II – *Kinematics*

0- Some words on kinematics

Kinematics is a branch of physics that studies the motion of objects without considering the causes behind that motion. It focuses on describing the position, velocity, and acceleration of particles or objects in motion. By studying kinematics, scientists can analyze the relationship between these variables and understand how an object moves and changes over time.

One of the fundamental concepts in kinematics is displacement. Displacement refers to the change in position of an object or particle from its initial position to its final position. It is a vector quantity as it has both magnitude and direction. By calculating the displacement, one can determine how far an object has moved and in which direction it has traveled.

Another important concept in kinematics is velocity. Velocity is the rate at which an object moves in a certain direction. It is calculated by dividing the displacement of an object by the time taken to travel that distance. Velocity is also a vector quantity and is dependent on both the magnitude and direction of displacement. It provides information about the speed of an object and the direction in which it is moving.

Overall, kinematics plays a pivotal role in understanding the basics of motion. By studying displacement and velocity, scientists can analyze an object's movement and describe it accurately. Whether it is calculating the displacement of a ball rolling down the slope or analyzing the velocity of a car in a race, kinematics helps provide insights into the motion of objects, enhancing our understanding of the physical world.

Kinematics is a branch of physics that deals with the motion of objects without considering what causes that motion. It focuses on describing the position, velocity, and acceleration of an object as it moves through space and time. Kinematics helps us understand how objects move and allows us to predict their future positions and velocities based on their initial conditions.

When studying kinematics, it is essential to know the basic terms used to describe motion. The position of an object refers to its location relative to a chosen reference point. Velocity is the rate at which an object's position changes, while acceleration is the rate at which its velocity changes. It is important to note that velocity and acceleration are vector quantities which means they have both a magnitude and a direction.

To analyze motion, kinematics uses mathematical equations and graphs. The three equations of motion, often referred to as the kinematic equations, are commonly used to solve various kinematics problems. The equations involve the initial and final velocities, acceleration, displacement, and time intervals. Graphs, such as position-time or velocity-time graphs, can provide a visual representation of an object's motion, enabling us to analyze its behavior more easily.

In summary, kinematics is a fundamental concept in physics that helps us understand the motion of objects. It involves studying the position, velocity, and acceleration of objects without considering the forces that cause them to move. By utilizing mathematical equations and graphs, kinematics allows us to predict and analyze the motion of objects accurately. Understanding kinematics is crucial for further exploring more complex topics in physics, such as dynamics and mechanics in general.

1- Concept of frame of reference

Let $(\vec{i},\vec{j},\vec{k})$ be an orthonormal basis, placed at a point chosen as the origin, which is used to locate a point " **M** ". It constitutes a reference frame. (***Frame of Reference* = *origin* + *basis***)

- If this point M is moving, it depends on time.

$$\overrightarrow{OM}(t) = \vec{r}(t) = x(t)\,\vec{i} + y(t)\,\vec{j} + z(t)\,\vec{k} \qquad t \text{ : is time}$$

- The Concept of motion is relative according to the observer (rest, moving independently or with the mobile M). The observer is the witness of time.

" *To describe the motion of a material point, a reference frame is necessary, that we affect (bind or link) an observer* to it, *which leads us to define a frame of reference* "

For example, the observer on earth say that path of moon is almost circular. But the observer siting on the sun sees the trajectory of the moon (same object) is a line wave path.

A reference frame is a platform from where a physical phenomenon, such as motion, is being observed

2- Equation of motion and trajectory equation

The change in position of the point, produce a motion. That motion is characterized by several parameters which are the displacement, distance, velocity and acceleration.

2.1- Position vector

The motion of a particle is described in some frame of reference. Starting first with locating it (position vector), then give its nature.

In an orthonormal coordinate system (cartesian) $(\boldsymbol{O}, \vec{\boldsymbol{\imath}}, \vec{\boldsymbol{\jmath}}, \overrightarrow{\boldsymbol{k}})$ the position vector is given by:

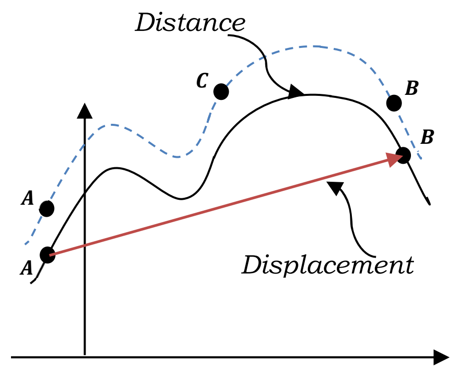
$$\overrightarrow{\boldsymbol{OM}} = \vec{\boldsymbol{r}} = \boldsymbol{x}\,\vec{\boldsymbol{\imath}} + \boldsymbol{y}\,\vec{\boldsymbol{\jmath}} + \boldsymbol{z}\,\overrightarrow{\boldsymbol{k}}$$

2.2- Displacement and distance

2.2-1: Displacement

The shortest distance joining the points **A** and **B** of the curve *i.e.*, the line $\boldsymbol{AB}$ which called displacement. It expresses how far is **B** from **A.**

Notice that the line $\boldsymbol{AB}$ has a direction from $\boldsymbol{A}$ to $\boldsymbol{B}$.



