UNIVERSITY MOHAMED BOUDIAF OF M'SILA

FACULTY OF TECHNOLOGY

DEPARTEMENT BASE COMMON ST

Physics 1

Mechanics of the material point

Intended for students of

1st Year LMD License

ST and SM domain

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A- Errors, measurements and representation

1- Unit

- All physical quantities are quantified, these quantities are characterized by units that are suitable for their measurements.
- In the international system (MKSA), we have 7 main units, the rest follows from that.

- M ⟶ meter (length).
- K ⟶ kilogram (mass).
- S ⟶ second (time).
- A ⟶ Ampère (electrical intensity).
- N ⟶ mole (number of particles)
- K ⟶ Kelvin (temperature)
- Cd ⟶ Candela (luminous Intensity)

2- Scientific notation

When quantifying physical quantities, some of them are very large or too small, for this, notation is used to write them. which is called scientific notation

$$\boldsymbol{v.10^n} \begin{cases} * \ \boldsymbol{v: real\ number} \quad \boldsymbol{1 \leq v \leq 9} \\ * \ \boldsymbol{n: Integer\ number} \end{cases}$$

Example:

- The earth mass: " 6 followed by 24 zeros " ⟶ $\boldsymbol{m = 6\ \ 10^{24} kg}$
- The electron mass: " 9.11 preceded by 30 zero " ⟶ $\boldsymbol{m = 9.11\ \ 10^{-31} kg}$

3- Measurement, errors and significant figures

3.1 Measurements

There are two types of measures

a- Direct measurements

This is the operation of reading or sampling directly from the measuring instrument (length, time, current, ...).

b- Indirect measurements

The desired quantity is expressed mathematically as a function of other quantities measured directly (area, volume, density, ...)

3.2 Errors

a- Notions of error and uncertainty

- Error:

Is the difference between the real and measured value of the physical quantity. This difference can be positive or negative.

There are two types of errors:

- *systematic errors*:

Those repeated each time in the same way (error of the instrument, ...)

- *Incidental errors*:

Those that appear each time but in a random or unpredictable way (reading, temperature change, ...)

- Uncertainty: is the maximum absolute value that the error can take.

b- Determination of uncertainty

- If "$\boldsymbol{x}$" is the real value of the physical quantity, while the measured value of the same quantity is "$\boldsymbol{x_0}$", then the error is:

$$\boldsymbol{e} = \boldsymbol{x} - \boldsymbol{x_0}$$

Note: The error may be negative or positive ($\boldsymbol{e} < 0 \; ou \; e > 0$)

- The absolute value of the error is: the absolute error

$$\boldsymbol{\delta x} = |\boldsymbol{e}| = |\boldsymbol{x} - \boldsymbol{x_0}|$$

- The absolute uncertainty is given by:

$$\Delta x = max(\delta x)$$

Note: we always have $\Delta x \geq \delta x$

- If the error is positive ($e > 0$) :

  $|x - x_0| = x - x_0 \implies \Delta x \geq \delta x = x - x_0 \implies x \leq x_0 + \Delta x$
- If the error is negative ($e < 0$) :

  $|x - x_0| = -(x - x_0) \implies \Delta x \geq \delta x = x_0 - x \implies x \geq x_0 - \Delta x$
- The real value can finally be written:

$$\boxed{x = x_0 \pm \Delta x}$$

- Determination of uncertainty
  - If the quantity is measured directly, the error made is on the smallest digit of the instrument. (Graduated rule in millimeters: the error made is in the mm).
  - If the quantity is given by indirect measurement, the error is expressed as a function of the errors of the quantities measured directly ($x = F(a, b, c..)$)

* <u>Sum</u>:

$$x = a + b + c + \cdots$$

$$\Delta x = \Delta a + \Delta b + \Delta c + \cdots$$

* <u>Product</u>:

$$x_0 = a.b.c$$

$$\Delta x = (b.c)\Delta a + (a.c)\Delta b + (a.b)\Delta c$$

And $\frac{\Delta x}{x_0} = \frac{\Delta a}{a} + \frac{\Delta b}{b} + \frac{\Delta c}{c} \Rightarrow \Delta x = \left(\frac{\Delta a}{a} + \frac{\Delta b}{b} + \frac{\Delta c}{c}\right) x_0$

finally: $x = x_0 \pm \Delta x$

Examples:

1°- Perimeter of a rectangle: L is the length and l is the width

$$P = 2.(L + l) \Rightarrow \Delta P = 2(\Delta L + \Delta l)$$

2°- Surface of this rectangle:

$$S = L.l \quad \Rightarrow \quad \Delta S = l.\Delta L + L.\Delta l \quad \Rightarrow \quad \frac{\Delta S}{S} = \frac{\Delta L}{L} + \frac{\Delta l}{l}$$

$$\Rightarrow \Delta S = \left(\frac{\Delta L}{L} + \frac{\Delta l}{l}\right) S$$

4- Signifiant figures

During the measurement, we write the quantified quantity in scientific notation, the figures that express this quantity are said to be *significant*.

Note: "13" and "13.0" have the same value, but their meanings are different i.e., the error of the second is 10 times less than the first

Generally:

- Non-zero figures are always significant (3.1415 $\longrightarrow$ 5 significant digits).
- All zeros that come at the end are significant (0.4500 $\longrightarrow$ 4 significant digits).
- The zeros between the significant digits are significant (0.104 $\longrightarrow$ 3 significant digits).
- The zeros used to move the comma are not meaningful (0.00125=1.25 10^{-5} $\longrightarrow$ 3 significant digits).

Some rules on significant numbers

5- Data and graphs

5.1- Data

These are the values that a physical quantity can take in different states

5.2- Graphs

The dependence that exists between two or more physical quantities is expressed by a function that can be represented by a curve or a graph.

There are several types of functions:

- Linear functions:

$y = ax + b$, express the dependence between $y\ et\ x$.

- Quadratic functions:

$\boldsymbol{y = ax^2 + bx + c}$ (Parabola of the 2nd order as well as that of the 3rd order and so on)

- Inverse functions:

$$\boldsymbol{y = \frac{k}{x}}$$

- Exponential and logarithmic functions:

$\boldsymbol{y = ae^{u(x)}}$, $\boldsymbol{y = \ln(v(x))}$ $où$ $\boldsymbol{u(x)}$ et $\boldsymbol{v(x)}$ are any numeric functions

- Circular or trigonometric functions:

$\boldsymbol{y = a.sin[u(x)]}$, $\boldsymbol{y = b.cos[u(x)]}$, $\boldsymbol{y = tg[u(x)]}$ ...

- Hyperbolic functions:

$\boldsymbol{y = a.sinh[u(x)]}$, $\boldsymbol{y = b.cosh[u(x)]}$, $\boldsymbol{y = tgh[u(x)]}$ ...

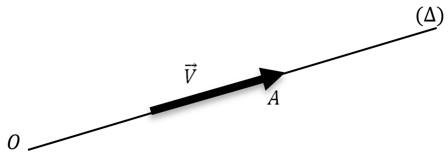
- Special functions.

B- <u>Vectors</u>

1- Notion of vector

1.1- Definition:

A vector is a mathematical entity that represents an element of a vector space $\mathbb{E}^3$ associated with an affine space (point), $\mathbb{R}^3$ where a direction, modulus, and point of application are defined.



- "$\boldsymbol{O}$" point of application

- "Δ" line of action

- In the orthonormal basis$(\vec{\boldsymbol{\imath}}, \vec{\boldsymbol{\jmath}}, \vec{\boldsymbol{k}})$,) and in Euclidean geometry:

The modulus of the vector $\vec{\boldsymbol{V}}$ is:

$$\vec{\boldsymbol{V}} = |\overrightarrow{\boldsymbol{OA}}| = \sqrt{x^2 + y^2 + z^2}$$

- From <u>O to A</u> is the direction

1.2-Types of vectors

1.2.1- Free vector

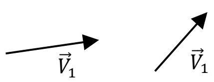
It is a vector where the application point can be transferred to any point in space.

1.2.2- Sliding vector

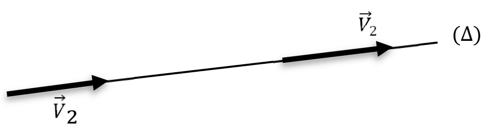
It is a vector where the application point can move along its line of action

1.2.3- Bound vector

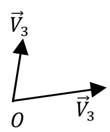
It is a vector where the point of application is fixed and defined by the coordinates of its origin



Free vector



Sliding vector



Bound vector

2- Operation on vectors

2.1- Sum of vectors (resultant):

Relative to an orthonormal$(\vec{\imath},\vec{\jmath},\vec{k})$ basis, the sum of two vectors is a vector, where the components are added two to two respectively

$$\vec{V}_1 = x_1\vec{\imath} + y_1\vec{\jmath} + z_1\vec{k} \qquad \textbf{and} \qquad \vec{V}_2 = x_2\vec{\imath} + y_2\vec{\jmath} + z_2\vec{k}$$

$$\Rightarrow \quad \vec{V} = \vec{V}_1 + \vec{V}_2 = (x_1 + x_2)\vec{\imath} + (y_1 + y_2)\vec{\jmath} + (z_1 + z_2)\vec{k}$$

<u>Note:</u>

For multiple vectors, the sum of the respective components added together represents the components of the resultant vector.

$$\vec{V} = \vec{V}_1 + \vec{V}_2 + \cdots + \vec{V}_n = (x_1 + x_2 + \cdots + x_n)\vec{\imath} + (y_1 + y_2 + \cdots + y_n)\vec{\jmath} + (z_1 + z_2 + \cdots + z_n)\vec{k}$$

$$\vec{V} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$$

$$\begin{cases} x = \sum_{l=1}^{n} x_l \\ y = \sum_{l=1}^{n} y_l \\ z = \sum_{l=1}^{n} z_l \end{cases}$$

2.2- Product of vectors:

a- Scalar product and projection:

The scalar product of two vectors $\vec{V}_1$ and $\vec{V}_2$, is a scalar denoted $\vec{V}_1 \circ \vec{V}_2$, which is equal to the sum of the products of the corresponding components taken pairwise.

$$\vec{V}_1 = x_1\vec{i} + y_1\vec{j} + z_1\vec{k} \quad \text{and} \quad \vec{V}_2 = x_2\vec{i} + y_2\vec{j} + z_2\vec{k}$$

$$\Rightarrow \qquad V = \vec{V}_1 \circ \vec{V}_2 = (x_1.x_2) + (y_1.y_2) + (z_1.z_2)$$

Note:

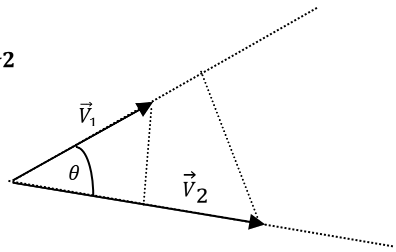
- For the unit vectors of the orthonormal basis, we have:

$$\begin{cases} \vec{i} \circ \vec{i} = \vec{j} \circ \vec{j} = \vec{k} \circ \vec{k} = 1 \\ \vec{i} \circ \vec{j} = \vec{i} \circ \vec{k} = \vec{j} \circ \vec{k} = 0 \end{cases}$$

- The square of the modulus of the vector is:

$$\vec{V} \circ \vec{V} = (x.x) + (y.y) + (z.z) = x^2 + y^2 + z^2 = V^2$$

$$\Rightarrow |\vec{V}| = V = \sqrt{x^2 + y^2 + z^2}$$



- The scalar product can also be defined as follows:

$$\vec{V}_1 \circ \vec{V}_2 = |\vec{V}_1|.|\vec{V}_2| \cos(\vec{V}_1, \vec{V}_2) = |\vec{V}_1|.|\vec{V}_2| \cos(\theta)$$

- The square of the modulus of a vector can be given by:

$$\vec{V}_1 \circ \vec{V}_1 = |\vec{V}_1|.|\vec{V}_2| \cos(\vec{V}_1, \vec{V}_1) = V_1^2$$

<u>Properties</u>:

- The scalar product is commutative

$$\vec{V}_1 \circ \vec{V}_2 = \vec{V}_2 \circ \vec{V}_1$$

- The scalar product is distributive with respect to addition

$$\vec{V}_1 \circ (\vec{V}_2 + \vec{V}_3) = \vec{V}_1 \circ \vec{V}_2 + \vec{V}_1 \circ \vec{V}_3$$

- The scalar product geometrically represents the projection of one vector onto the direction of another

$$\begin{cases} \vec{V} \circ \vec{i} = (x\vec{i} + y\vec{j} + z\vec{k}) \circ \vec{i} = x \\ \vec{V} \circ \vec{j} = (x\vec{i} + y\vec{j} + z\vec{k}) \circ \vec{j} = y \\ \vec{V} \circ \vec{k} = (x\vec{i} + y\vec{j} + z\vec{k}) \circ \vec{k} = z \end{cases}$$

- The scalar product is zero if:

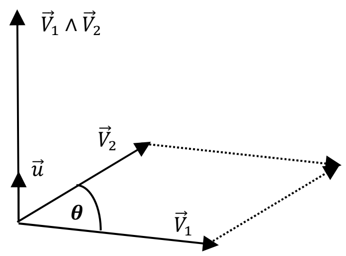
$$|\vec{V}_1| = 0, |\vec{V}_2| = 0 \text{ or } \vec{V}_1 \perp \vec{V}_2$$

b- Vector product and oriented surface:

The cross product of two vectors, $\vec{V}_1$ and $\vec{V}_2$, is a vector denoted $\vec{V}_1 \wedge \vec{V}_2$ and given by:

$$\vec{V}_1 \wedge \vec{V}_2 = \begin{vmatrix} \vec{i} & \vec{j} & k \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = (y_1.z_2 - y_2.z_1)\vec{i} - (x_1.z_2 - x_2.z_1)\vec{j} + (x_1.y_2 - x_2.y_1)\vec{k}$$

Also defined as follows:



$$\vec{V}_1 \wedge \vec{V}_2 = |\vec{V}_1|.|\vec{V}_2| \sin(\vec{V}_1, \vec{V}_2)\,\vec{u} = |\vec{V}_1|.|\vec{V}_2| \sin(\theta)\,\vec{u}$$

$\vec{u}$: is a unit vector

$\vec{u} \perp (\vec{V}_1 et \vec{V}_2)$

Properties:

- The vector product is noncommutative (anticommutative)

$$\vec{V}_1 \wedge \vec{V}_2 = -\vec{V}_2 \wedge \vec{V}_1$$

- The vector product is distributive with respect to the addition

$$\vec{V}_1 \wedge (\vec{V}_2 + \vec{V}_3) = \vec{V}_1 \wedge \vec{V}_2 + \vec{V}_1 \wedge \vec{V}_3$$

- The resulting vector of the cross product is always perpendicular to the operand vectors.
- The vector product obeys the rule of circular permutation

$$\begin{cases} \vec{i} \wedge \vec{j} = \vec{k} \\ \vec{j} \wedge \vec{k} = \vec{i} \\ \vec{k} \wedge \vec{i} = \vec{j} \end{cases} \quad \text{and } \vec{i} \wedge \vec{i} = \vec{j} \wedge \vec{j} = \vec{k} \wedge \vec{k} = 0$$

- The vector product is zero if:

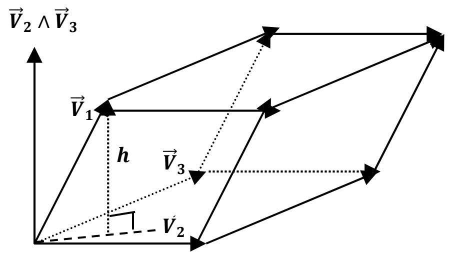
$$|\vec{V}_1| = 0\ , |\vec{V}_2| = 0 \text{ or } \vec{V}_1 \parallel \vec{V}_2$$

- The cross product geometrically represents the area of the oriented surface formed by operand vectors.

c- Triple product:

❖ <u>The scalar triple product</u>

The scalar triple product, is a scalar defined as:



$$\vec{V}_1 \circ (\vec{V}_2 \wedge \vec{V}_3) = W$$

<u>Properties</u>:

- The scalar triple product is invariant by cyclic permutation

$$\vec{V}_1 \circ (\vec{V}_2 \wedge \vec{V}_3) = \vec{V}_3 \circ (\vec{V}_1 \wedge \vec{V}_2) = \vec{V}_2 \circ (\vec{V}_3 \wedge \vec{V}_1)$$

- The scalar triple product is zero if:

$|\vec{V}_1| = 0\ |\vec{V}_2| = 0 |\vec{V}_3| = 0,$ or $\vec{V}_1, \vec{V}_2$ and $\vec{V}_3$ are coplanar

- Geometrically, the scalar triple product represents the volume formed by the operand vectors.

