac{1}{\sqrt{LC}}$

generalized coordinate ↑ charge

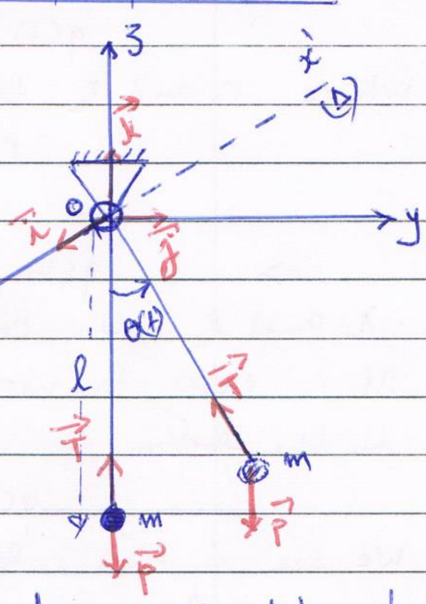
$\Rightarrow T_0 = 2\pi\sqrt{LC}$

if the I.C are: $q(0) = q_0$ and $\dot{q}(0) = i(0) = 0$ then we obtain:

$$q(t) = q_0 \cos\left(\frac{1}{\sqrt{LC}} t\right)$$

II-6 Example of a mechanical system _ simple pendulum

a simple pendulum is a mechanical system consisting of a rod of length l and negligeable mass, hinged by its one end at a fixed point labeled O, and carryes on point mass by at the other end (lower end). The system can rotate freely in the Oyz plane around an axis $\Delta(xx')$ which is perpendicular to it. When the pendulum is shifted from its equilibrium position, which corresponds to the verticale downward position it starts to oscillate. Two methods are considered for deriving the equation of vibrations: Newton's method and Lagrange's method.



1) Newton's method

a) At equilibrium we have: $\vec{T} + \vec{P} = 0 \Rightarrow \boxed{\vec{T} + m\vec{g} = 0}$ e.c

b) In motion, and applying the angular momentum theorem (the relation is projected on the axis Δ):

$$J_{/\Delta} \cdot \ddot{\theta} = \Sigma \mathcal{M}_{/\Delta}(\vec{P}) + \Sigma \mathcal{M}_{/\Delta}(\vec{T}) = \overrightarrow{Om} \wedge \vec{P} + \overrightarrow{Om} \wedge \vec{T} /_{\Delta}$$

$$= l(-\cos\theta \vec{k} + \sin\theta \vec{j}) \times (-mg\vec{k})\Big|_{\Delta} = -mgl\sin\theta$$

where $J_{/\Delta} = ml^2$

so: $\boxed{ml^2\ddot{\theta} + mgl\sin\theta = 0}$

It's a non lenear diff equation

for small oscillations ($\theta \ll$) this equation can be linearized if we admit that $\sin\theta \simeq \theta$ and then we obtain:

$$ml^2\ddot{\theta} + mgl\theta = 0 \Rightarrow \ddot{\theta} + \frac{g}{l}\theta = 0$$

It's a linear diff equation of type: $\ddot{q} + \omega_0^2 q = 0$ (13)
then the solution is sinusoidal of angular frequency:
$$\omega_0 = \sqrt{g/l} \implies \theta(t) = \theta_0 \cos\left(\sqrt{g/l}\, t + \varphi\right)$$

2) Lagrange's method
The simple pendulum is of single degree of freedom (two constraints $\Rightarrow s = 3 - 2 = 1$)
then the system admit a single generalized coordinate (q) which can be chosen as the angle θ made by the rod whith respect to the vertical; $\theta = q(t)$ this method involves calculating the kinetic and potential energy,

so: $T = \frac{1}{2} J_{/\Delta} \dot{\theta}^2 = \boxed{\frac{1}{2} m l^2 \dot{\theta}^2}$

U: how to calculate potential energy?
. the first step is to calculat the generalized force:
$$f_\theta = \frac{\delta W(\vec{P} \text{ and } \vec{T})}{\partial \theta} = \frac{\delta W(\vec{P})}{\partial \theta} + \underbrace{\frac{\delta W(\vec{T})}{\partial \theta}}_{=0} = \frac{\delta W(\vec{P})}{\partial \theta}$$
$$\Rightarrow f_\theta = m\vec{g} \cdot \frac{\delta \vec{Om}}{\partial \theta} = -mgl\,(\sin\theta\, \vec{k}\cdot\vec{k} + \cos\theta\, \vec{j}\cdot\vec{k}) = -mgl \sin\theta$$