- Displacement: vector $\overrightarrow{\boldsymbol{AB}}$ (Red line)

- Distance: length of the curve ACB (Blue Dashed)

So, the displacement:

- *It is a vector quantity and is independent of the choice of origin*
- *It is unique for any kind of motion between two points*
- *It is always concealing (cover) about the actual track followed by the particle's motion between any two points, i.e. It doesn't give information about a path.*
- *It can be positive, negative and even be zero.*
- *The magnitude of the displacement is always less than or equal to the distance for particle's motion between two points*
- *A body may have finite distance travelled for zero displacement*

2.2-2: Distance

The distance express how long is the path from A to B passing through a point C.

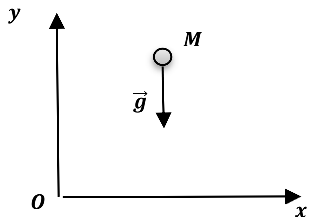
- *The distance is a scalar quantity*

- *The distance is always positive i.e., it only increases.*

- *The distance is always greater or equal to the magnitude of the displacement.*

2.3- Equation of motion

The equation of motion expresses the manner of change of motion or how the motion is changing in time by giving its kinematic parameters which are the displacement, velocity and acceleration.



Example: Free fall

$$y(t) = -\frac{1}{2}gt^2 + v_0 t + y_0$$

The distance y traveled by the point M is given as a function of time.

y(t) is the time equation of motion

Note:

The coordinates of the point, M $(x(t)\,,y(t)z(t))$, are the parametric equations

2.4- Trajectory equation (Path equation)

- Since the vector position $\overrightarrow{OM}$ changes, i.e., the point M change its position as time is varying, then we have:

$$\begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases}$$

($x(t)$, $y(t)$, and $z(t)$) are called parametric equations of motion.

- ***The trajectory (path) is the curve which traces the locations occupied by the mobile in space during the variations of time (as the time is changing).***

- To find the equation of the trajectory, we eliminate the time from the parametric equations, and find the form: $f(x, y, z) = 0$

Example: Motion in the plane

$$\begin{cases} x = acos(\omega t) \\ y = asin(\omega t) \end{cases} \Rightarrow \begin{cases} x^2 = a^2cos^2(\omega t) \\ y^2 = a^2sin^2(\omega t) \end{cases} \Rightarrow x^2 + y^2 = a^2$$

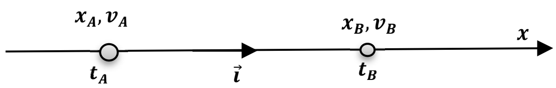
This is an equation of circle with radius $R = a$ centered on the point C (0,0) called the center of this circle.

3- Concept of velocity and speed

3.1-1: Average velocity

The average velocity is the ratio of the displacement between two points A and B to the travel time without taking into account the nature of the motion (the way in which the section AB is traveled).

- **In one direction (one dimension)**



Let the motion along the straight-line "ox ". *The point "A" is the initial position and "B" is the final point, so the average velocity is defined as* $<\vec{v} \geq \vec{v}_{moy}$ *such that:*

$$<\vec{v} \geq \vec{v}_{moy} = \frac{x_f - x_i}{t_f - t_i}\vec{i} = \frac{x_B - x_A}{t_B - t_A}\vec{i} = \frac{\Delta x}{\Delta t}\vec{i}$$

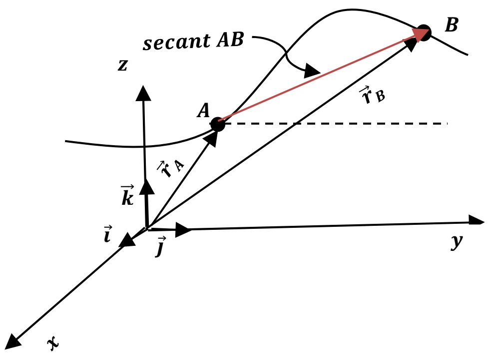
- **In all space (three dimensions)**:

The initial point is: $A(x_A, y_A, z_A)$

and the final point is: $B(x_B, y_B, z_B)$

The displacement is then: $\Delta\vec{r} = \vec{r}_B - \vec{r}_A$

So, the average velocity is:



$$<\vec{v}> = \vec{v}_{moy} = \frac{\Delta\vec{r}}{\Delta t} = \frac{\vec{r}_f - \vec{r}_i}{t_f - t_i} \Rightarrow <\vec{v}> = \vec{v}_{moy} = \frac{x_B - x_A}{t_B - t_A}\vec{i} + \frac{y_B - y_A}{t_B - t_A}\vec{j} + \frac{z_B - z_A}{t_B - t_A}\vec{k}$$

Note:

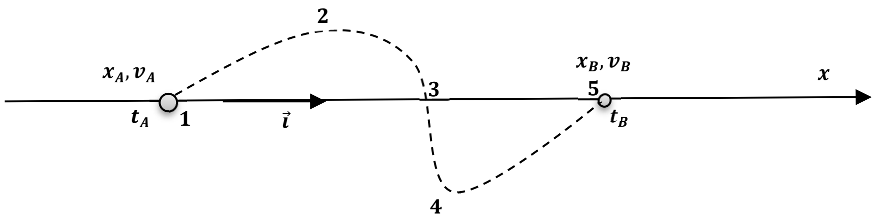
The average velocity, geometrically, is the slope of the secant that join the final and the initial positions in the curve which represents the variation of position with time (x-t curve).