❖ <u>The vector triple product</u>

The vector triple product is a vector defined by the following relation:

$$\vec{V}_1 \wedge (\vec{V}_2 \wedge \vec{V}_3) = (\vec{V}_1 \circ \vec{V}_3)\vec{V}_2 - (\vec{V}_1 \circ \vec{V}_2)\vec{V}_3 = \alpha\vec{V}_2 + \beta\vec{V}_3 = \vec{W}$$

<u>Remark:</u>

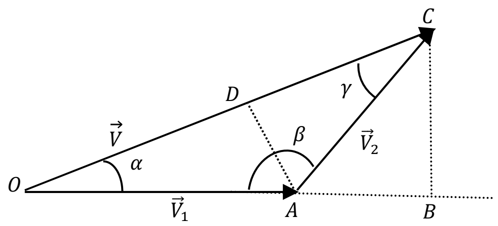
The multiplication of a vector by a scalar is a vector (it is a homothety)

$$\lambda\vec{V} = \vec{W}$$

3- Rule of sines

$$\vec{V} = \vec{V}_1 + \vec{V}_2$$

$$|\vec{V}| = \sqrt{(\vec{V}_1 + \vec{V}_2) \circ (\vec{V}_1 + \vec{V}_2)} = \sqrt{|V_1| + |V_2| + 2|V_1| \circ |V_2|}$$



- The triangles ABC and OBC give:

$$\begin{cases} sin(\alpha) = \dfrac{BC}{OC} \\ sin(\pi - \beta) = \dfrac{BC}{AC} \end{cases} \Rightarrow \quad OC.sin(\alpha) = AC.sin(\beta) \quad \Rightarrow \frac{|\vec{V}|}{sin(\beta)} = \frac{|\vec{V}_2|}{sin(\alpha)}$$

- The triangles OAD and give:ACD

$$\begin{cases} \sin(\alpha) = \dfrac{AD}{OA} \\ \sin(\gamma) = \dfrac{AD}{AC} \end{cases} \Rightarrow \quad OA.\sin(\alpha) = AC.\sin(\gamma) \quad \Rightarrow \frac{|\vec{V}_1|}{sin(\gamma)} = \frac{|\vec{V}_2|}{sin(\alpha)}$$

$$\Rightarrow \frac{|\vec{V}|}{sin(\beta)} = \frac{|\vec{V}_2|}{sin(\alpha)} = \frac{|\vec{V}_1|}{sin(\gamma)}$$

4- Derived from a vector

In a Cartesian orthonormal basis, the vector is expressed $\vec{a}$ by:

$$\vec{a} = x\vec{i} + y\vec{j} + z\vec{k}$$

If it is variable, its derivative comes down to differentiating these components.

$$\frac{d\vec{a}}{dt} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k}$$

- The derivative of the sum of the vectors is equal to the sum of the derivatives of these vectors

$$\frac{d(\vec{a}+\vec{b})}{dt}=\frac{d\vec{a}}{dt}+\frac{d\vec{b}}{dt}$$

- The derivative of the product of the vectors is equal to

$$\frac{d(\vec{a}\circ\vec{b})}{dt}=\vec{b}\circ\frac{d\vec{a}}{dt}+\vec{a}\circ\frac{d\vec{b}}{dt} \qquad \textit{for the scalar product}$$

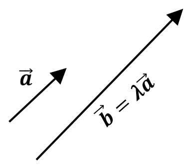
$$\frac{d(\vec{a}\wedge\vec{b})}{dt}=\vec{a}\wedge\frac{d\vec{b}}{dt}+\frac{d\vec{a}}{dt}\ \wedge\ \vec{b} \qquad \textit{for the cross product}$$

I - Coordinate systems

1- Introduction

- Two vectors are linearly dependent if one vector can be expressed in terms of the other.

$\vec{b} = \lambda\vec{a}$ where "λ" is a real



vectors linearly dependant

- Two vectors are linearly independent if any of the vectors cannot be expressed in terms of the other.

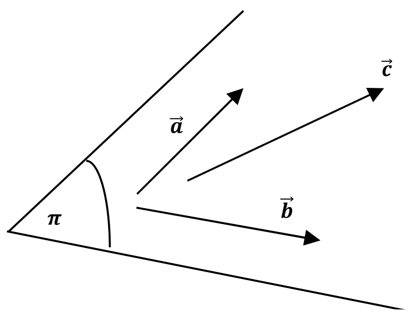
Remarks:

- In a plane, a vector can be expressed as a linear combination of two linearly independent vectors.

$$\vec{c} = \alpha\vec{a} + \beta\vec{b}$$

- The case can be generalized to three dimensions and more

$$\vec{v} = \alpha\vec{a} + \beta\vec{b} + \gamma\vec{c} + \cdots$$



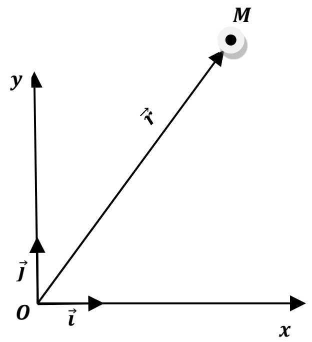
vectors lineairly independant

- The three vectors $\vec{a}, \vec{b}, \vec{c}$ form a basis if they are linearly independent.
    - If they are pairwise orthogonal, they form an orthogonal basis.
    - If they are normalized, the basis is called orthonormal.

2- Representation in the plan

2.1- Cartesian (Rectangular)coordinates $[(x, y) \rightarrow (\vec{i}, \vec{j})]$

In the plane we choose an orthonormal basis $(\vec{i}, \vec{j})$ where the coordinates of the point "M" are (x, y)



Location of "M" :

The point M position is given by the vector $\overrightarrow{OM}$ such that:

$$\overrightarrow{OM} = \vec{r} = x\vec{i} + y\vec{j}$$

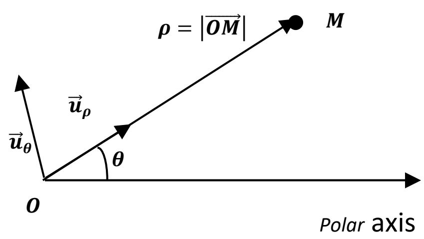
The module is:

$$|\overrightarrow{OM}| = |\vec{r}| = \sqrt{x^2 + y^2}$$

2.2- Polar coordinates$[(\rho, \theta) \rightarrow (\vec{u}_\rho, \vec{u}_\theta)]$

If we choose a local base $(\vec{u}_\rho, \vec{u}_\theta)$. "$O$" taken arbitrarily as the pole. The unit vector $\vec{u}_\rho$ is oriented along the vector $\overrightarrow{OM}$. The direction passing through the pole "O" is the polar axis, taken as a reference to define the angle (coordinate) "θ". The other coordinate "ρ" is the magnitude of the vector $\overrightarrow{OM}$.

$$\overrightarrow{OM} = \rho\vec{u}_\rho$$

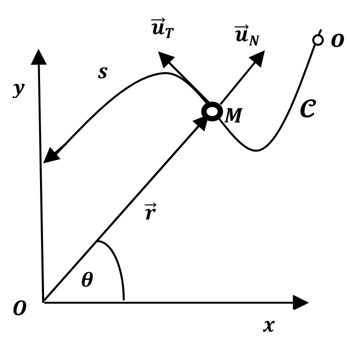


The module is:

$$|\overrightarrow{OM}| = \rho$$

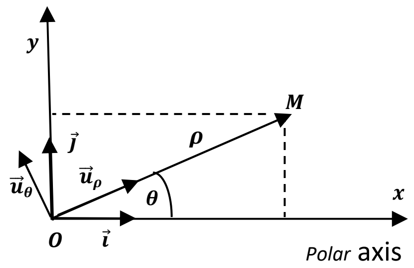
2.3- Intrinsic coordinates $[(\vec{u}_N, \vec{u}_T)]$

We cannot represent the point in the intrinsic coordinate system unless we know the curve "C" of the trajectory, which is taken as the axis. Equipped with an origin, the distance $\widehat{oM}$ is denoted as "s".



$$\widehat{oM} = s \quad \text{and} \quad \overrightarrow{OM} = \vec{r}$$

2.4- Relationship between the coordinates of the different systems



- In Cartesian coordinates: $\overrightarrow{OM} = \vec{r} = x\vec{i} + y\vec{j}$

- In Polar coordinates: $\overrightarrow{OM} = \rho\vec{u}_\rho$

- If we make a choice such that the polar axis is superimposed with the $\overrightarrow{ox}$ axis

We will have:

$$\begin{cases} \vec{u}_\rho = cos(\theta)\,\vec{i} + sin(\theta)\vec{j} \\ \vec{u}_\theta = -sin(\theta)\,\vec{i} + cos(\theta)\vec{j} \end{cases}$$

Then: $$\overrightarrow{OM} = \vec{r} = x\vec{i} + y\vec{j} = \rho\vec{u}_\rho = \rho\, cos(\theta)\,\vec{i} + \rho\, sin(\theta)\,\vec{j}$$

By comparison we will get:

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases} \Leftrightarrow \begin{cases} \rho = \sqrt{x^2 + y^2} \\ \theta = arctg(y/x) \end{cases}$$

Note:

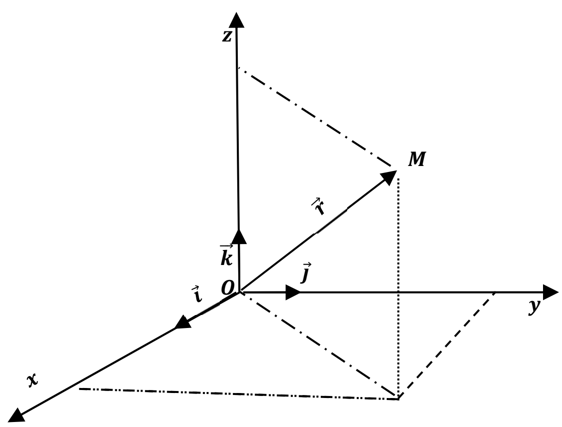
Polar coordinates and intrinsic coordinates should not be merge (confused).

3- Representation in space

3.1- Cartesian (Rectangular)coordinates $[(x, y, z) \rightarrow (\vec{i}, \vec{j}, \vec{k})]$

In space, the location of the point "M" is expressed by the (x, y, z) coordinates in an orthonormal basis $(\vec{i}, \vec{j}, \vec{k})$. in such a way that:

$$\overrightarrow{OM} = \vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$



$\overrightarrow{OM}$: is the position vector of the point M

The module is:

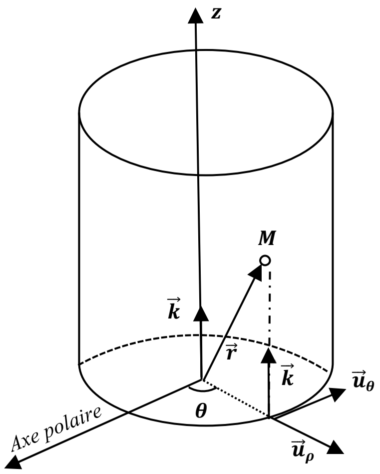
$$|\overrightarrow{OM}| = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$

x : is the projection of $\overrightarrow{OM}$ on the direction $\vec{i}$

y : is the projection of $\overrightarrow{OM}$ on the direction $\vec{j}$

z : is the projection of $\overrightarrow{OM}$ on the direction $\vec{k}$

3.2- Coordinates cylindrical $[(\rho, \theta, z) \rightarrow (\vec{u}_\rho, \vec{u}_\theta, \vec{k})]$



To locate a point "M" in space, instead of using a Cartesian system, other systems can be used. Among these, the cylindrical system. In this system, we imagine that point "M" is on the surface of a cylinder with axis $\overrightarrow{OZ}$, radius ρ, and "some" base.

The projection of $\overrightarrow{OM}$, on the base of the cylinder is located by (ρ, θ).

So $\quad \overrightarrow{OM} = \vec{r} = \rho\vec{u}_\rho + z\vec{k}$

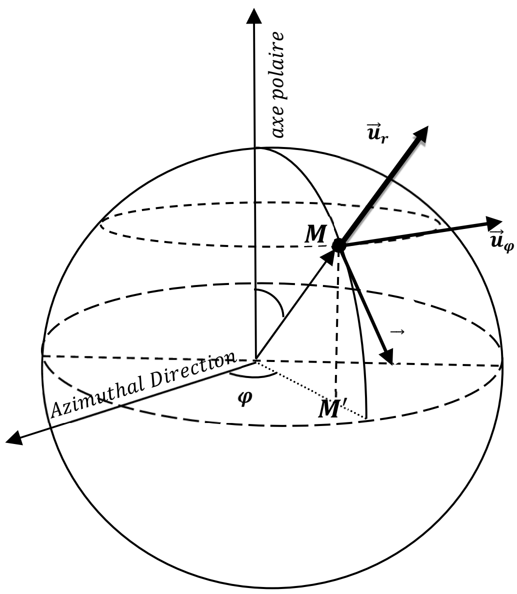
And $\quad |\overrightarrow{OM}| = |\vec{r}| = \sqrt{\rho^2 + z^2}$

3.3- Spherical coordinates $[(r, \theta, \varphi) \rightarrow (\vec{u}_r, \vec{u}_\theta, \vec{u}_\varphi)]$

Another system allows us to locate a point "M" in space. In this system, it is imagined that point "M" is on the surface of a sphere with radius "r" and center "O" . This center is taken as the origin, and called pole. It is located in the equatorial plane.

In spherical coordinates, a point "M" is characterized by the linear variable "r", and the angular variables "φ, θ".

- "θ" polar angle: Angle between the polar axis taken arbitrarily and the direction $\overrightarrow{OM}$.
  "O" is the center of this sphere.
- The projection of "M" on the Equatorial plane is "M' ". It is located by the azimuthal angle "φ" with respect to an arbitrary direction axis (azimuthal direction) in that plane.



θ

$$\overrightarrow{OM} = \vec{r} = |\vec{r}|\vec{u}_r$$

- $\vec{u}_r$: radial unit vector (in the direction of the radius $\overrightarrow{OM}$)
- $\vec{u}_\theta$: unit vector tangent to the great circle (all circles of radius $\overrightarrow{OM}$).
- $\vec{u}_\varphi$: unit vector tangent to parallels (circles parallel to the equator).