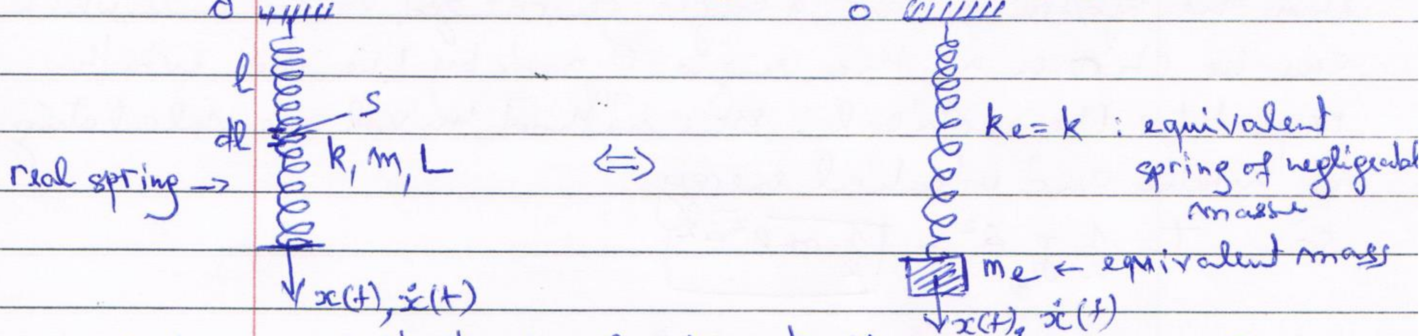
f_θ is a force deriving from the potential $U \Rightarrow$
$$f_\theta = -\frac{dU}{d\theta} \implies +dU = +mgl(\sin\theta) \Rightarrow$$
$$\int_{U(0)}^{U(\theta)} dU = \int_0^\theta mgl \sin\theta\, d\theta \implies U(\theta) = mgl(1 - \cos\theta) + U(0)$$

If we choise $\theta = 0°$ as the origine of potential then we can put that: $U(0) = 0$
at last we have: $\boxed{U(\theta) = mgl(1 - \cos\theta)}$
the Lagrange's function is:
$$\mathcal{L}(\theta, \dot{\theta}) = T - U = \frac{1}{2} m l^2 \dot{\theta}^2 - mgl(1 - \cos\theta)$$
for small angles ($\theta \ll$): $\cos\theta = 1 - \frac{\theta^2}{2}$
and so: $\mathcal{L}(\theta, \dot{\theta}) = \frac{1}{2} m l^2 \dot{\theta}^2 - \frac{1}{2} mgl\,\theta^2$
the Lagrange's equation for a single degree of freedom is
$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right] - \frac{\partial \mathcal{L}}{\partial \theta} = 0 \implies \ddot{\theta} + \frac{g}{l}\theta = 0$$ the same diff equation

II-7 - simple equivalent model

A vibratory system consisting of multiple elements can be substituted by a simple model provided that its vibrations are the same as those in the original system, thus having the same frequency. Here below some examples how to search the simple equivalent model:

a- Equivalent mass of a spring with non-negligeable mass



We take an infinitesimal element dl from the real spring, which is around of a certain point S, which is located at a distance l from the suspension point of the spring (point O)

i.e: $OS = l$

It means that $0 \leq l \leq L$

if the spring is homogeneous then the linear mass is:

$$\bar{\rho} = \frac{m}{L}$$

so the mass of the element dl is: $m_{dl} = \bar{\rho}\, dl = \frac{m}{L} dl$

If we accept that the velocity of a point s from the spring varies linearly with l, then:

$0 \leq \dot{x}_s \leq x(t)$ whereas $\dot{x}_s = \frac{l}{L} \dot{x}(t)$

so the kinetic energy of S is: (element dl)

$$dT_s = \frac{1}{2} m_{dl} \dot{x}_s^2 = \frac{1}{2} \frac{m}{L^3} \dot{x}^2 l^2 dl$$

so the kinetic energy of the spring as all will be:

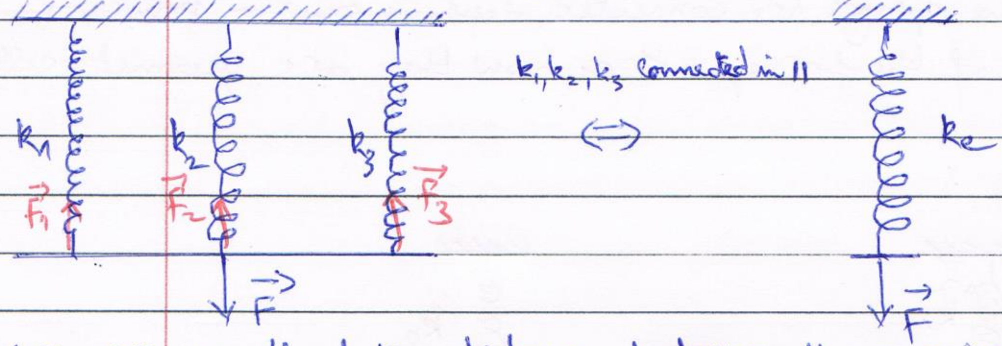
$$T = \int_0^L \frac{1}{2} \frac{m}{L^3} \dot{x}^2 l^2 dl = \frac{1}{2} \frac{m}{3} \dot{x}^2$$