3.1-2: Average speed

The average speed is the ratio of the distance traveled between two points A and B to the travel time of duration of trip

$$< S > = S_{moy} = \frac{Distance\ covered}{time\ taken}$$

The path A≡1-2-3-4-5≡B has the distance **d**, so the average speed defined as the distance of the journey on the time duration of the trip



If a particle starts from $'A \equiv 1'$ to the point $'B'$. Let $'d_{12}'$ be the distance covered by the particle to go from position $'1'$ to $'2'$, and $'d_{23}'$ that covered from position $'2'$ to $'3'$ and so on, until this particle arrives to the final position $'5 \equiv B'$. The average speed is:

$$S_{moy} = \frac{Distance\ covered}{time\ taken} = \frac{d_{12} + d_{23} + \ldots + d_{45}}{t_{12} + t_{23} + \ldots + t_{45}}$$

3.1-2-1: Average speed in case when the time is divided in equal intervals

Let the actual path from A to B, be divided in several intervals not equal, each traversed with in the same lapse of time but with different speeds. To compute the average speed, we proceed as follows:

$$S_{moy} = \frac{Distance\ covered}{time\ taken} = \frac{d_1 + d_2 + \ldots + d_n}{t_1 + t_2 + \ldots + t_n} = \frac{\sum_1^n d_i}{\sum_1^n t_i}$$

But $d_1 = s_1 . t_1$, …., $d_1 = s_n . t_n$

Since the lapse of time are equal: $t_1 = t_2 = \ldots = t_n = {}^{t}/_{n}$ with $'t'$ the time taken during the trip

$$S_{moy} = \frac{v_1(\frac{t}{n}) + v_2(\frac{t}{n}) + \ldots + v_n(\frac{t}{n})}{t_1 + t_2 + \ldots + t_n} = \frac{\frac{t}{n}(v_1 + v_2 + \ldots + v_n)}{t} = \frac{\sum_1^n v_i}{n}$$

So, we observe that when an interval is divided into n equal time parts, then the average speed $'S_{moy}'$ is simply the arithmetic mean of the speeds in the respective intervals.

$$S_{moy} = \frac{1}{n}\sum_{1}^{n} v_i$$

3.1-2-2: Average speed in case when the length is divided in equal intervals

In the same manner, the distance will be divided in equal intervals

$d_1 = d_2 = \ldots = d_n = {}^{d}/_{n}$,

But $d_1 = v_1 t_1$ and $d_2 = v_2 t_2$, …, $d_n = v_n t_n$

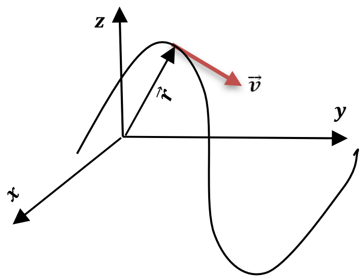
So $t_1 = \frac{d_1}{v_1} = \frac{d/n}{v_1}$, …, $t_n = \frac{d_n}{v_n} = \frac{d/n}{v_n}$

$$S_{moy} = \frac{d_1 + d_2 + \ldots + d_n}{t_1 + t_2 + \ldots + t_n} = \frac{{}^{d}/_{n} + {}^{d}/_{n} + \ldots + {}^{d}/_{n}}{\frac{d_1}{v_1} + \ldots + \frac{d_n}{v_n}} = \frac{d}{\frac{d}{nv_1} + \ldots + \frac{d}{nv_n}} = \frac{n}{\frac{1}{v_1} + \ldots + \frac{1}{v_n}}$$

The average speed $'S_{moy}'$ is simply n time the reciprocal of the harmonic mean of the speeds in the respective intervals.

3.2- Instantaneous velocity and instantaneous speed

Instantaneous velocity is the velocity that the material point will have at every moment on the trip. The velocity, in an infinitely small lapse of time, in the corresponding infinitesimal displacement, doesn't change.



$$\vec{v}(t) = \lim_{\Delta t \to 0}\left(\frac{\Delta\vec{r}}{\Delta t}\right) = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k}$$

$$\vec{v}(t) = v_x\,\vec{i} + v_y\,\vec{j} + v_z\,\vec{k}$$

The magnitude of the velocity is given by:

$$|\vec{v}(t)| = \sqrt{v_x^2 + v_y^2 + v_z^2}$$

Note:

1- The instantaneous speed is equal to the magnitude of the instantaneous velocity

2- *The instantaneous velocity, geometrically, is the slope of the tangent to the curve that represents the change in position with time (x-t curve).*

4- Concept of acceleration

4.1- Average acceleration

- Acceleration is the rate of change of velocity over time.