3.4- Relationship between the coordinates of the different systems

3.4- 1 Relationship between Cartesian coordinates and cylindrical coordinates

- In Cartesian coordinates: $\overrightarrow{OM} = \vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$
- In cylindrical coordinates: $\overrightarrow{OM} = \rho\vec{u}_\rho + z\vec{k}$

With $\vec{u}_\rho = \cos\theta\,\vec{i} + \sin\theta\,\vec{j}$

$$\overrightarrow{OM} = \vec{r} = x\vec{i} + y\vec{j} + z\vec{k} = \rho\cos\theta\,\vec{i} + \rho\sin\theta\,\vec{j} + z\vec{k}$$

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \\ z = z \end{cases} \Leftrightarrow \begin{cases} \rho = \sqrt{x^2 + y^2} \\ \theta = arctg(y/x) \end{cases}$$

3.4- 2 Relationship between Cartesian and spherical coordinates

- In Cartesian coordinates: $\overrightarrow{OM} = \vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$
- In spherical coordinates: $\overrightarrow{OM} = |\vec{r}|\vec{u}_r = r\vec{u}_r$

With $\vec{u}_r = \sin\theta\cos\varphi\,\vec{i} + \sin\theta\sin\varphi\,\vec{j} + \cos\theta\,\vec{k}$

So:

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases} \Leftrightarrow \begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \varphi = arctg(y/x) \\ \theta = arcos\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) \end{cases}$$

II – *Kinematics*

0- Some words on kinematics

Kinematics is a branch of physics that studies the motion of objects without considering the causes behind that motion. It focuses on describing the position, velocity, and acceleration of particles or objects in motion. By studying kinematics, scientists can analyze the relationship between these variables and understand how an object moves and changes over time.

One of the fundamental concepts in kinematics is displacement. Displacement refers to the change in position of an object or particle from its initial position to its final position. It is a vector quantity as it has both magnitude and direction. By calculating the displacement, one can determine how far an object has moved and in which direction it has traveled.

Another important concept in kinematics is velocity. Velocity is the rate at which an object moves in a certain direction. It is calculated by dividing the displacement of an object by the time taken to travel that distance. Velocity is also a vector quantity and is dependent on both the magnitude and direction of displacement. It provides information about the speed of an object and the direction in which it is moving.

Overall, kinematics plays a pivotal role in understanding the basics of motion. By studying displacement and velocity, scientists can analyze an object's movement and describe it accurately. Whether it is calculating the displacement of a ball rolling down the slope or analyzing the velocity of a car in a race, kinematics helps provide insights into the motion of objects, enhancing our understanding of the physical world.

Kinematics is a branch of physics that deals with the motion of objects without considering what causes that motion. It focuses on describing the position, velocity, and acceleration of an object as it moves through space and time. Kinematics helps us understand how objects move and allows us to predict their future positions and velocities based on their initial conditions.

When studying kinematics, it is essential to know the basic terms used to describe motion. The position of an object refers to its location relative to a chosen reference point. Velocity is the rate at which an object's position changes, while acceleration is the rate at which its velocity changes. It is important to note that velocity and acceleration are vector quantities which means they have both a magnitude and a direction.

To analyze motion, kinematics uses mathematical equations and graphs. The three equations of motion, often referred to as the kinematic equations, are commonly used to solve various kinematics problems. The equations involve the initial and final velocities, acceleration, displacement, and time intervals. Graphs, such as position-time or velocity-time graphs, can provide a visual representation of an object's motion, enabling us to analyze its behavior more easily.

In summary, kinematics is a fundamental concept in physics that helps us understand the motion of objects. It involves studying the position, velocity, and acceleration of objects without considering the forces that cause them to move. By utilizing mathematical equations and graphs, kinematics allows us to predict and analyze the motion of objects accurately. Understanding kinematics is crucial for further exploring more complex topics in physics, such as dynamics and mechanics in general.

1- Concept of frame of reference

Let $(\vec{i},\vec{j},\vec{k})$ be an orthonormal basis, placed at a point chosen as the origin, which is used to locate a point " **M** ". It constitutes a reference frame. (***Frame of Reference = origin + basis***)

- If this point M is moving, it depends on time.

$$\overrightarrow{OM}(t) = \vec{r}(t) = x(t)\,\vec{i} + y(t)\,\vec{j} + z(t)\,\vec{k} \qquad t : \text{is time}$$

- The Concept of motion is relative according to the observer (rest, moving independently or with the mobile M). The observer is the witness of time.

" *To describe the motion of a material point, a reference frame is necessary, that we affect (bind or link) an observer* to it, *which leads us to define a frame of reference* "

For example, the observer on earth say that path of moon is almost circular. But the observer siting on the sun sees the trajectory of the moon (same object) is a line wave path.

A reference frame is a platform from where a physical phenomenon, such as motion, is being observed

2- Equation of motion and trajectory equation

The change in position of the point, produce a motion. That motion is characterized by several parameters which are the displacement, distance, velocity and acceleration.

2.1- Position vector

The motion of a particle is described in some frame of reference. Starting first with locating it (position vector), then give its nature.

In an orthonormal coordinate system (cartesian) $(O, \vec{i}, \vec{j}, \vec{k})$ the position vector is given by:

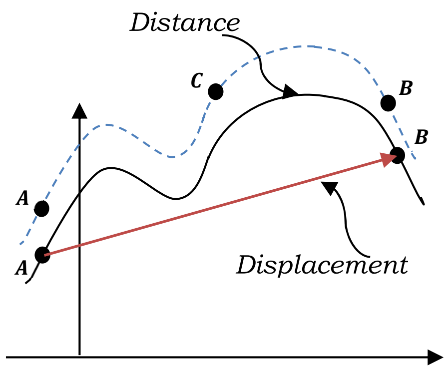
$$\overrightarrow{OM} = \vec{r} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k}$$

2.2- Displacement and distance

2.2-1: Displacement

The shortest distance joining the points **A** and **B** of the curve *i.e.*, the line AB which called displacement. It expresses how far is **B** from **A**.

Notice that the line AB has a direction from A to B.



- Displacement: vector $\overrightarrow{AB}$ (Red line)

- Distance: length of the curve ACB (Blue Dashed)

So, the displacement:

- *It is a vector quantity and is independent of the choice of origin*
- *It is unique for any kind of motion between two points*
- *It is always concealing (cover) about the actual track followed by the particle's motion between any two points, i.e. It doesn't give information about a path.*
- *It can be positive, negative and even be zero.*
- *The magnitude of the displacement is always less than or equal to the distance for particle's motion between two points*
- *A body may have finite distance travelled for zero displacement*

2.2-2: Distance

The distance express how long is the path from $\boldsymbol{A}$ to $\boldsymbol{B}$ passing through a point $\boldsymbol{C}$.

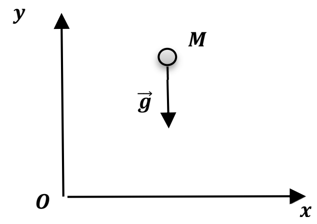
- *The distance is a scalar quantity*

- *The distance is always positive i.e., it only increases.*

- *The distance is always greater or equal to the magnitude of the displacement.*

2.3- Equation of motion

The equation of motion expresses the manner of change of motion or how the motion is changing in time by giving its kinematic parameters which are the displacement, velocity and acceleration.



Example: Free fall

$$y(t) = -\frac{1}{2}gt^2 + v_0t + y_0$$

The distance $\boldsymbol{y}$ traveled by the point M is given as a function of time.

y(t) is the time equation of motion

Note:

The coordinates of the point, M $(x(t)\,,\, y(t) z(t))$, are the parametric equations

2.4- Trajectory equation (Path equation)

- Since the vector position $\overrightarrow{OM}$ changes, i.e., the point $\boldsymbol{M}$ change its position as time is varying, then we have:

$$\begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases}$$

$(x(t), y(t),$ and $z(t)$ are called parametric equations of motion.

- ***The trajectory (path) is the curve which traces the locations occupied by the mobile in space during the variations of time (as the time is changing).***

- To find the equation of the trajectory, we eliminate the time from the parametric equations, and find the form: $f(x, y, z) = 0$

Example: Motion in the plane

$$\begin{cases} x = acos(\omega t) \\ y = asin(\omega t) \end{cases} \Longrightarrow \begin{cases} x^2 = a^2cos^2(\omega t) \\ y^2 = a^2sin^2(\omega t) \end{cases} \Longrightarrow x^2 + y^2 = a^2$$

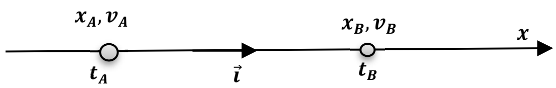
This is an equation of circle with radius $R = a$ centered on the point C (0,0) called the center of this circle.

3- Concept of velocity and speed

3.1-1: Average velocity

The average velocity is the ratio of the displacement between two points A and B to the travel time without taking into account the nature of the motion (the way in which the section AB is traveled).

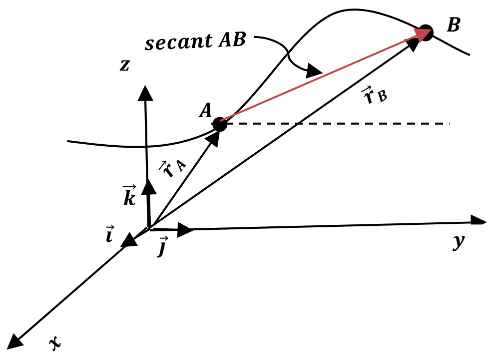
- **In one direction (one dimension)**



Let the motion along the straight-line "ox ". *The point "A" is the initial position and "B" is the final point, so the average velocity is defined as* $< \vec{v} \geq \vec{v}_{moy}$ *such that:*

$$< \vec{v} \geq \vec{v}_{moy} = \frac{x_f - x_i}{t_f - t_i}\vec{i} = \frac{x_B - x_A}{t_B - t_A}\vec{i} = \frac{\Delta x}{\Delta t}\vec{i}$$

- **In all space (three dimensions)**:



The initial point is: $A(x_A, y_A, z_A)$

and the final point is: $B(x_B, y_B, z_B)$

The displacement is then: $\Delta\vec{r} = \vec{r}_B - \vec{r}_A$

So, the average velocity is:

$$< \vec{v} > = \vec{v}_{moy} = \frac{\Delta\vec{r}}{\Delta t} = \frac{\vec{r}_f - \vec{r}_i}{t_f - t_i} \Longrightarrow < \vec{v} > = \vec{v}_{moy} = \frac{x_B - x_A}{t_B - t_A}\vec{i} + \frac{y_B - y_A}{t_B - t_A}\vec{j} + \frac{z_B - z_A}{t_B - t_A}\vec{k}$$

Note:

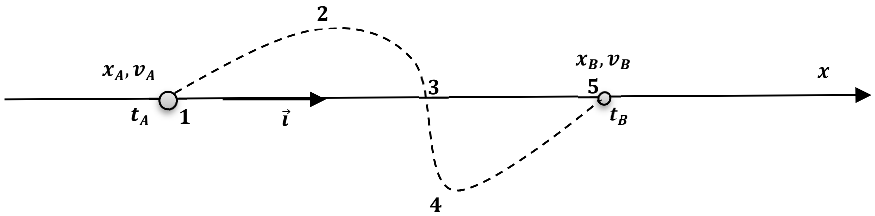
The average velocity, geometrically, is the slope of the secant that join the final and the initial positions in the curve which represents the variation of position with time (x-t curve).

3.1-2: Average speed

The average speed is the ratio of the distance traveled between two points A and B to the travel time of duration of trip

$$< S > = S_{moy} = \frac{Distance\ covered}{time\ taken}$$

The path A≡1-2-3-4-5≡B has the distance **d**, so the average speed defined as the distance of the journey on the time duration of the trip



If a particle starts from $'A \equiv 1'$ to the point $'B'$. Let $'d_{12}'$ be the distance covered by the particle to go from position $'1'$ to $'2'$, and $'d_{23}'$ that covered from position $'2'$ to $'3'$ and so on, until this particle arrives to the final position $'5 \equiv B'$. The average speed is:

$$S_{moy} = \frac{Distance\ covered}{time\ taken} = \frac{d_{12} + d_{23} + \ldots + d_{45}}{t_{12} + t_{23} + \ldots + t_{45}}$$

3.1-2-1: Average speed in case when the time is divided in equal intervals

Let the actual path from A to B, be divided in several intervals not equal, each traversed with in the same lapse of time but with different speeds. To compute the average speed, we proceed as follows:

$$S_{moy} = \frac{Distance\ covered}{time\ taken} = \frac{d_1 + d_2 + \ldots + d_n}{t_1 + t_2 + \ldots + t_n} = \frac{\sum_1^n d_i}{\sum_1^n t_i}$$

But $d_1 = s_1 . t_1$, …., $d_1 = s_n . t_n$

Since the lapse of time are equal: $t_1 = t_2 = \ldots = t_n = {}^{t}/_{n}$ with $'t'$ the time taken during the trip

$$S_{moy} = \frac{v_1(\frac{t}{n}) + v_2(\frac{t}{n}) + \ldots + v_n(\frac{t}{n})}{t_1 + t_2 + \ldots + t_n} = \frac{\frac{t}{n}(v_1 + v_2 + \ldots + v_n)}{t} = \frac{\sum_1^n v_i}{n}$$

So, we observe that when an interval is divided into n equal time parts, then the average speed $'S_{moy}'$ is simply the arithmetic mean of the speeds in the respective intervals.

$$S_{moy} = \frac{1}{n}\sum_{1}^{n} v_i$$

3.1-2-2: Average speed in case when the length is divided in equal intervals

In the same manner, the distance will be divided in equal intervals

$d_1 = d_2 = \ldots = d_n = {}^{d}/_{n}$,

But $d_1 = v_1 t_1$ and $d_2 = v_2 t_2$, …, $d_n = v_n t_n$

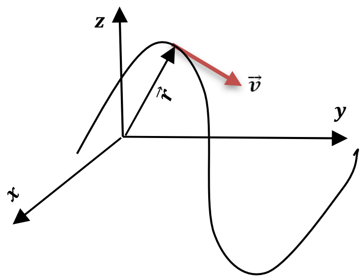
So $t_1 = \frac{d_1}{v_1} = \frac{d/n}{v_1}$, …, $t_n = \frac{d_n}{v_n} = \frac{d/n}{v_n}$

$$S_{moy} = \frac{d_1 + d_2 + \ldots + d_n}{t_1 + t_2 + \ldots + t_n} = \frac{d/n + d/n + \ldots + d/n}{\frac{d_1}{v_1} + \ldots + \frac{d_n}{v_n}} = \frac{d}{\frac{d}{nv_1} + \ldots + \frac{d}{nv_n}} = \frac{n}{\frac{1}{v_1} + \ldots + \frac{1}{v_n}}$$

The average speed $'S_{moy}'$ is simply n time the reciprocal of the harmonic mean of the speeds in the respective intervals.

3.2- Instantaneous velocity and instantaneous speed

Instantaneous velocity is the velocity that the material point will have at every moment on the trip. The velocity, in an infinitely small lapse of time, in the corresponding infinitesimal displacement, doesn't change.



$$\vec{v}(t) = \lim_{\Delta t \to 0}\left(\frac{\Delta \vec{r}}{\Delta t}\right) = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k}$$

$$\vec{v}(t) = v_x\,\vec{i} + v_y\,\vec{j} + v_z\,\vec{k}$$

The magnitude of the velocity is given by:

$$|\vec{v}(t)| = \sqrt{v_x^2 + v_y^2 + v_z^2}$$

Note:

1- The instantaneous speed is equal to the magnitude of the instantaneous velocity

2- *The instantaneous velocity, geometrically, is the slope of the tangent to the curve that represents the change in position with time (x-t curve).*

4- Concept of acceleration

4.1- Average acceleration

- Acceleration is the rate of change of velocity over time.