in the other side the kinetic energy of the equivalent system is

$$T_{eq} = \frac{1}{2} m_e \dot{x}^2 \Longrightarrow m_e = \frac{m}{3}$$

Conclusion: the simple equivalent model consisting of a spring of negligeable mass and the same stifness $k_e = k$ and an equivalent mass $m_e = m/3$ suspended at the free end of the spring

b) Equivalent spring for springs in parallel



if we assume that the distance between the springs is negligible, then they will have the same elongation Δx.

if we apply a force $\vec{F}$ to the common point of the springs from their free side then we write:

$$\vec{F} = -\vec{F_1} - \vec{F_2} - \vec{F_3} \Rightarrow \|\vec{F}\| = k_1 \Delta x + k_2 \Delta x + k_3 \Delta x$$
$$= (k_1 + k_2 + k_3) \Delta x \ldots$$

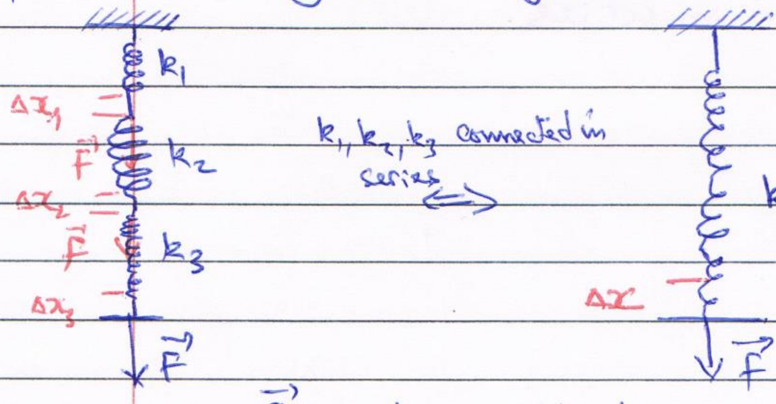
for the equivalent spring we have:

$$\|\vec{F}\| = k_e \Delta x \ldots$$

by identification of the two equations we obtain:

$$\boxed{k_e = k_1 + k_2 + k_3 + \ldots}$$

c) Equivalent spring for springs in series



The same force $\vec{F}$ is transmitted to each of the springs k_1, k_2 et k_3 ... in series and consequently, the force $\vec{F}$ causes different elongations; Δx_1, Δx_2 and Δx_3, ... successively. Then we have:

$$F = k_1 \Delta x_1 = k_2 \Delta x_2 = k_3 \Delta x_3 = \ldots$$

In the equivalent spring the equivalent elongation Δx is

$$\Delta x = \Delta x_1 + \Delta x_2 + \Delta x_3 \Rightarrow F = k_e \Delta x$$

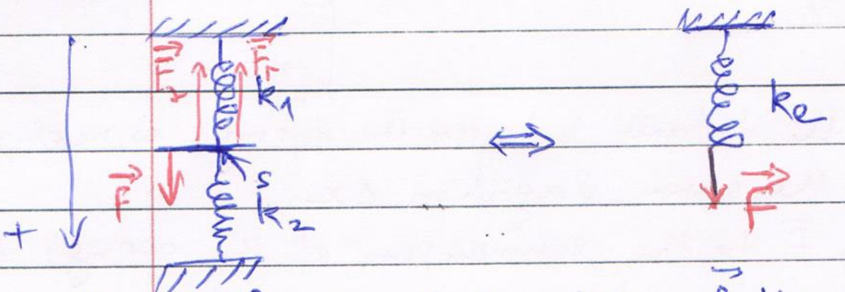
then we get: $$\frac{F}{k_e} = \frac{F}{k_1} + \frac{F}{k_2} + \frac{F}{k_3} \Rightarrow \boxed{\frac{1}{k_e} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} + \ldots}$$

(16)

Important note

The way the springs are connected depends on the point of application of the force, not on how they are geometrically connected.

Example:



under the influence of the force $\vec{F}$ the point S is moved by Δx downward. then the spring k_1 is elongated by Δx and K_e is compressed by Δx

So:

$$\vec{F} = \vec{f_1} + \vec{f_2} \Rightarrow (-k_1 \Delta x - k_2 \Delta x)\vec{i} = -(k_1+k_2)\Delta x \vec{i}$$

for the equivalent spring we have:

$$\vec{F} = -k_e \Delta x \vec{i}$$

$$\Rightarrow k_e = k_1 + k_2$$

then the two springs are in parallel, although they are geometrically connected in series.