- The average acceleration is the rate of change in velocity between the initial "A" and final points B, regardless of how the path is traversed.

- In one direction only

$$<\vec{a}> = \vec{a}_{moy} = \frac{\Delta\vec{v}}{\Delta t} = \frac{\vec{v}_f - \vec{v}_i}{t_f - t_i} = \frac{v_B - v_A}{t_B - t_A}\vec{i}$$

- I**n all space (three dimensions)**

The initial velocity is: $\vec{v}_A(v_{xA}, v_{yA}, v_{zA})$

The final velocity is: $\vec{v}_B(v_{xB}, v_{yB}, v_{zB})$

The variation of velocity is then: $\Delta\vec{v} = \vec{v}_B - \vec{v}_A$

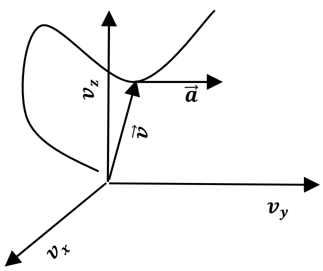
So, the average acceleration is:

$$<\vec{a}> = \vec{a}_{moy} = \frac{\Delta\vec{v}}{\Delta t} = \frac{\vec{v}_f - \vec{v}_i}{t_f - t_i}$$

$$\vec{a}_{moy} = \frac{v_{xB} - v_{xA}}{t_B - t_A}\vec{i} + \frac{v_{yB} - v_{yA}}{t_B - t_A}\vec{j} + \frac{v_{zB} - v_{zA}}{t_B - t_A}\vec{k}$$

4.2- Instantaneous acceleration

Instantaneous acceleration is the rate of change of velocity in time at each moment. In an infinitely small lapse of time, on the corresponding infinitesimal change in velocity, the acceleration doesn't change.



$$\vec{a}(t) = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt} = \frac{dv_x}{dt}\vec{\imath} + \frac{dv_y}{dt}\vec{\jmath} + \frac{dv_z}{dt}\vec{k}$$

- The hodograph of motion is the curve described by the end of the velocity vector

- **Note**:

The instantaneous acceleration, geometrically, is the slope of the curve (hodograph) which represents the variation of the velocity over time.

$$\vec{a}(t) = \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right) = \frac{d^2\vec{r}}{dt^2} = \frac{d^2x}{dt^2}\vec{\imath} + \frac{d^2y}{dt^2}\vec{\jmath} + \frac{d^2z}{dt^2}\vec{k}$$

$$\vec{a}(t) = a_x\vec{\imath} + a_y\vec{\jmath} + a_z\vec{k}$$

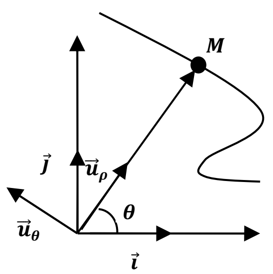
5- Position, velocity and acceleration in the different coordinate systems

5.1- Derivative of unit vectors

- Polar basis $(\vec{u}_\rho, \vec{u}_\theta)$

$\theta(t)$ and $\rho(t)$ change in time $\vec{u}_\rho, \vec{u}_\theta$ changes also and are written in the Cartesian base as follows:

$$\begin{cases} \vec{u}_\rho = cos(\theta)\,\vec{\imath} + sin(\theta)\,\vec{\jmath} \\ \vec{u}_\theta = -sin(\theta)\,\vec{\imath} + cos(\theta)\,\vec{\jmath} \end{cases}$$



$\vec{u}_\rho$ and $\vec{u}_\theta$ are a composite function, so we apply the chain rule

- If we have a function $f = F(u(x))$ which depend on the variable u who depends also on the other variable x. Then the derivative of this this function with respect to the variable x is given by:

$$df = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x}$$

$$\Rightarrow \quad \begin{cases} \frac{d\vec{u}_\rho}{dt} = \frac{d\vec{u}_\rho}{d\theta}\frac{d\theta}{dt} = [-sin(\theta)\,\vec{\imath} + cos(\theta)\,\vec{\jmath}]\frac{d\theta}{dt} \\ \frac{d\vec{u}_\theta}{dt} = \frac{d\vec{u}_\theta}{d\theta}\frac{d\theta}{dt} = [-cos(\theta)\,\vec{\imath} - sin(\theta)\,\vec{\jmath}]\frac{d\theta}{dt} \end{cases}$$

$$\Rightarrow \begin{cases} \frac{d\vec{u}_\rho}{dt} = \dot{\theta}\left[\cos\left(\theta+\frac{\pi}{2}\right)\vec{\imath} + \sin\left(\theta+\frac{\pi}{2}\right)\vec{\jmath}\right] \\ \frac{d\vec{u}_\theta}{dt} = \dot{\theta}\left[-\sin\left(\theta+\frac{\pi}{2}\right)\vec{\imath} + \cos\left(\theta+\frac{\pi}{2}\right)\vec{\jmath}\right] \end{cases} \quad \Rightarrow \quad \begin{cases} \dot{\vec{u}}_\rho = \frac{d\vec{u}_\rho}{dt} = \dot{\theta}\vec{u}_\theta \\ \dot{\vec{u}}_\theta = \frac{d\vec{u}_\theta}{dt} = -\dot{\theta}\,\vec{u}_\rho \end{cases}$$