- The average acceleration is the rate of change in velocity between the initial "A" and final points B, regardless of how the path is traversed.

- In one direction only

$$< \vec{a} > = \vec{a}_{moy} = \frac{\Delta\vec{v}}{\Delta t} = \frac{\vec{v}_f - \vec{v}_i}{t_f - t_i} = \frac{v_B - v_A}{t_B - t_A}\vec{i}$$

- I**n all space (three dimensions)**

The initial velocity is: $\vec{v}_A(v_{xA}, v_{yA}, v_{zA})$

The final velocity is: $\vec{v}_B(v_{xB}, v_{yB}, v_{zB})$

The variation of velocity is then: $\Delta\vec{v} = \vec{v}_B - \vec{v}_A$

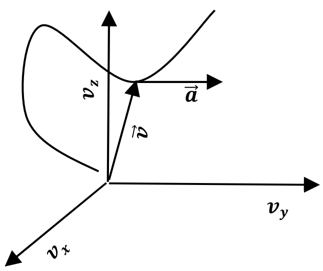
So, the average acceleration is:

$$< \vec{a} > = \vec{a}_{moy} = \frac{\Delta\vec{v}}{\Delta t} = \frac{\vec{v}_f - \vec{v}_i}{t_f - t_i}$$

$$\vec{a}_{moy} = \frac{v_{xB} - v_{xA}}{t_B - t_A}\vec{i} + \frac{v_{yB} - v_{yA}}{t_B - t_A}\vec{j} + \frac{v_{zB} - v_{zA}}{t_B - t_A}\vec{k}$$

4.2- Instantaneous acceleration

Instantaneous acceleration is the rate of change of velocity in time at each moment. In an infinitely small lapse of time, on the corresponding infinitesimal change in velocity, the acceleration doesn't change.



$$\vec{a}(t) = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt} = \frac{dv_x}{dt}\vec{\imath} + \frac{dv_y}{dt}\vec{\jmath} + \frac{dv_z}{dt}\vec{k}$$

- The hodograph of motion is the curve described by the end of the velocity vector

- **Note**:

The instantaneous acceleration, geometrically, is the slope of the curve (hodograph) which represents the variation of the velocity over time.

$$\vec{a}(t) = \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right) = \frac{d^2\vec{r}}{dt^2} = \frac{d^2x}{dt^2}\vec{\imath} + \frac{d^2y}{dt^2}\vec{\jmath} + \frac{d^2z}{dt^2}\vec{k}$$

$$\vec{a}(t) = a_x\vec{\imath} + a_y\vec{\jmath} + a_z\vec{k}$$

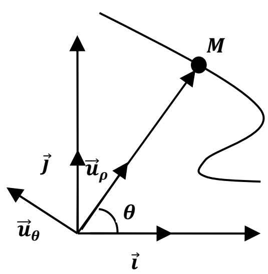
5- Position, velocity and acceleration in the different coordinate systems

5.1- Derivative of unit vectors

- Polar basis $(\vec{u}_\rho, \vec{u}_\theta)$

$\theta(t)$ and $\rho(t)$ change in time $\vec{u}_\rho, \vec{u}_\theta$ changes also and are written in the Cartesian base as follows:

$$\begin{cases} \vec{u}_\rho = cos(\theta)\,\vec{\imath} + sin(\theta)\,\vec{\jmath} \\ \vec{u}_\theta = -sin(\theta)\,\vec{\imath} + cos(\theta)\,\vec{\jmath} \end{cases}$$



$\vec{u}_\rho$ and $\vec{u}_\theta$ are a composite function, so we apply the chain rule

- If we have a function $f = F(u(x))$ which depend on the variable u who depends also on the other variable x. Then the derivative of this this function with respect to the variable x is given by:

$$df = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x}$$

$$\Rightarrow \quad \begin{cases} \frac{d\vec{u}_\rho}{dt} = \frac{d\vec{u}_\rho}{d\theta}\frac{d\theta}{dt} = [-sin(\theta)\,\vec{\imath} + cos(\theta)\,\vec{\jmath}]\frac{d\theta}{dt} \\ \frac{d\vec{u}_\theta}{dt} = \frac{d\vec{u}_\theta}{d\theta}\frac{d\theta}{dt} = [-cos(\theta)\,\vec{\imath} - sin(\theta)\,\vec{\jmath}]\frac{d\theta}{dt} \end{cases}$$

$$\Rightarrow \begin{cases} \frac{d\vec{u}_\rho}{dt} = \dot{\theta}\left[\cos\left(\theta+\frac{\pi}{2}\right)\vec{\imath}+\sin\left(\theta+\frac{\pi}{2}\right)\vec{\jmath}\right] \\ \frac{d\vec{u}_\theta}{dt} = \dot{\theta}\left[-\sin\left(\theta+\frac{\pi}{2}\right)\vec{\imath}+\cos\left(\theta+\frac{\pi}{2}\right)\vec{\jmath}\right] \end{cases} \quad \Rightarrow \quad \begin{cases} \dot{\vec{u}}_\rho = \frac{d\vec{u}_\rho}{dt} = \dot{\theta}\,\vec{u}_\theta \\ \dot{\vec{u}}_\theta = \frac{d\vec{u}_\theta}{dt} = -\dot{\theta}\,\vec{u}_\rho \end{cases}$$

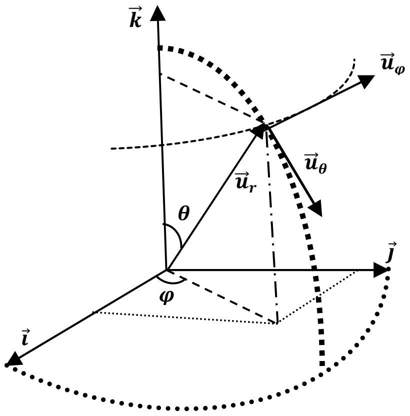
To find the derivative of a unit vector, we make a rotation anti-clockwise of $+\pi/2$

Note:

The cylindrical basis gives similar results as the polar basis by adding the* z *coordinate.

- Spherical base $\vec{u}_r, \vec{u}_\theta, \vec{u}_\varphi$

From the figure, the unit vectors $\vec{u}_r$, $\vec{u}_\theta$ and $\vec{u}_\varphi$ are given by:



$$\begin{cases} \vec{u}_r = \sin\theta\cos\varphi\,\vec{\imath} + \sin\theta\sin\varphi\,\vec{\jmath} + \cos\theta\,\vec{k} \\ \vec{u}_\theta = \cos\theta\cos\varphi\,\vec{\imath} + \cos\theta\sin\varphi\,\vec{\jmath} - \sin\theta\,\vec{k} \\ \vec{u}_\varphi = -\sin\varphi\,\vec{\imath} + \cos\varphi\,\vec{\jmath} \end{cases}$$

If we use the differential form of the function that depend on many variables:

$$f(x,y,z) \quad \Rightarrow \quad df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

$$\Rightarrow \quad \frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

$$\vec{u}_r = f(\theta,\varphi) \text{ and } \vec{u}_r = h(\theta,\varphi) \quad \Rightarrow \begin{cases} d\vec{u}_r = \frac{\partial \vec{u}_r}{\partial\theta}.d\theta + \frac{\partial \vec{u}_r}{\partial\varphi}.d\varphi \\ d\vec{u}_\theta = \frac{\partial \vec{u}_\theta}{\partial\theta}.d\theta + \frac{\partial \vec{u}_\theta}{\partial\varphi}.d\varphi \\ d\vec{u}_\varphi = \frac{d\vec{u}_\varphi}{d\varphi}.d\varphi \end{cases}$$

Finally

$$\Rightarrow \begin{cases} \dot{\vec{u}}_r = \frac{d\vec{u}_r}{dt} = \frac{\partial \vec{u}_r}{\partial\theta}.\frac{d\theta}{dt} + \frac{\partial \vec{u}_r}{\partial\varphi}.\frac{d\varphi}{dt} = \dot{\theta}\,\vec{u}_\theta + \dot{\varphi}\sin\theta\,\vec{u}_\varphi \\ \dot{\vec{u}}_\theta = \frac{d\vec{u}_\theta}{dt} = \frac{\partial \vec{u}_\theta}{\partial\theta}.\frac{d\theta}{dt} + \frac{\partial \vec{u}_\theta}{\partial\varphi}.\frac{d\varphi}{dt} = -\dot{\theta}\,\vec{u}_r + \dot{\varphi}\cos\theta\,\vec{u}_\varphi \\ \dot{\vec{u}}_\varphi = \frac{d\vec{u}_\varphi}{dt} = \frac{d\vec{u}_\varphi}{d\varphi}.\frac{d\varphi}{dt} = -\dot{\varphi}(\sin\theta\,\vec{u}_r + \cos\theta\,\vec{u}_\theta) \end{cases}$$

$$\Rightarrow \begin{cases} \dot{\vec{u}}_r = \dot{\theta}\,\vec{u}_\theta + \dot{\varphi}\sin\theta\,\vec{u}_\varphi \\ \dot{\vec{u}}_\theta = -\dot{\theta}\,\vec{u}_r + \dot{\varphi}\cos\theta\,\vec{u}_\varphi \\ \dot{\vec{u}}_\varphi = -\dot{\varphi}(\sin\theta\,\vec{u}_r + \cos\theta\,\vec{u}_\theta) \end{cases}$$

5.2- Polar coordinates

a- Position vector

As we have already seen that the position vector in the polar basis is:

$$\overrightarrow{OM} = \vec{r} = \rho\vec{u}_\rho \Rightarrow |\overrightarrow{OM}| = \rho$$

b- velocity vector

According to the definition:

$$\vec{v}(\,t\,) = \frac{d\overrightarrow{OM}}{dt} = \frac{d\vec{r}}{dt} = \frac{d(\rho\vec{u}_\rho)}{dt} = \dot{\rho}\vec{u}_\rho + \rho\frac{d\vec{u}_\rho}{dt}$$

But $\frac{d\vec{u}_\rho}{dt} = \dot{\theta}\,\vec{u}_\theta$

Then $\vec{v}(\mathbf{t}) = \dot{\rho}\,\vec{u}_\rho + \rho\dot{\theta}\,\vec{u}_\theta = \vec{v}_\rho + \vec{v}_\theta$

Where $\begin{cases} |\vec{v}_\rho| = \dot{\rho} \\ |\vec{v}_\theta| = \rho\dot{\theta} \end{cases}$

$$\Rightarrow \quad |\vec{v}(\mathbf{t})| = \sqrt{v_\rho^2 + v_\theta^2} = \sqrt{\dot{\rho}^2 + (\rho\dot{\theta})^2}$$

c- acceleration vector

According to the definition: $\vec{\dot{u}}_\rho$ $\vec{\dot{u}}_\theta$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right) = \frac{d}{dt}(\dot{\rho}\,\vec{u}_\rho + \rho\dot{\theta}\,\vec{u}_\theta)$$

$$\Rightarrow \quad \vec{a} = \ddot{\rho}\,\vec{u}_\rho + \dot{\rho}\vec{\dot{u}}_\rho + \dot{\rho}\dot{\theta}\,\vec{u}_\theta + \rho\ddot{\theta}\,\vec{u}_\theta + \rho\dot{\theta}\,\vec{\dot{u}}_\theta$$

$$\vec{a} = (\ddot{\rho} - \rho\dot{\theta}^2)\,\vec{u}_\rho + (2\dot{\rho}\dot{\theta} + \rho\ddot{\theta})\,\vec{u}_\theta = \vec{a}_\rho + \vec{a}_\theta$$

Where $\begin{cases} |\vec{a}_\rho| = (\ddot{\rho} - \rho\dot{\theta}^2) \\ |\vec{a}_\theta| = (2\dot{\rho}\dot{\theta} + \rho\ddot{\theta}) \end{cases}$

$$\Rightarrow \quad |\vec{a}(\mathbf{t})| = \sqrt{a_\rho^2 + a_\theta^2} = \sqrt{(\ddot{\rho} - \rho\dot{\theta}^2)^2 + (2\dot{\rho}\dot{\theta} + \rho\ddot{\theta})^2}$$

5.3- Intrinsic coordinates (Natural coordinates)

The coordinate is the distance "$\boldsymbol{s}$ "traveled along the trajectory such that:

$$s = \widehat{oM}$$

The position vector is given by:

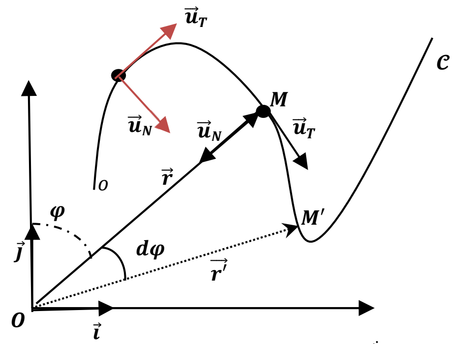
$$\overrightarrow{OM} = \vec{r} = x\,\vec{\imath} + y\,\vec{\jmath} \qquad .$$

If the mobile moves from the point M to the point M', then

$$\vec{r}' = \vec{r} + \overrightarrow{MM'} \qquad \Rightarrow \qquad \vec{r}' - \vec{r} = \overrightarrow{MM'} = \overrightarrow{dr} = dx\,\vec{\imath} + dy\,\vec{\jmath}$$

The segment "$\boldsymbol{ds}$"of the curve is related to the variation of Cartesian coordinates in such a way that:

$$ds = \sqrt{dx^2 + dy^2} = \left|\overrightarrow{dr}\right|$$



b- velocity vector

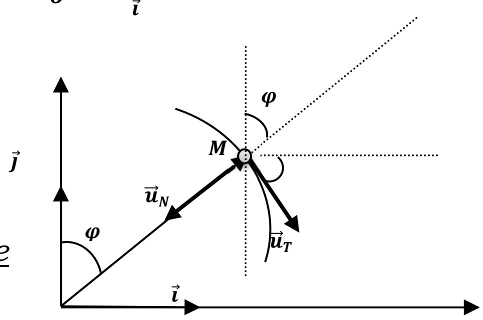
According to the definition:

$$\vec{v}(\mathrm{t}) = \frac{d\overrightarrow{OM}}{dt} = \frac{d\vec{r}}{dt} = \frac{\overrightarrow{MM'}}{dt} = \frac{\left|\overrightarrow{MM'}\right|}{dt}\vec{u}_T$$

Since the limit $\quad ds = \left|\overrightarrow{MM'}\right| = \left|\overrightarrow{dr}\right|$

Then: $\quad \vec{v}(t) = \frac{\left|\overrightarrow{MM'}\right|}{dt}\vec{u}_T = \frac{ds}{dt}\vec{u}_T = v.\vec{u}_T$

<u>*The velocity vector is oriented along the tangent to the curve*</u>



c- acceleration vector

According to the definition:

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right) = \frac{d}{dt}(v.\vec{u}_T) = \frac{d}{dt}(v).\vec{u}_T + v.\frac{d}{dt}(\vec{u}_T)$$

According to the previous figure “ $\vec{u}_T$ " is tangent to the curve and “ $\vec{u}_N$ "oriented towards the concavity

$$\begin{cases} \vec{u}_T = \cos\varphi\,\vec{\imath} - \sin\varphi\,\vec{\jmath} \\ \vec{u}_N = -\sin\varphi\,\vec{\imath} - \cos\varphi\,\vec{\jmath} \end{cases}$$

$$\Rightarrow \begin{cases} \dot{\vec{u}}_T = \frac{d\vec{u}_T}{dt} = -\frac{d\varphi}{dt}(sin\varphi\,\vec{i} + cos\varphi\,\vec{j}) = \dot{\varphi}\,\vec{u}_N \\ \dot{\vec{u}}_N = \frac{d\vec{u}_N}{dt} = -\frac{d\varphi}{dt}(cos\varphi\vec{i} - sin\varphi\vec{j}) = -\dot{\varphi}\,\vec{u}_T \end{cases}$$