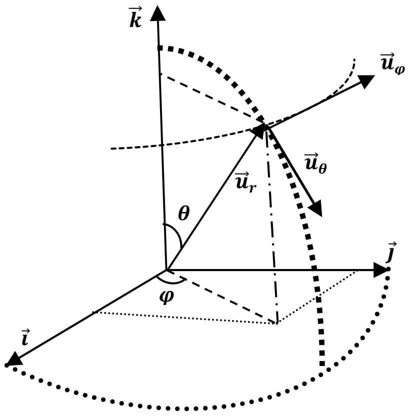
To find the derivative of a unit vector, we make a rotation anti-clockwise of $+\pi/2$

Note:

The cylindrical basis gives similar results as the polar basis by adding the z coordinate.

- Spherical base $\vec{u}_r, \vec{u}_\theta, \vec{u}_\varphi$

From the figure, the unit vectors $\vec{u}_r$, $\vec{u}_\theta$ and $\vec{u}_\varphi$ are given by:



$$\begin{cases} \vec{u}_r = \sin\theta\cos\varphi\,\vec{\imath} + \sin\theta\sin\varphi\,\vec{\jmath} + \cos\theta\,\vec{k} \\ \vec{u}_\theta = \cos\theta\cos\varphi\,\vec{\imath} + \cos\theta\sin\varphi\,\vec{\jmath} - \sin\theta\,\vec{k} \\ \vec{u}_\varphi = -\sin\varphi\,\vec{\imath} + \cos\varphi\,\vec{\jmath} \end{cases}$$

If we use the differential form of the function that depend on many variables:

$$f(x,y,z) \quad \Rightarrow \quad df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

$$\Rightarrow \quad \frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

$\vec{u}_r = f(\theta,\varphi)$ and $\vec{u}_r = h(\theta,\varphi)$

$$\Rightarrow \begin{cases} d\vec{u}_r = \frac{\partial\vec{u}_r}{\partial\theta}.d\theta + \frac{\partial\vec{u}_r}{\partial\varphi}.d\varphi \\ d\vec{u}_\theta = \frac{\partial\vec{u}_\theta}{\partial\theta}.d\theta + \frac{\partial\vec{u}_\theta}{\partial\varphi}.d\varphi \\ d\vec{u}_\varphi = \frac{d\vec{u}_\varphi}{d\varphi}.d\varphi \end{cases}$$

Finally

$$\Rightarrow \begin{cases} \dot{\vec{u}}_r = \frac{d\vec{u}_r}{dt} = \frac{\partial\vec{u}_r}{\partial\theta}.\frac{d\theta}{dt} + \frac{\partial\vec{u}_r}{\partial\varphi}.\frac{d\varphi}{dt} = \dot{\theta}\,\vec{u}_\theta + \dot{\varphi}\sin\theta\,\vec{u}_\varphi \\ \dot{\vec{u}}_\theta = \frac{d\vec{u}_\theta}{dt} = \frac{\partial\vec{u}_\theta}{\partial\theta}.\frac{d\theta}{dt} + \frac{\partial\vec{u}_\theta}{\partial\varphi}.\frac{d\varphi}{dt} = -\dot{\theta}\,\vec{u}_r + \dot{\varphi}\cos\theta\,\vec{u}_\varphi \\ \dot{\vec{u}}_\varphi = \frac{d\vec{u}_\varphi}{dt} = \frac{d\vec{u}_\varphi}{d\varphi}.\frac{d\varphi}{dt} = -\dot{\varphi}(\sin\theta\,\vec{u}_r + \cos\theta\,\vec{u}_\theta) \end{cases}$$

$$\Rightarrow \begin{cases} \dot{\vec{u}}_r = \dot{\theta}\,\vec{u}_\theta + \dot{\varphi}\sin\theta\,\vec{u}_\varphi \\ \dot{\vec{u}}_\theta = -\dot{\theta}\,\vec{u}_r + \dot{\varphi}\cos\theta\,\vec{u}_\varphi \\ \dot{\vec{u}}_\varphi = -\dot{\varphi}(\sin\theta\,\vec{u}_r + \cos\theta\,\vec{u}_\theta) \end{cases}$$

5.2- Polar coordinates

a- Position vector

As we have already seen that the position vector in the polar basis is:

$$\overrightarrow{OM} = \vec{r} = \rho\vec{u}_\rho \Rightarrow |\overrightarrow{OM}| = \rho$$

b- velocity vector

According to the definition:

$$\vec{v}(\,t\,) = \frac{d\overrightarrow{OM}}{dt} = \frac{d\vec{r}}{dt} = \frac{d(\rho\vec{u}_\rho)}{dt} = \dot{\rho}\vec{u}_\rho + \rho\frac{d\vec{u}_\rho}{dt}$$

But $\frac{d\vec{u}_\rho}{dt} = \dot{\theta}\,\vec{u}_\theta$

Then $\vec{v}(\mathrm{t}) = \dot{\rho}\,\vec{u}_\rho + \rho\dot{\theta}\,\vec{u}_\theta = \vec{v}_\rho + \vec{v}_\theta$