Then:

$$\vec{a} = \frac{d}{dt}(v).\vec{u}_T + v.\dot{\varphi}\,\vec{u}_N$$

But $\widehat{MM'} = ds = \rho d\varphi$

ρ is a curvature radius

$$\frac{ds}{dt} = \rho\frac{d\varphi}{dt} = v$$

$$\Rightarrow \ \dot{\varphi} = v/\rho$$

Finally

$$\vec{a} = \frac{dv}{dt}.\vec{u}_T + v.\frac{v}{\rho}\vec{u}_N = \frac{dv}{dt}.\vec{u}_T + \frac{v^2}{\rho}\vec{u}_N = \vec{a}_T + \vec{a}_N$$

$$\vec{a} = \vec{a}_T + \vec{a}_N$$

$$\begin{cases} \vec{a}_T = \ \textit{is the tangential component due to the variation of the speed modulus} \\ \vec{a}_N = \ \textit{is the normal component due to the variation in the direction of the speed} \end{cases}$$

5.4- Cylindrical coordinates

a- Position vector

The position vector is given by:

$$\overrightarrow{OM} = \rho\,\vec{u}_\rho + z\,\vec{k} \Rightarrow \left|\overrightarrow{OM}\right| = \sqrt{\rho^2 + z^2}$$

b- velocity vector

Based on definition:

$$\vec{v}(\mathbf{t}) = \frac{d\overrightarrow{OM}}{dt} = \frac{d\vec{r}}{dt} = \frac{d(\rho\,\vec{u}_\rho + z\,\vec{k})}{dt} = \dot{\rho}\,\vec{u}_\rho + \rho\frac{d\vec{u}_\rho}{dt} + \dot{z}\,\vec{k}$$

But $\frac{d\vec{u}_\rho}{dt} = \dot{\theta}\,\vec{u}_\theta$

$$\vec{v}(t) = \dot{\rho}\,\vec{u}_\rho + \rho\dot{\theta}\,\vec{u}_\theta + \dot{z}\,\vec{k} = \vec{v}_\rho + \vec{v}_\theta + \vec{v}_z$$

With $\begin{cases} |\vec{v}_\rho| = \dot{\rho} \\ |\vec{v}_\theta| = \rho\dot{\theta} \\ |\vec{v}_z| = \dot{z} \end{cases}$

$$\Rightarrow \quad |\vec{v}(\mathrm{t})| = \sqrt{v_\rho^2 + v_\theta^2 + v_z^2} = \sqrt{\dot{\rho}^2 + \left(\rho\dot{\theta}\right)^2 + \dot{z}^2}$$

c- acceleration vector

According to the definition:

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right) = \frac{d}{dt}\left(\dot{\rho}\,\vec{u}_\rho + \rho\dot{\theta}\,\vec{u}_\theta + \dot{z}\,\vec{k}\right)$$

$$\Rightarrow \qquad \vec{a} = \ddot{\rho}\,\vec{u}_\rho + \dot{\rho}\,\dot{\vec{u}}_\rho + \dot{\rho}\dot{\theta}\,\vec{u}_\theta + \rho\ddot{\theta}\,\vec{u}_\theta + \rho\dot{\theta}\,\dot{\vec{u}}_\theta + \ddot{z}\,\vec{k}$$

$$\vec{a} = \left(\ddot{\rho} - \rho\dot{\theta}^2\right)\vec{u}_\rho + \left(2\dot{\rho}\dot{\theta} + \rho\ddot{\theta}\right)\vec{u}_\theta + \ddot{z}\,\vec{k}$$

$\vec{a} = \vec{a}_\rho + \vec{a}_\theta + \vec{a}_z$ With $\begin{cases} |\vec{a}_\rho| = \left(\ddot{\rho} - \rho\dot{\theta}^2\right) \\ |\vec{a}_\theta| = \left(2\dot{\rho}\dot{\theta} + \rho\ddot{\theta}\right) \\ |\vec{a}_z| = \ddot{z} \end{cases}$

$$\Rightarrow \qquad |\vec{a}(\mathrm{t})| = \sqrt{a_\rho^2 + a_\theta^2 + a_z^2} = \sqrt{\left(\ddot{\rho} - \rho\dot{\theta}^2\right)^2 + \left(2\dot{\rho}\dot{\theta} + \rho\ddot{\theta}\right)^2 + \ddot{z}^{\,2}}$$

5.5- Spherical coordinates

a- Position vector

The position vector is given by**:**

$$\overrightarrow{OM} = r\,\vec{u}_r \quad \Rightarrow \quad \left|\overrightarrow{OM}\right| = r$$

b- velocity vector

According to the definition:

$$\vec{v}(t) = \frac{d\overrightarrow{OM}}{dt} = \frac{d\vec{r}}{dt} = \dot{r}\,\vec{u}_r + r\,\dot{\vec{u}}_r$$

But $\quad \dot{\vec{u}}_r = \dot{\theta}\,\vec{u}_\theta + \dot{\varphi}\,sin\theta\,\vec{u}_\varphi.$

$$\Longrightarrow \quad \vec{v}(\mathrm{t}) = \dot{r}\,\vec{u}_r + r\dot{\theta}\,\vec{u}_\theta + r\dot{\varphi}\,sin\theta\,\vec{u}_\varphi = \vec{v}_r + \vec{v}_\theta + \vec{v}_\varphi$$

$$\text{With} \quad \begin{cases} |\vec{v}_r| = \dot{r} \\ |\vec{v}_\theta| = r\dot{\theta} \\ |\vec{v}_\varphi| = r\dot{\varphi}sin\theta \end{cases}$$

$$\Longrightarrow \quad |\vec{v}(\mathrm{t})| = \sqrt{v_r^2 + v_\theta^2 + v_\varphi^2} = \sqrt{\dot{r}^2 + \left(r\dot{\theta}\right)^2 + (r\dot{\varphi}sin\theta)^2}$$

c- acceleration vector

According to the definition:

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right) = \frac{d}{dt}\left(\dot{r}\,\vec{u}_r + r\dot{\theta}\,\vec{u}_\theta + r\dot{\varphi}\,sin\theta\,\vec{u}_\varphi\right)$$

$$\Longrightarrow \vec{a} = \ddot{r}\vec{u}_r + \dot{r}\dot{\vec{u}}_r + \dot{r}\dot{\theta}\vec{u}_\theta + r\ddot{\theta}\vec{u}_\theta + r\dot{\theta}\,\dot{\vec{u}}_\theta + \dot{r}\dot{\varphi}\,sin\theta\,\vec{u}_\varphi + r\ddot{\varphi}\,sin\theta\,\vec{u}_\varphi + r\dot{\varphi}\dot{\theta}\,cos\theta\,\vec{u}_\varphi + r\dot{\varphi}\,sin\theta\,\dot{\vec{u}}_\varphi$$

Knowing also that:

$$-\ \dot{\vec{u}}_\theta = \frac{d\vec{u}_\theta}{dt} = \frac{\partial\vec{u}_\theta}{\partial\theta}.\frac{d\theta}{dt} + \frac{\partial\vec{u}_\theta}{\partial\varphi}.\frac{d\varphi}{dt} = -\dot{\theta}\,\vec{u}_r + \dot{\varphi}cos\theta\,\vec{u}_\varphi$$

$$-\ \dot{\vec{u}}_\varphi = \frac{d\vec{u}_\varphi}{dt} = \frac{d\vec{u}_\varphi}{d\theta}.\frac{d\theta}{dt} = -\dot{\varphi}(sin\theta\,\vec{u}_r + cos\theta\,\vec{u}_\theta)$$

$$\Longrightarrow \vec{a} = \left(\ddot{r} - r\dot{\theta}^2 - r\dot{\varphi}^2sin^2\theta\right)\vec{u}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\varphi}^2sin\theta cos\theta\right)\vec{u}_\theta + \left(r\ddot{\varphi}sin\theta + 2r\dot{\varphi}\dot{\theta}cos\theta + 2\dot{r}\dot{\varphi}sin\theta\right)\vec{u}_\varphi$$

$$\vec{a} = \vec{a}_r + \vec{a}_\theta + \vec{a}_\varphi \qquad \text{With} \qquad \begin{cases} |\vec{a}_r| = \ddot{r} - r\dot{\theta}^2 - r\dot{\varphi}^2sin^2\theta \\ |\vec{a}_\theta| = r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\varphi}^2sin\theta cos\theta \\ |\vec{a}_\varphi| = r\ddot{\varphi}sin\theta + 2r\dot{\varphi}\dot{\theta}cos\theta + 2\dot{r}\dot{\varphi}sin\theta \end{cases}$$

Since $\quad |\vec{a}(\mathrm{t})| = \sqrt{a_r^2 + a_\theta^2 + a_\varphi^2} \qquad \Longrightarrow$

$$|\vec{a}| = \sqrt{\left(\ddot{r} - r\dot{\theta}^2 - r\dot{\varphi}^2sin^2\theta\right)^2 + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\varphi}^2sin\theta cos\theta\right)^2 + \left(r\ddot{\varphi}sin\theta + 2r\dot{\varphi}\dot{\theta}cos\theta + 2\dot{r}\dot{\varphi}sin\theta\right)^2}$$

7-Special motions

In general, we meet in nature 3 types of motion

- The translational motion.
- The rotational motion.
- Vibrational motion

We will limit ourselves to certain particular motions of each type.

6.1-Rectilinear motion

When the trajectory of the mobile **M** is a line, the motion is said to be rectilinear.

a- Uniform rectilinear motion (motion at constant velocity)

If the motion is done at constant velocity $(\vec{a} = \vec{0})$, the motion is said to be uniform

$\Delta v = v - v_0$ Since v is constant

$\Rightarrow \quad v = v_0$ and $a = \frac{\Delta v}{\Delta t} = 0$

The route or the path can be obtained as follows:

$$v = \frac{\Delta x}{\Delta t} \quad \Rightarrow \quad \Delta x = x_f - x_i = v(t_f - t_i)$$

If $x_i = 0$ and $t_i = 0$ then

$$\boxed{x = vt} \ .$$

This is the equation of uniform rectilinear motion.

The same result is given by the integral form because:

$$v = {dx}/{dt} \Rightarrow \quad dx = vdt$$

$$\int_{x_i}^{x_f} dx = \int_{t_i}^{t_f} vdx$$

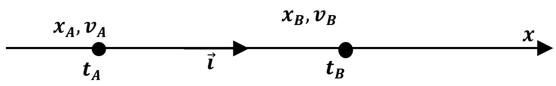
$$\Rightarrow \quad x_f - x_i = v(t_f - t_i)$$

b- Uniformly varied rectilinear motion (motion done at constant acceleration)

If the motion has (is done with) a constant acceleration, it is said to be uniformly varying.

Example: free fall

- Let a motion that done on to the direction $\overrightarrow{ox}$



$$\vec{a}_{moy} = \frac{\Delta\vec{v}}{\Delta t} = \frac{\vec{v}_f - \vec{v}_i}{t_f - t_i} = \frac{v_B - v_A}{t_B - t_A}\vec{i}$$

When the motion is uniformly varying $a = constante$

$$a_{moy} = a \Longrightarrow \Delta v = v_B - v_A = a(t_B - t_A)$$

If $t_B = t$ any time and $t_A = t_0$

as $v_B = v$ and $v_A = v_0$

also $x_B = x$ and $x_A = x_0$

Then:

$$v_B = v = a(t - t_0) + v_0$$

Likewise,

$$\vec{v}_{moy} = \frac{\Delta\vec{r}}{\Delta t} = \frac{x_f - x_i}{t_f - t_i}\vec{i} = \frac{x_B - x_A}{t_B - t_A}\vec{i}$$

$$\Longrightarrow \qquad v_{moy} = \frac{x - x_0}{t - t_0}$$

We know that the average value is given by:

$$X_{moy} = \bar{X} = \frac{\sum_{l=1}^{n} x_l}{n} \Longrightarrow v_{moy} = \frac{v + v_0}{2}$$

Hence $\boldsymbol{v_{moy}} = \frac{\boldsymbol{x - x_0}}{\boldsymbol{\Delta t}} = \frac{\boldsymbol{v + v_0}}{\boldsymbol{2}}$ t

Since $\boldsymbol{v = a(t - t_0) + v_0}$

$$\Longrightarrow \quad \boldsymbol{x - x_0 = \frac{a(t - t_0) + v_0}{2}(t - t_0) + \frac{v_0}{2}(t - t_0)}$$

$$\boldsymbol{x = \frac{a}{2}(t - t_0)^2 + v_0(t - t_0) + x_0}$$

It is the equation of uniformly varying rectilinear motion

Note: The same result can be found using the integral form.

If the acceleration is constant

$$\boldsymbol{a} = \frac{dv}{dt} \Longrightarrow \boldsymbol{dv = adt} \Longrightarrow \int_{v_0}^{v} \boldsymbol{dv} = \int_{t_0}^{t} \boldsymbol{adt}$$

$$\Longrightarrow \quad \boldsymbol{v - v_0 = a(t - t_0)} \quad \Longrightarrow \quad \boldsymbol{v = v_0 + a(t - t_0)}$$

Since $\boldsymbol{v = \frac{dx}{dt}} \quad \Longrightarrow \quad \boldsymbol{dx = vdt}$

$$\Longrightarrow \int_{x_0}^{x} \boldsymbol{dx} = \int_{t_0}^{t} \boldsymbol{vdt} = \int_{t_0}^{t} \boldsymbol{[v_0 + a(t - t_0)]dt}$$

$$\boldsymbol{x - x_0 = \frac{1}{2}a(t - t_0)^2 + v_0(t - t_0)}$$

$$\boldsymbol{x = \frac{1}{2}a(t - t_0)^2 + v_0(t - t_0) + x_0}$$

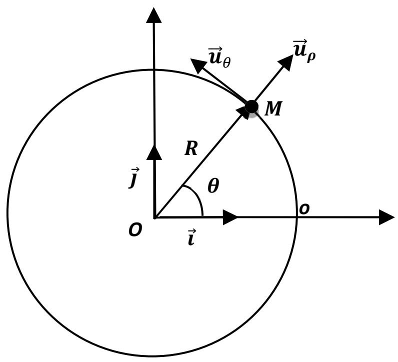
7.2- Curvilinear motion

When the trajectory of the mobile **M** is any curve, the movement is said to be curvilinear.

7.2.1- Circular motion

The trajectory of the mobile M is a circle

a-Uniform circular motion ($\dot{\theta} = \omega$ is constant)



when the motion of the mobile **M** is done at constant angular velocity.

The distance traveled is the arc $\widehat{oM} = s$, it is expressed as a function of radius R and the angle θ as follows:

$$s = R\theta$$

The position vector is:

$$\overrightarrow{OM} = \vec{r} = R\,\vec{u}_\rho$$

- **Angular velocity and linear velocity**

  - The angular velocity is given by the derivative of the angle θ with respect to time. $\frac{d\theta}{dt} = \dot{\theta} = \omega$
  - The linear velocity is given by the derivative of a displacement with respect to time.

$$\frac{d\vec{r}}{dt} = \vec{v} = \frac{d(R\vec{u}_\rho)}{dt} = \dot{R}\,\vec{u}_\rho + R\,\dot{\vec{u}}_\rho$$

since R is constant (circular motion). $\Longrightarrow \quad \dot{R} = 0$

we have $\quad \dot{\vec{u}}_\rho = \dot{\theta}\vec{u}_\theta$

$$\Longrightarrow \qquad \vec{v} = R\,\dot{\vec{u}}_\rho = R\dot{\theta}\,\vec{u}_\theta$$

So $\qquad |\vec{v}| = v = R\dot{\theta} = R\omega\ ,$

And $\frac{|d\vec{r}|}{dt} = |\vec{v}| = \frac{ds}{dt}$

The linear velocity is tangential to the curve (to the trajectory)

Angular acceleration and linear acceleration

- The angular acceleration is given by the derivative of the angular velocity $\dot{\theta}$ with respect to time.

$$\varepsilon = \ddot{\theta} = \frac{d(\dot{\theta})}{dt}$$

For a uniform circular motion $\varepsilon = o$

- Linear acceleration is given by the derivative of velocity with respect to time.

$$\frac{d\vec{v}}{dt} = \vec{a} = \frac{d(R\dot{\theta}\,\vec{u}_\theta)}{dt} = \dot{R}\dot{\theta}\,\vec{u}_\theta + R\ddot{\theta}\,\vec{u}_\theta + R\dot{\theta}\,\dot{\vec{u}}_\theta$$

Since R and $\dot{\theta}$ are constants (uniform circular motion)

$\Longrightarrow \quad \dot{R} = 0$ and $\ddot{\theta} = 0$

We have also $\dot{\vec{u}}_\theta = -\dot{\theta}\vec{u}_\rho$

$\Longrightarrow \quad \vec{a} = R\dot{\theta}(-\dot{\theta}\vec{u}_\rho) = -R\dot{\theta}^2\vec{u}_\rho$

The linear acceleration is radial and directed towards the center (centripetal)

b- Uniformly varied circular motion ($\varepsilon = \ddot{\theta}$ is constant)

We have $\varepsilon = \ddot{\theta} = \frac{d\dot{\theta}}{dt} \quad \Longrightarrow d\dot{\theta} = \varepsilon dt$

$$\Longrightarrow \quad \int_{\theta_i}^{\theta_f} d\dot{\theta} = \int_{t_i}^{t_f} \epsilon dt \quad \Longrightarrow \quad \dot{\theta}_f - \dot{\theta}_i = \varepsilon(t_f - t_i)$$

If we take $t_i = t_0$ as an initial time and $t_f = t$ any time during the motion, with,

$\dot{\theta}_f = \dot{\theta}$; $\dot{\theta}_i = \dot{\theta}_0$

then:

$$\dot{\theta} = \varepsilon(t - t_0) + \dot{\theta}_0$$

We have:

$$\frac{d\theta}{dt} = \dot{\theta} = \omega$$

$$\Rightarrow \qquad d\theta = \omega dt \qquad \Rightarrow \int_{\theta_i}^{\theta_f} d\theta = \int_{t_i}^{t_f} \omega dt$$

$$\Rightarrow \qquad \theta_f - \theta_i = \frac{1}{2}\varepsilon\left(t_f - t_i\right)^2 + \dot{\theta}_i\left(t_f - t_i\right)$$

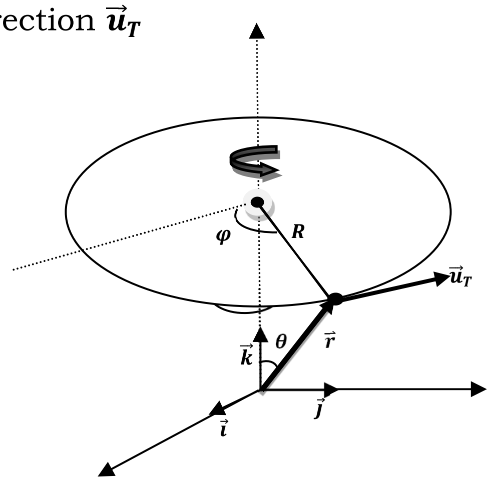
If we take: $\theta_f = \theta$, $\theta_i = \theta_0$

$$\Rightarrow \qquad \theta - \theta_0 = \frac{1}{2}\varepsilon\left(t_f - t_0\right)^2 + \dot{\theta}_0\left(t_f - t_0\right)$$

$$\Rightarrow \qquad \theta = \frac{1}{2}\varepsilon(t - t_0)^2 + \dot{\theta}_0(t - t_0) + \theta_0$$

➢ Vectorial expression between linear velocity and angular velocity

Since linear velocity has as magnitude $v = R\dot{\varphi}$ and direction $\vec{u}_T$



So: $\vec{v} = R\dot{\varphi}\vec{u}_T$

As shown in the figure:

$$\vec{\dot{\varphi}} = \omega\vec{k}$$

But:

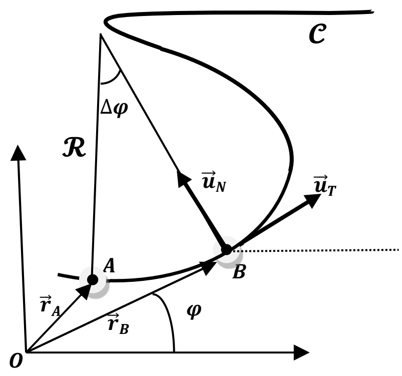
$$\begin{cases} v = R\dot{\varphi} = R\omega \\ |\vec{\omega}\wedge\vec{r}| = |\vec{\omega}||\vec{r}|sin\theta \\ R = |\vec{r}|sin\theta et \vec{u}_T \perp (\vec{\omega}, \vec{r}) \end{cases}$$

$$\Rightarrow \qquad \vec{v} = R\dot{\varphi}\vec{u}_T = \omega r sin\theta\vec{u}_T = \vec{\omega}\wedge\vec{r}$$

$$\vec{v} = \vec{\omega}\wedge\vec{r}$$

which is an important result

c – General motion



When the point moves from A to B, it traverses the arc

$$s = \widehat{AB}.$$

The position vector is given by $\vec{r}$, so the displacement is:

$$\Delta\vec{r} = \vec{r}_B - \vec{r}_A$$

For elementary variations $(A \to B)$, the sector dr overlay the magnitude of displacement dr

$$d\vec{r} = |d\vec{r}|\vec{u}_T = ds\vec{u}_T$$

- **Velocity**

We know that:

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{ds}{dt}\vec{u}_T = |\vec{v}|\,\vec{u}_T = v\,\vec{u}_T$$

Linear velocity is always tangential to the curve

- **Acceleration**

By definition: $\vec{a} = \frac{d\vec{v}}{dt}$

So $\vec{a} = \frac{d(v\vec{u}_T)}{dt} = \frac{dv}{dr}\vec{u}_T + v\frac{d\vec{u}_T}{dt}$

But $\begin{cases} \vec{u}_T = cos(\varphi)\,\vec{\imath} + sin(\varphi)\,\vec{\jmath} \\ \vec{u}_N = -sin(\varphi)\,\vec{\imath} + cos(\varphi)\,\vec{\jmath} \end{cases}$

Then

$$\Rightarrow \begin{cases} \dfrac{d\vec{u}_T}{dt} = \dfrac{d\vec{u}_T}{d\varphi}.\dfrac{d\varphi}{dt} = \dot{\varphi}[-sin(\varphi)\,\vec{\imath} + cos(\varphi)\,\vec{\jmath}] \\ \dfrac{d\vec{u}_N}{dt} = \dfrac{d\vec{u}_N}{d\varphi}.\dfrac{d\varphi}{dt} = -\dot{\varphi}[cos(\varphi)\,\vec{\imath} + sin(\varphi)\,\vec{\jmath}] \end{cases}$$

$$\Rightarrow \quad \begin{cases} \dot{\vec{u}}_T = \dot{\varphi}\,\vec{u}_N \\ \dot{\vec{u}}_N = -\dot{\varphi}\,\vec{u}_T \end{cases} \quad \Rightarrow \quad \vec{a} = \frac{dv}{dr}\vec{u}_T + v(\dot{\varphi}\,\vec{u}_N)$$

We have seen in the circular motion that:

$$v = \rho\frac{d\varphi}{dt} = \rho\dot{\varphi}$$

ρ is the radius of curvature of C $\quad \Rightarrow \quad \dot{\varphi} = \frac{v}{\rho}$

Finally:

$$\vec{a} = \frac{dv}{dr}\vec{u}_T + \frac{v^2}{\rho}\vec{u}_N = \vec{a}_T + \vec{a}_N$$

$$\begin{cases} a_T = \dfrac{dv}{dr} \ \textit{due to the variation in the modulus of the velocity vector: tangential acceleration} \\ a_N = \dfrac{v^2}{\rho} \ \textit{due to the variation in the direction of the velocity: normal acceleration vector} \end{cases}$$

7.3- Harmonic motion

If the motion of the particle is along a line with back and forth the motion is said to be rectilinear harmonic.

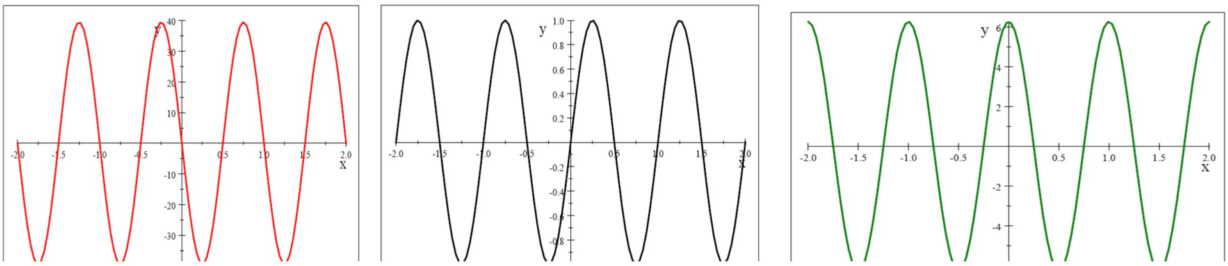
- **Temporary equation**

The temporary equation of motion is a circular function of form:

$$x(t) = x_0\sin(\omega t + \varphi)$$

where
- $x0$ is the *amplitude* of motion
- $\omega = \frac{2\pi}{T}$ is the *pulsation of motion*; T is the *periode of motion*
- $\omega t + \varphi$ *is the phse of the motion*
- φ *is the initial* phase

$\ddot{x}(t)$ $x(t)$ $\dot{x}(t)$



➢ Velocity

We know that the velocity is given by the derivative of the position vector with respect to time.

$$v = \frac{dx}{dt} = \omega x_0 \cos(\omega t + \varphi) = \omega x_0 \sin\left(\omega t + \varphi + \frac{\pi}{2}\right)$$

The phase difference between velocity and abscissa is $\frac{\pi}{2}$. They are said to be in quadrature

➢ Acceleration

We know that acceleration is given by the derivative of velocity with respect to time.

$$a = \frac{dv}{dt} = -\omega^2 x_0 \sin(\omega t + \varphi) = \omega^2 x_0 \sin(\omega t + \varphi + \pi) = -\omega^2 x$$

The phase difference between velocity and acceleration is "π". They are said to be in phase opposition.

<u>Remark:</u>

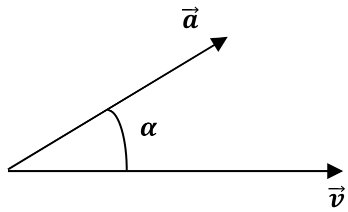
From the expression of acceleration one can deduce the equation of harmonic motion.

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = -\omega^2 x \Rightarrow \ddot{x} + \omega^2 x = o$$

It is a second-order differential equation

In general case

$\frac{dv^2}{dt} = 2\vec{v} \circ \frac{d\vec{v}}{dt} = 2\vec{v} \circ \vec{a} = 2|\vec{v}||\vec{a}|cos\alpha$



- If there is movement, speed $v \neq 0$

* Uniform motion:

$$\frac{dv^2}{dt} = 0 \Rightarrow \begin{cases} a = 0 \\ \alpha = \pm\frac{\pi}{2} \end{cases}$$

$$\Rightarrow \begin{cases} a = 0 & rectilinear\ uniforme\ motion \\ \alpha = \pm\frac{\pi}{2} & circular\ uniforme\ motion\ \vec{v} \perp \vec{a} \end{cases}$$

* Uniform varied motion

- The movement is accelerated if the norm of speed is an increasing function of time

$$\frac{dv^2}{dt} > 0 \Rightarrow 2|\vec{v}||\vec{a}|cos\alpha > 0$$

$$\Rightarrow cos\alpha > 0 \Rightarrow 0 < \alpha < \frac{\pi}{2}$$

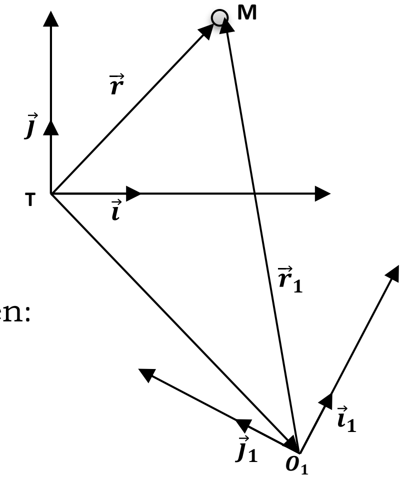
- Movement is delayed if:

$$\frac{dv^2}{dt} < 0 \Rightarrow 2|\vec{v}||\vec{a}|cos\alpha < 0$$

$$\Rightarrow cos\alpha < 0 \Rightarrow \frac{\pi}{2} < \alpha < \pi$$

8 Relative motion

8.1- Change of basis- System transform



* In an orthonormal basis $(\boldsymbol{O}, \vec{\boldsymbol{\imath}}, \vec{\boldsymbol{\jmath}})$, the vector $\overrightarrow{\boldsymbol{OM}}$ is written as:

$$\overrightarrow{OM} = \vec{r} = x\,\vec{\imath} + y\,\vec{\jmath}$$

* In another orthonormal basis $(\boldsymbol{O_1}, \vec{\boldsymbol{\imath}}_1, \vec{\boldsymbol{\jmath}}_1)$, the vector $\overrightarrow{\boldsymbol{O_1M}}$ is written:

$$\overrightarrow{O_1M} = \vec{r}_1 = x_1\vec{\imath}_1 + y_1\vec{\jmath}_1$$

Question: How to write the coordinates of one basis according to the other basis?

The relationship between the two position vectors is

$$\overrightarrow{OM} = \overrightarrow{OO_1} + \overrightarrow{O_1M}$$

$$\Longrightarrow \qquad x\vec{\imath} + y\vec{\jmath} = (x_{O1}\vec{\imath} + y_{O1}\vec{\jmath}) + (x_1\vec{\imath}_1 + y_1\vec{\jmath}_1)$$

The passage of $\boldsymbol{OM}$ to $\mathbf{O_1}\boldsymbol{M}$ is called basis change (transform)

8.2- Motion of a reference frame $\mathcal{R}_1$ with respect to reference frame $\mathcal{R}$

Let $(\boldsymbol{O}, \vec{\boldsymbol{\imath}}, \vec{\boldsymbol{\jmath}}, \vec{\boldsymbol{k}})$ and $(\boldsymbol{O_1}, \vec{\boldsymbol{\imath}}_1, \vec{\boldsymbol{\jmath}}_1, \vec{\boldsymbol{k}}_1)$ be two orthonormal bases assigned to both $\boldsymbol{\mathcal{R}}$ and R_1 which are fixed and mobile reference frame respectively.