Where $\begin{cases} |\vec{v}_\rho| = \dot{\rho} \\ |\vec{v}_\theta| = \rho\dot{\theta} \end{cases}$

$$\Rightarrow \quad |\vec{v}(\mathrm{t})| = \sqrt{v_\rho^2 + v_\theta^2} = \sqrt{\dot{\rho}^2 + \left(\rho\dot{\theta}\right)^2}$$

c- acceleration vector

According to the definition: $\dot{\vec{u}}_\rho$ $\dot{\vec{u}}_\theta$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right) = \frac{d}{dt}\left(\dot{\rho}\,\vec{u}_\rho + \rho\dot{\theta}\,\vec{u}_\theta\right)$$

$$\Rightarrow \quad \vec{a} = \ddot{\rho}\,\vec{u}_\rho + \dot{\rho}\dot{\vec{u}}_\rho + \dot{\rho}\dot{\theta}\,\vec{u}_\theta + \rho\ddot{\theta}\,\vec{u}_\theta + \rho\dot{\theta}\,\dot{\vec{u}}_\theta$$

$$\vec{a} = \left(\ddot{\rho} - \rho\dot{\theta}^2\right)\vec{u}_\rho + \left(2\dot{\rho}\dot{\theta} + \rho\ddot{\theta}\right)\vec{u}_\theta = \vec{a}_\rho + \vec{a}_\theta$$

Where $\begin{cases} |\vec{a}_\rho| = \left(\ddot{\rho} - \rho\dot{\theta}^2\right) \\ |\vec{a}_\theta| = \left(2\dot{\rho}\dot{\theta} + \rho\ddot{\theta}\right) \end{cases}$

$$\Rightarrow \quad |\vec{a}(\mathrm{t})| = \sqrt{a_\rho^2 + a_\theta^2} = \sqrt{\left(\ddot{\rho} - \rho\dot{\theta}^2\right)^2 + \left(2\dot{\rho}\dot{\theta} + \rho\ddot{\theta}\right)^2}$$

5.3- Intrinsic coordinates (Natural coordinates)

The coordinate is the distance "$\boldsymbol{s}$ "traveled along the trajectory such that:

$$s = \widehat{oM}$$

The position vector is given by:

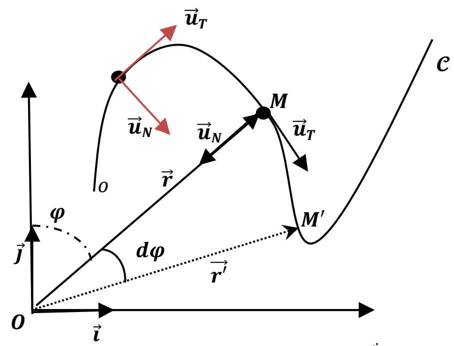
$$\overrightarrow{OM} = \vec{r} = x\,\vec{\imath} + y\,\vec{\jmath} \qquad .$$

If the mobile moves from the point M to the point M', then

$$\vec{r}' = \vec{r} + \overrightarrow{MM'} \qquad \Rightarrow \qquad \vec{r}' - \vec{r} = \overrightarrow{MM'} = \overrightarrow{dr} = dx\,\vec{\imath} + dy\,\vec{\jmath}$$

The segment "$\boldsymbol{ds}$"of the curve is related to the variation of Cartesian coordinates in such a way that:

$$ds = \sqrt{dx^2 + dy^2} = \left|\overrightarrow{dr}\right|$$



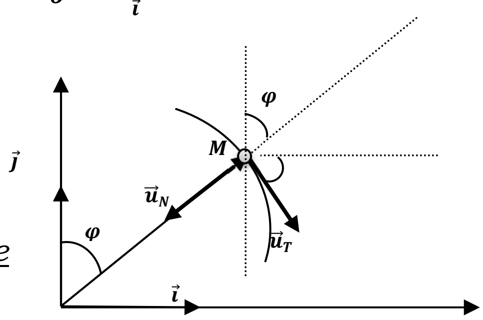
b- velocity vector

According to the definition:

$$\vec{v}(\mathrm{t}) = \frac{d\overrightarrow{OM}}{dt} = \frac{d\vec{r}}{dt} = \frac{\overrightarrow{MM'}}{dt} = \frac{\left|\overrightarrow{MM'}\right|}{dt}\vec{u}_T$$

Since the limit $\quad ds = \left|\overrightarrow{MM'}\right| = \left|\overrightarrow{dr}\right|$

Then: $\quad \vec{v}(t) = \frac{\left|\overrightarrow{MM'}\right|}{dt}\vec{u}_T = \frac{ds}{dt}\vec{u}_T = v.\vec{u}_T$



The velocity vector is oriented along the tangent to the curve

c- acceleration vector

According to the definition:

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right) = \frac{d}{dt}(v.\vec{u}_T) = \frac{d}{dt}(v).\vec{u}_T + v.\frac{d}{dt}(\vec{u}_T)$$