8.2.1- Position vector

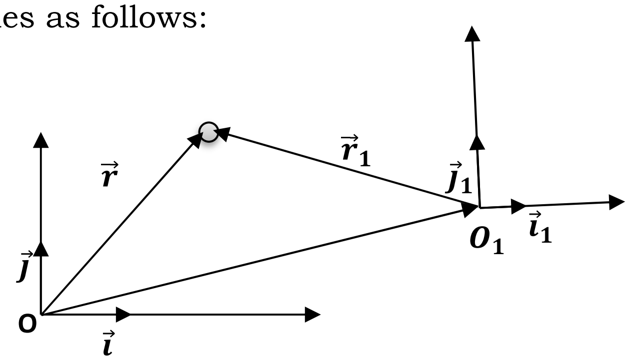
Position vectors are written in both reference frames as follows:

- In the fixed frame of reference

$$\overrightarrow{OM}_{/\mathcal{R}} = \vec{r} = x\,\vec{\imath} + y\,\vec{\jmath} + z\,\vec{k}$$

- In the mobile frame of reference

$$\overrightarrow{O_1M}_{/\mathcal{R}_1} = \vec{r}_1 = x_1\vec{\imath}_1 + y_1\vec{\jmath}_1 + z_1\vec{k}$$



The relationship between the two position vectors is:

$$\vec{r} = \overrightarrow{OM}_{/\mathcal{R}} = \overrightarrow{OO}_{1/\mathcal{R}} + \overrightarrow{O_1M}_{/\mathcal{R}_1} = \overrightarrow{OO}_{1/\mathcal{R}} + \vec{r}_{1/\mathcal{R}_1}$$

$$\Rightarrow \quad x\vec{i} + y\vec{j} + z\vec{k} = \left(x_{O1}\vec{i} + y_{O1}\vec{j} + z_{O1}\vec{k}\right) + \left(x_1\vec{i}_1 + y_1\vec{j}_1 + z_1\vec{k}_1\right)$$

8.2.2- Velocity

Remained the transport theorem:

For two reference frames $\mathcal{R}$ and $\mathcal{R}_1$ Let $\vec{\omega}$ the angular velocity of $\mathcal{R}_1$ with respect to $\mathcal{R}$.The derivative of a vector $\vec{A}$ with respect to $\mathcal{R}$ is:

$$\frac{d\vec{A}_{/\mathcal{R}}}{dt} = \frac{d\vec{A}_{/\mathcal{R}_1}}{dt} + \vec{\omega} \wedge \vec{A}_{/\mathcal{R}_1}$$

According to the definition:

$$\vec{v}_M = \frac{d\vec{r}}{dt} = \frac{d\overrightarrow{OM}_{/\mathcal{R}}}{dt} = \frac{d\overrightarrow{OO}_{1/\mathcal{R}}}{dt} + \frac{d\overrightarrow{O_1M}_{/\mathcal{R}}}{dt} = \frac{d\overrightarrow{OO}_{1/\mathcal{R}}}{dt} + \frac{d\overrightarrow{O_1M}_{/\mathcal{R}}}{dt_1} . \frac{dt_1}{dt}$$

In the case of low speeds, time is considered to be absolute, i.e.

$$t = t_1 \quad \Rightarrow \quad dt = dt_1$$

$$\vec{v}_M = \frac{d\overrightarrow{OO}_{1/\mathcal{R}}}{dt} + \frac{d\overrightarrow{O_1M}_{/\mathcal{R}}}{dt_1}$$

$$\vec{v}_M = \frac{d\left(x_{O1}\vec{i} + y_{O1}\vec{j} + z_{O1}\vec{k}\right)}{dt} + \frac{d\left(x_1\vec{i}_1 + y_1\vec{j}_1 + z_1\vec{k}_1\right)}{dt}$$

$$\vec{v}_M = \left(\frac{dx_{O1}}{dt}\vec{i} + \frac{dy_{O1}}{dt}\vec{j} + \frac{dz_{O1}}{dt}\vec{k}\right) + \left(\frac{dx_1}{dt}\vec{i}_1 + \frac{dy_1}{dt}\vec{j}_1 + \frac{dz_1}{dt}\vec{k}_1\right) + \left(x_1\frac{d\vec{i}_1}{dt} + y_1\frac{d\vec{j}_1}{dt} + z_1\frac{d\vec{k}_1}{dt}\right)$$

The moving basis is in translation and rotation with an angular velocity $\vec{\omega}$ with respect to the fixed basis.

But the derivative of a vector with respect to time is:

$\frac{d\vec{A}}{dt} = \vec{\omega} \wedge \vec{A}$ for any vector $\vec{A}$ of constant magnitude

For any unitary vector $'\vec{u}'$: $\frac{d\vec{u}}{dt} = \vec{\omega} \wedge \vec{u}$

$$\Rightarrow \quad \frac{d\vec{i}}{dt} = \vec{\omega} \wedge \vec{i} \ , \quad \frac{d\vec{j}}{dt} = \vec{\omega} \wedge \vec{j} \quad \text{and} \quad \frac{d\vec{k}}{dt} = \vec{\omega} \wedge \vec{k}$$

The velocity of the point M is written as follows:

$$\vec{v}_{M/\mathcal{R}} = \dot{x}_{O1}\vec{i} + \dot{y}_{O1}\vec{j} + \dot{z}_{O1}\vec{k} + \dot{x}_1\vec{i}_1 + \dot{y}_1\vec{j}_1 + \dot{z}_1\vec{k}_1 + x_1(\vec{\omega} \wedge \vec{i}_1) + y_1(\vec{\omega} \wedge \vec{j}_1) + z_1(\vec{\omega} \wedge \vec{k}_1)$$

$$\vec{v}_{M/\mathcal{R}} = \dot{x}_{O1}\vec{i} + \dot{y}_{O1}\vec{j} + \dot{z}_{O1}\vec{k} + \dot{x}_1\vec{i}_1 + \dot{y}_1\vec{j}_1 + \dot{z}_1\vec{k}_1 + (\vec{\omega} \wedge x_1\vec{i}_1) + (\vec{\omega} \wedge y_1\vec{j}_1) + (\vec{\omega} \wedge z_1\vec{k}_1)$$

Since the vector product is distributive with respect to addition, we will have:

$$\vec{v}_{M/\mathcal{R}} = (\dot{x}_{O1}\vec{i} + \dot{y}_{O1}\vec{j} + \dot{z}_{O1}\vec{k}) + (\dot{x}_1\vec{i}_1 + \dot{y}_1\vec{j}_1 + \dot{z}_1\vec{k}_1) + \vec{\omega} \wedge (x_1\vec{i}_1 + y_1\vec{j}_1 + z_1\vec{k}_1)$$

$$\vec{v}_{M/\mathcal{R}} = \vec{v}_{M/\mathcal{R}_1} + \vec{v}_{O1/\mathcal{R}} + \vec{\omega} \wedge \overrightarrow{O_1M} = \vec{v}_{M/\mathcal{R}_1} + \vec{v}_{O1/\mathcal{R}} + \vec{\omega} \wedge \vec{r}_1$$

$$\vec{v}_{M/\mathcal{R}} = \vec{v}_a = \vec{v}_r + \vec{v}_e$$

This is the law of velocities composition

$$\vec{v}_{M/\mathcal{R}} = \vec{v}_a$$

Is the absolute velocity, i.e., the **velocity** of the point M with respect to the fixed reference frame $\mathcal{R}(O, \vec{i}, \vec{j}, \vec{k})$.

$$\vec{v}_{M/\mathcal{R}_1} = \vec{v}_r$$

Is the **relative** velocity, i.e., the velocity of the point M with respect to the mobile reference frame $\mathcal{R}_1(O_1, \vec{i}_1, \vec{j}_1, \vec{k}_1)$.

$$\vec{v}_{O1/\mathcal{R}} + \vec{\omega} \wedge \vec{r}_1 = \vec{v}_e$$

Is transport velocity, i.e., the **velocity of** the point with respect to the fixed reference frame, M assuming that this point is fixed in the mobile reference frame (The velocity of the mobile reference frame with respect to the fixed one)

8.2.3- Acceleration vector

According to the definition:

$$\vec{a}_M = \frac{d\vec{v}_{M/\mathcal{R}}}{dt} = \frac{d^2(\overrightarrow{OM}_{/\mathcal{R}})}{dt^2}$$

$$\vec{a}_M = \frac{d\left(\vec{v}_{M/\mathcal{R}_1} + \vec{v}_{O1/\mathcal{R}} + \vec{\omega} \wedge \overrightarrow{O_1M}\right)}{dt} = \frac{d\left(\vec{v}_{M/\mathcal{R}_1}\right)}{dt} + \frac{d\left(\vec{v}_{O1/\mathcal{R}}\right)}{dt} + \frac{d\left(\vec{\omega} \wedge \overrightarrow{O_1M}\right)}{dt}$$

$$\begin{cases} \frac{d\left(\vec{v}_{M/\mathcal{R}_1}\right)}{dt} = \frac{d\left(\dot{x}_1\vec{i}_1 + \dot{y}_1\vec{j}_1 + \dot{z}_1\vec{k}_1\right)}{dt} = \left(\ddot{x}_1\vec{i}_1 + \ddot{y}_1\vec{j}_1 + \ddot{z}_1\vec{k}_1\right) + \vec{\omega} \wedge \left(\dot{x}_1\vec{i}_1 + \dot{y}_1\vec{j}_1 + \dot{z}_1\vec{k}_1\right) \\ \frac{d\left(\vec{v}_{O1/\mathcal{R}}\right)}{dt} = \frac{d\left(\dot{x}_{O1}\vec{i} + \dot{y}_{O1}\vec{j} + \dot{z}_{O1}\vec{k}\right)}{dt} = \ddot{x}_{O1}\vec{i} + \ddot{y}_{O1}\vec{j} + \ddot{z}_{O1}\vec{k} \\ \frac{d\left(\vec{\omega} \wedge \vec{r}_1\right)}{dt} = \dot{\vec{\omega}} \wedge \vec{r}_1 + \vec{\omega} \wedge \left[\left(\dot{x}_1\vec{i}_1 + \dot{y}_1\vec{j}_1 + \dot{z}_1\vec{k}_1\right) + \vec{\omega} \wedge \left(x_1\vec{i}_1 + y_1\vec{j}_1 + z_1\vec{k}_1\right)\right] \end{cases}$$

$$\begin{cases} \frac{d\left(\vec{v}_{M/\mathcal{R}_1}\right)}{dt} = \vec{a}_{M/\mathcal{R}_1} + \vec{\omega} \wedge \vec{v}_r \\ \frac{d\left(\vec{v}_{O1/\mathcal{R}}\right)}{dt} = \vec{a}_{O1/\mathcal{R}} \\ \frac{d\left(\vec{\omega} \wedge \vec{r}_1\right)}{dt} = \dot{\vec{\omega}} \wedge \vec{r}_1 + \vec{\omega} \wedge \left[\vec{v}_r + \vec{\omega} \wedge \vec{r}_1\right] \end{cases}$$

Finally:

$$\vec{a}_M = \vec{a}_{M/\mathcal{R}_1} + \vec{a}_{O1/\mathcal{R}} + \dot{\vec{\omega}} \wedge \vec{r}_1 + \vec{\omega} \wedge \left(\vec{\omega} \wedge \vec{r}_1\right) + 2\vec{\omega} \wedge \vec{v}_r$$

$$\vec{a} = \vec{a}_r + \vec{a}_e + \vec{a}_c$$

$\vec{a}$: Is the absolute acceleration, i.e., the acceleration of the point M with respect to the fixed reference frame $\mathcal{R}(O, \vec{i}, \vec{j}, \vec{k})$.

$\vec{a}_r$: Is the **relative acceleration**, i.e. The acceleration of the point M with respect to the mobile frame of reference $\mathcal{R}_1(O_1, \vec{i}_1, \vec{j}_1, \vec{k}_1)$

$\vec{a}_e$: Is **the transport acceleration**,

$\vec{a}_c$: Is **the Coriolis acceleration.** This acceleration cancels out if:

- $\vec{\omega} = \vec{0}$ Movement is a pure translation
- $\vec{v}_r = \vec{0}$ The point is fixed in the moving coordinate system M
- $\vec{\omega} \parallel \vec{v}_r$ The rotation is in a plane perpendicular to the displacement of in the M moving coordinate system

8.3- Special case

8.3.1 -"$\mathcal{R}_1$" in translation with respect to "$\mathcal{R}$"

a- Translation at constant velocity:

In this case: $\vec{\omega} = \vec{0}$ and the acceleration of the point O_1 is zero $\frac{d(\vec{v}_{O1/\mathcal{R}})}{dt} = \vec{0}$,

then:

$$\vec{v}_a = \vec{v}_r + \vec{v}_e = \vec{v}_r + \vec{v}_{O1/\mathcal{R}}$$

The transport velocity is that of the moving coordinate system.

$$\boxed{\vec{a} = \vec{a}_r}$$

Note: *In this case Newton's laws are the same in both referential "$\mathcal{R}$" and "$\mathcal{R}_1$" , they are called Galilean referential*

b - Translation at variable velocity

In this case: $\vec{\omega} = \vec{0}$ and the acceleration of the point O_1 is not zero

$$\frac{d\left(\vec{v}_{O1/\mathcal{R}}\right)}{dt} \neq \vec{0}$$

$$\vec{v}_a = \vec{v}_r + \vec{v}_e = \vec{v}_r + \vec{v}_{O1/\mathcal{R}} \text{ and } \quad \vec{a} = \vec{a}_r + \vec{a}_{O1/\mathcal{R}}$$

Note:

We see that the absolute acceleration is increased by the acceleration of the origin of the moving coordinate system. The frame of reference is not Galilean, Newton's 2nd law is not valid but will be corrected.

8.3.2- "$\mathcal{R}_1$" in rotation with respect to "$\mathcal{R}$"

a- Rotation at constant angular velocity: $\vec{\omega} = Constante$

In this case: $\vec{\omega} = Cste \quad \Longrightarrow \quad \frac{d\vec{\omega}}{dt} = \dot{\vec{\omega}} = \vec{0}$

And $\vec{a}_{O1/\mathcal{R}} = \frac{d(\vec{v}_{O1/\mathcal{R}})}{dt} = \vec{0}$; $\quad \vec{v}_{O1/\mathcal{R}} = \vec{0}$ (Only rotation)

The absolute velocity is:

$$\vec{v}_a = \vec{v}_r + \vec{v}_e = \vec{v}_r + \vec{\omega} \wedge \vec{r}_1$$

And the absolute acceleration is given by:

$$\vec{a} = \vec{a}_r + \vec{a}_e + \vec{a}_c = \vec{a}_r + \vec{\omega} \wedge (\vec{\omega} \wedge \vec{r}_1) + 2\vec{\omega} \wedge \vec{v}_r$$

b- Variable angular velocity

In this case: $\vec{v}_{O1/\mathcal{R}} = \vec{0}$ (Only rotation)

And $\vec{a}_{O1/\mathcal{R}} = \frac{d(\vec{v}_{O1/\mathcal{R}})}{dt} = \vec{0}$

Then:

$$\vec{v}_a = \vec{v}_r + \vec{v}_e = \vec{v}_r + \vec{\omega} \wedge \vec{r}_1$$

And

$$\vec{a} = \vec{a}_r + \vec{a}_e + \vec{a}_c = \vec{a}_r + \left[\dot{\vec{\omega}} \wedge \vec{r}_1 + \vec{\omega} \wedge (\vec{\omega} \wedge \vec{r}_1)\right] + 2\vec{\omega} \wedge \vec{v}_r$$

III- Dynamics

1- Introduction

1-1-Definition:

Dynamics (kinetics) is the study of motion by taking into account the causes that generate it

1-2-Inertial Frame of Reference (Galilean)

In the case of relative motion, the reference frames have been defined as "$\mathcal{R}$", and "$\mathcal{R}_1$", one is assumed to be absolute (fixed), the other is mobile. But the question for "$\mathcal{R}$", it is fixed with respect to what? As a result, it is assumed that a frame of reference is fixed according to the problem under study where the laws of physics become simpler.
The frame of reference in which an isolated (free) object maintains its state of motion(constant velocity) is a privileged reference frame called an inertial frame.