According to the previous figure " $\vec{u}_T$ " is tangent to the curve and " $\vec{u}_N$ "oriented towards the concavity

$$\begin{cases} \vec{u}_T = \cos\varphi\,\vec{\imath} - \sin\varphi\,\vec{\jmath} \\ \vec{u}_N = -\sin\varphi\,\vec{\imath} - \cos\varphi\,\vec{\jmath} \end{cases}$$

$$\Rightarrow \begin{cases} \dot{\vec{u}}_T = \frac{d\vec{u}_T}{dt} = -\frac{d\varphi}{dt}(sin\varphi\,\vec{i} + cos\varphi\,\vec{j}) = \dot{\varphi}\,\vec{u}_N \\ \dot{\vec{u}}_N = \frac{d\vec{u}_N}{dt} = -\frac{d\varphi}{dt}(cos\varphi\vec{i} - sin\varphi\vec{j}) = -\dot{\varphi}\,\vec{u}_T \end{cases}$$

Then:

$$\vec{a} = \frac{d}{dt}(v).\vec{u}_T + v.\dot{\varphi}\,\vec{u}_N$$

But $\widehat{MM'} = ds = \rho d\varphi$

ρ is a curvature radius

$$\frac{ds}{dt} = \rho\frac{d\varphi}{dt} = v$$

$$\Rightarrow \ \dot{\varphi} = v/\rho$$

Finally

$$\vec{a} = \frac{dv}{dt}.\vec{u}_T + v.\frac{v}{\rho}\vec{u}_N = \frac{dv}{dt}.\vec{u}_T + \frac{v^2}{\rho}\vec{u}_N = \vec{a}_T + \vec{a}_N$$

$$\vec{a} = \vec{a}_T + \vec{a}_N$$

$$\begin{cases} \vec{a}_T = \ \textit{is the tangential component due to the variation of the speed modulus} \\ \vec{a}_N = \ \textit{is the normal component due to the variation in the direction of the speed} \end{cases}$$

5.4- Cylindrical coordinates

a- Position vector

The position vector is given by:

$$\overrightarrow{OM} = \rho\,\vec{u}_\rho + z\,\vec{k} \Longrightarrow \left|\overrightarrow{OM}\right| = \sqrt{\rho^2 + z^2}$$

b- velocity vector

Based on definition:

$$\vec{v}(\mathbf{t}) = \frac{d\overrightarrow{OM}}{dt} = \frac{d\vec{r}}{dt} = \frac{d\left(\rho\,\vec{u}_\rho + z\,\vec{k}\right)}{dt} = \dot{\rho}\,\vec{u}_\rho + \rho\frac{d\vec{u}_\rho}{dt} + \dot{z}\,\vec{k}$$

But $\frac{d\vec{u}_\rho}{dt} = \dot{\theta}\,\vec{u}_\theta$

$$\vec{v}(t) = \dot{\rho}\,\vec{u}_\rho + \rho\dot{\theta}\,\vec{u}_\theta + \dot{z}\,\vec{k} = \vec{v}_\rho + \vec{v}_\theta + \vec{v}_z$$

With $\begin{cases} |\vec{v}_\rho| = \dot{\rho} \\ |\vec{v}_\theta| = \rho\dot{\theta} \\ |\vec{v}_z| = \dot{z} \end{cases}$

$$\Rightarrow \quad |\vec{v}(\mathrm{t})| = \sqrt{v_\rho^2 + v_\theta^2 + v_z^2} = \sqrt{\dot{\rho}^2 + \left(\rho\dot{\theta}\right)^2 + \dot{z}^2}$$

c- acceleration vector

According to the definition:

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right) = \frac{d}{dt}\left(\dot{\rho}\,\vec{u}_\rho + \rho\dot{\theta}\,\vec{u}_\theta + \dot{z}\,\vec{k}\right)$$

$$\Rightarrow \qquad \vec{a} = \ddot{\rho}\,\vec{u}_\rho + \dot{\rho}\,\dot{\vec{u}}_\rho + \dot{\rho}\dot{\theta}\,\vec{u}_\theta + \rho\ddot{\theta}\,\vec{u}_\theta + \rho\dot{\theta}\,\dot{\vec{u}}_\theta + \ddot{z}\,\vec{k}$$

$$\vec{a} = \left(\ddot{\rho} - \rho\dot{\theta}^2\right)\vec{u}_\rho + \left(2\dot{\rho}\dot{\theta} + \rho\ddot{\theta}\right)\vec{u}_\theta + \ddot{z}\,\vec{k}$$

$$\vec{a} = \vec{a}_\rho + \vec{a}_\theta + \vec{a}_z \qquad \text{With} \qquad \begin{cases} |\vec{a}_\rho| = \left(\ddot{\rho} - \rho\dot{\theta}^2\right) \\ |\vec{a}_\theta| = \left(2\dot{\rho}\dot{\theta} + \rho\ddot{\theta}\right) \\ |\vec{a}_z| = \ddot{z} \end{cases}$$

$$\Rightarrow \qquad |\vec{a}(\mathrm{t})| = \sqrt{a_\rho^2 + a_\theta^2 + a_z^2} = \sqrt{\left(\ddot{\rho} - \rho\dot{\theta}^2\right)^2 + \left(2\dot{\rho}\dot{\theta} + \rho\ddot{\theta}\right)^2 + \ddot{z}^{\,2}}$$

5.5- Spherical coordinates

a- Position vector

The position vector is given by**:**

$$\overrightarrow{OM} = r\,\vec{u}_r \quad \Rightarrow \quad \left|\overrightarrow{OM}\right| = r$$

b- velocity vector

According to the definition:

$$\vec{v}(t) = \frac{d\overrightarrow{OM}}{dt} = \frac{d\vec{r}}{dt} = \dot{r}\,\vec{u}_r + r\,\dot{\vec{u}}_r$$

But $\dot{\vec{u}}_r = \dot{\theta}\,\vec{u}_\theta + \dot{\varphi}\,sin\theta\,\vec{u}_\varphi$.

$$\Longrightarrow \quad \vec{v}(\mathrm{t}) = \dot{r}\,\vec{u}_r + r\dot{\theta}\,\vec{u}_\theta + r\dot{\varphi}\,sin\theta\,\vec{u}_\varphi = \vec{v}_r + \vec{v}_\theta + \vec{v}_\varphi$$

$$\text{With} \begin{cases} |\vec{v}_r| = \dot{r} \\ |\vec{v}_\theta| = r\dot{\theta} \\ |\vec{v}_\varphi| = r\dot{\varphi}sin\theta \end{cases}$$

$$\Longrightarrow \quad |\vec{v}(\mathrm{t})| = \sqrt{v_r^2 + v_\theta^2 + v_\varphi^2} = \sqrt{\dot{r}^2 + (r\dot{\theta})^2 + (r\dot{\varphi}sin\theta)^2}$$

c- acceleration vector

According to the definition:

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right) = \frac{d}{dt}\left(\dot{r}\,\vec{u}_r + r\dot{\theta}\,\vec{u}_\theta + r\dot{\varphi}\,sin\theta\,\vec{u}_\varphi\right)$$

$$\Longrightarrow \vec{a} = \ddot{r}\vec{u}_r + \dot{r}\dot{\vec{u}}_r + \dot{r}\dot{\theta}\vec{u}_\theta + r\ddot{\theta}\vec{u}_\theta + r\dot{\theta}\,\dot{\vec{u}}_\theta + \dot{r}\dot{\varphi}\,sin\theta\,\vec{u}_\varphi + r\ddot{\varphi}\,sin\theta\,\vec{u}_\varphi + r\dot{\varphi}\dot{\theta}\,cos\theta\,\vec{u}_\varphi + r\dot{\varphi}\,sin\theta\,\dot{\vec{u}}_\varphi$$

Knowing also that:

$$-\ \dot{\vec{u}}_\theta = \frac{d\vec{u}_\theta}{dt} = \frac{\partial\vec{u}_\theta}{\partial\theta}.\frac{d\theta}{dt} + \frac{\partial\vec{u}_\theta}{\partial\varphi}.\frac{d\varphi}{dt} = -\dot{\theta}\,\vec{u}_r + \dot{\varphi}cos\theta\,\vec{u}_\varphi$$

$$-\ \dot{\vec{u}}_\varphi = \frac{d\vec{u}_\varphi}{dt} = \frac{d\vec{u}_\varphi}{d\theta}.\frac{d\theta}{dt} = -\dot{\varphi}(sin\theta\,\vec{u}_r + cos\theta\,\vec{u}_\theta)$$

$$\Longrightarrow \vec{a} = \left(\ddot{r} - r\dot{\theta}^2 - r\dot{\varphi}^2sin^2\theta\right)\vec{u}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\varphi}^2sin\theta cos\theta\right)\vec{u}_\theta + \left(r\ddot{\varphi}sin\theta + 2r\dot{\varphi}\dot{\theta}cos\theta + 2\dot{r}\dot{\varphi}sin\theta\right)\vec{u}_\varphi$$

$$\vec{a} = \vec{a}_r + \vec{a}_\theta + \vec{a}_\varphi \qquad \text{With} \begin{cases} |\vec{a}_r| = \ddot{r} - r\dot{\theta}^2 - r\dot{\varphi}^2sin^2\theta \\ |\vec{a}_\theta| = r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\varphi}^2sin\theta cos\theta \\ |\vec{a}_\varphi| = r\ddot{\varphi}sin\theta + 2r\dot{\varphi}\dot{\theta}cos\theta + 2\dot{r}\dot{\varphi}sin\theta \end{cases}$$

Since $|\vec{a}(\mathrm{t})| = \sqrt{a_r^2 + a_\theta^2 + a_\varphi^2} \quad \Longrightarrow$

$$|\vec{a}| = \sqrt{\left(\ddot{r} - r\dot{\theta}^2 - r\dot{\varphi}^2sin^2\theta\right)^2 + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\varphi}^2sin\theta cos\theta\right)^2 + \left(r\ddot{\varphi}sin\theta + 2r\dot{\varphi}\dot{\theta}cos\theta + 2\dot{r}\dot{\varphi}sin\theta\right)^2}$$