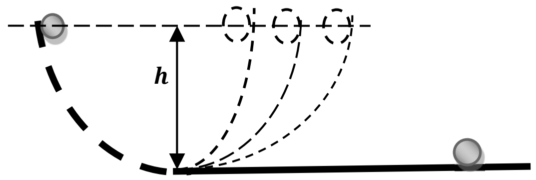
1.3- Observation:

- If a ball is dropped, from a height" h", into a smooth tank (frictionless), it goes down and up again at the same level " h" regardless of the slope.

- If the second side of the bowl is flattened, then it has been lowered, the ball follows a horizontal path and continues its path with a uniform rectilinear movement.

Result:

An isolated ball follows a uniform straight path.



2- Principle of inertia

In an inertial frame of reference (Galilean), a free body (isolated or not subjected to any external forces), continues to move in a straight line at a constant speed (uniform rectilinear motion) if it was already in motion, if it is in rest, it remains at rest.

Note: *The principle of inertia brings us closer to the concept of force.*

3- Mass and momentum

3.1- Mass

The greater the mass of a body, the more difficult to stop or move it.

Mass is the amount of matter in a body that characterizes its ability to resisting the change of motion (velocity), it characterizes its inertia.

3.2- Momentum

- For two bodies with the same velocity, it is easier to stop or move the one with the smaller mass.
- For two bodies with the same mass, it is easier to stop or move the one with the lower velocity.

3-2-1-Definition

The product of a body's mass by its velocity defines the **momentum** *denoted* "$\vec{P}$".

$$\vec{P} = m\vec{v} \quad [kg.m/s]$$

Note: *The principle of inertia can be stated as follows:*

<u>An isolated body of constant mass has a constant momentum</u>.

3-2-2-Momentum of a Particle System

Let be an isolated system consisting "n" of particles of respective velocities "$\vec{v}_1, \vec{v}_2, \vec{v}_3, \dots, \vec{v}_n$". We define the center of mass "G" whose vector position "$\vec{r}_G$" such that:

$$\vec{r}_G = \frac{\sum_{i=1}^{n} m_i \vec{r}_i}{\sum_{i=1}^{n} m_i}$$

$\vec{r}_i$: is the position vector for the i^{th} particle of mass "m_i"

Then:

$$\frac{d\vec{r}_G}{dt} = \vec{v}_G = \frac{\sum_{i=1}^{n} m_i \frac{d\vec{r}_i}{dt}}{\sum_{i=1}^{n} m_i} = \frac{\sum_{i=1}^{n} m_i \vec{v}_i}{\sum_{i=1}^{n} m_i}$$

$\sum_{i=1}^{n} m_i = M$ the total mass

Then:

$$\vec{v}_G = \frac{\sum_{i=1}^{n} \vec{P}_i}{\sum_{i=1}^{n} m_i} = \frac{\sum_{i=1}^{n} \vec{P}_i}{M} \quad \Rightarrow \quad \frac{\sum_{i=1}^{n} m_i \vec{v}_i}{\sum_{i=1}^{n} m_i} = \sum_{i=1}^{n} \vec{P}_i$$

Hence: the momentum (linear momentum) of the system

$$\vec{P} = M\vec{v}_G = \vec{P}_1 + \vec{P}_2 + \vec{P}_3 + \dots + \vec{P}_n = \sum_{i=1}^{n} \vec{P}_i$$

The momentum of system of "n" particles is the same as if all its mass were concentrated at its center of mass that whose velocity is $\vec{v}_G$.

3-2-3-Conservation of Momentum

a – Conservation of momentum

Let be a system consisting of two particles $[(m_1,\vec{v}_1)\,;(m_2,\vec{v}_2)]$ in interaction. Due to the change in their velocities, each of the particles follows a curvilinear path.

- at the moment "$t = t_0$" the two particles are in position A_1 and A_2
- at the moment "$t = t_1$" the two particles are in position B_1 and B_2

The position vector of the center of mass of the system is:

$$\vec{r}_G = \frac{\sum_{i=1}^{n} m_i\vec{r}_i}{\sum_{i=1}^{n} m_i} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$$

The momentum is:

- At "$t = t_0$" : $\vec{P} = m_1\vec{v}_1 + m_2\vec{v}_2$
- At "$t = t_1$" : $\vec{P'} = m_1\vec{v'}_1 + m_2\vec{v'}_2$

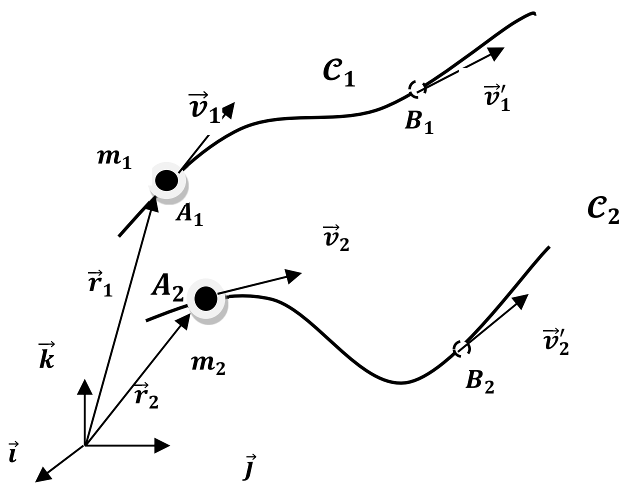
The velocity of the center of mass of the system is:

- At "$t = t_0$" :

$$\vec{v}_G = \frac{d\vec{r}_G}{dt} = \frac{\sum_{i=1}^{n} m_i\frac{d\vec{r}_i}{dt}}{\sum_{i=1}^{n} m_i} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1 + m_2}$$

- At "$t = t_1$" :

$$\vec{v'}_G = \frac{d\vec{r}_G}{dt} = \frac{\sum_{i=1}^{n} m_i\frac{d\vec{r'}_i}{dt}}{\sum_{i=1}^{n} m_i} = \frac{m_1\vec{v'}_1 + m_2\vec{v'}_2}{m_1 + m_2}$$



Since the system is isolated, the center of mass moves at a constant speed.

$$\vec{v}_G = \vec{v'}_G$$

- At "$t = t_0$" : $\vec{P} = M\vec{v}_G$
- At "$t = t_1$" : $\vec{P'} = M\vec{v'}_G$

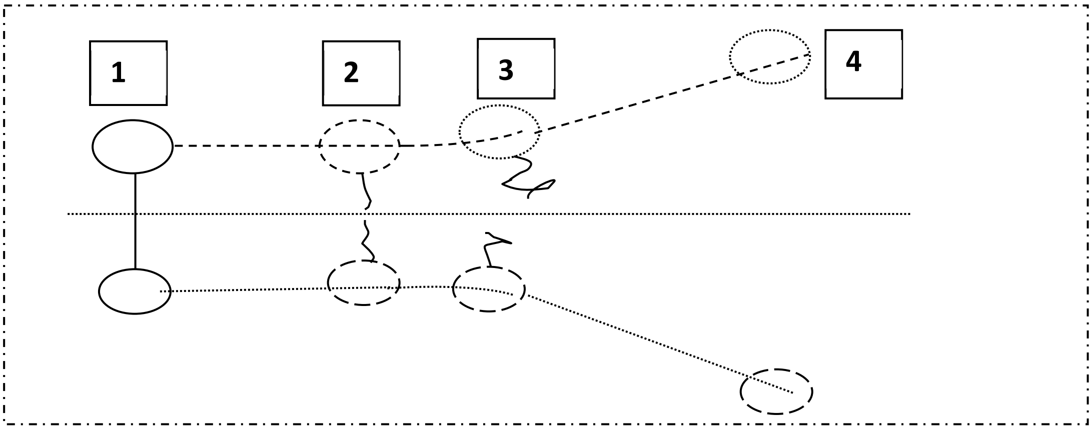
$$\vec{v}_G = \vec{v'}_G \quad\Rightarrow\quad \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1 + m_2} = \frac{m_1\vec{v'}_1 + m_2\vec{v'}_2}{m_1 + m_2}$$

$$m_1\vec{v}_1 + m_2\vec{v}_2 = M\vec{v}_G = m_1\vec{v'}_1 + m_2\vec{v'}_2 = M\vec{v'}_G$$

$$\Rightarrow \quad \vec{P} = \vec{P'}$$

b - Equality of changes in momentum

Let be two magnetic disks linked by a string and thrown on a blower table which constitutes an isolated system.



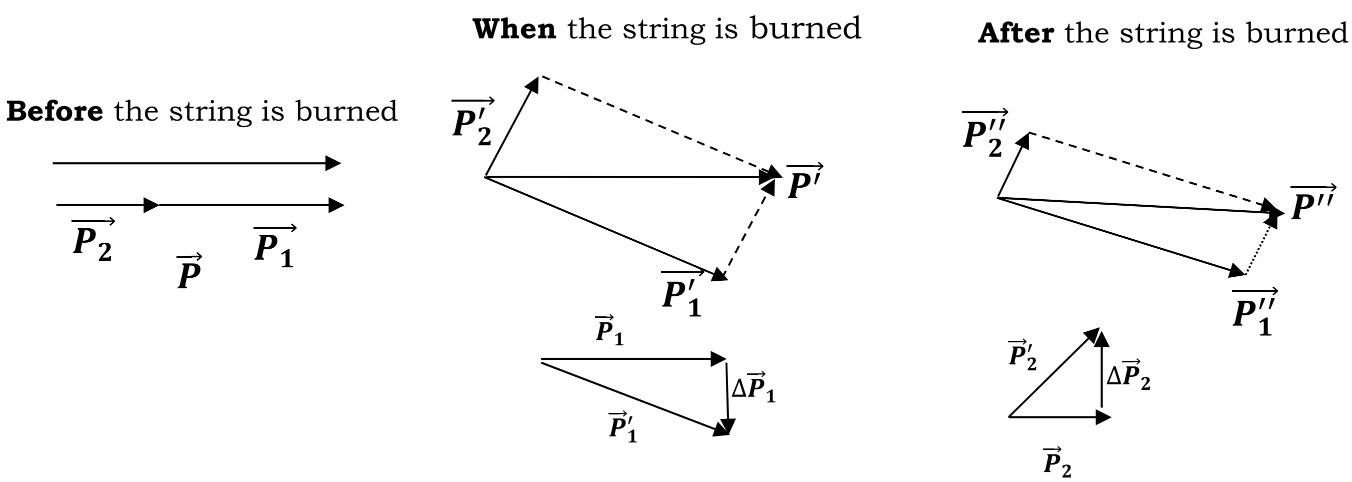
- **1** Position before burning the string

  The system is isolated, and the disks are still linked.
- **2** Position where the string is burned:

  The system is isolated and the disks begin to repel each other.
- **3** Position after the string is burned:

  The system is still isolated, but the disks become non-isolated and repel each other (interact) and change their velocities.
- **4** Position after a moment of disk separation:

  The system is still isolated, but the disks become free again and continue in a straight path.



Since the system is isolated, momentum is conserved, $\vec{P} = \vec{P'} = \vec{P''}$

but for the disks constituting this system are interacting, which changes their momentum $\vec{P_1}$ and $\vec{P}_2$.

Since: $\vec{P} = \vec{P'} \quad \Rightarrow \quad \vec{P}_1 + \vec{P}_2 = \vec{P'}_1 + \vec{P'}_2$

$$\Rightarrow \quad \vec{P'}_1 - \vec{P}_1 = \vec{P}_2 - \vec{P'}_2$$

The change in momentum is:

$$\Delta\vec{P} = \vec{P'} - \vec{P}$$

$$\Rightarrow \quad \Delta\vec{P}_1 = \vec{P'}_1 - \vec{P}_1 \quad \text{and} \quad \Delta\vec{P}_2 = \vec{P'}_2 - \vec{P}_2$$

$$\Rightarrow \quad \Delta\vec{P}_1 = -\Delta\vec{P}_2$$

i.e., the variations in momentum are equal and opposite

4- Newton's Laws

4.1: 1st Law: Law of Inertia

In an inertial frame of reference, the momentum of a free body is conserved, i.e., the body (system) is in uniform rectilinear motion or at rest depending on its initial state

4.2: 2nd Law: Fundamental Principle of Dynamics

This law is already mentioned, i.e., any change in velocity (or change in momentum) of an isolated (free) system is the result of an interaction that results in a ***force***.

The rate of change in momentum in an interval time produces the applied force.

$$\vec{F} = \sum_i \vec{F}_i^{ex} = \frac{\Delta\vec{P}}{\Delta t}$$

Where: $\begin{cases} \vec{F}: Net\ force \\ \vec{P}: \text{system momentun} \end{cases}$

In the limit case with an infinitesimal change:

$$\vec{F} = \lim_{\Delta t \to 0}\left(\frac{\Delta\vec{P}}{\Delta t}\right) = \frac{d\vec{P}}{dt}$$

Note: In the case where the mass of the system is constant, the 2^{nd}law becomes

$$\vec{F} = \sum_i \vec{F}_i^{ex} = \frac{d\vec{P}}{dt} = \frac{d(m\vec{v})}{dt} = m\frac{d\vec{v}}{dt}$$

$$\Rightarrow \qquad \vec{F} = m\vec{a}$$

4.3: 3rd Law: Law of Reciprocity (Law of Action and Reaction)

As already pointed out, the momentum exchanging during the interaction between two particles in the system are the same but opposite.

$$\vec{P} = \vec{P'} \Rightarrow \quad \vec{P}_1 + \vec{P}_2 = \vec{P'}_1 + \vec{P'}_2$$

$$\Rightarrow \quad \vec{P'}_1 - \vec{P}_1 = \vec{P}_2 - \vec{P'}_2$$

$$\Rightarrow \quad \Delta\vec{P}_1 = -\Delta\vec{P}_2$$

If:

$\vec{F}_{12}$**:** is the action of particle (**1**) on particle (**2**)

$\vec{F}_{21}$ **:** is the action of particle **(2)** on particle **(1)**

So: $\vec{F}_{12} = \frac{\Delta\vec{P}_2}{\Delta t}$and $\qquad \vec{F}_{21} = \frac{\Delta\vec{P}_1}{\Delta t}$Since $\qquad \Delta\vec{P}_1 = -\Delta\vec{P}_2 \Rightarrow$

at the limit: $\Delta\vec{P}_1 \to d\vec{P}_1$ and $\Delta\vec{P}_2 \to d\vec{P}_2$

$$\Rightarrow \quad \vec{F}_{12} = \frac{d\vec{P}_2}{dt} \quad \text{and} \quad \vec{F}_{21} = \frac{d\vec{P}_1}{dt}$$

$$\Delta\vec{P}_1 = -\Delta\vec{P}_2 \Rightarrow \quad d\vec{P}_1 = -d\vec{P}_2$$

$$\Rightarrow \quad \vec{F}_{21} = -\vec{F}_{12}$$

Result:

If one body exerts an effort on another, the latter reacts with an equal and opposite force

5- Some laws of force

According to the fundamental law of dynamics, we have:

$$\vec{F} = m\vec{a} = m\ddot{\vec{r}}$$

Where: $\vec{F} = \vec{F}\,(\dot{\vec{r}}, \vec{r}, t)$

5.1- Constant force

In this case, the net force is:

$$\vec{F} = \vec{F}\,(\dot{\vec{r}}, \vec{r}, t) = \vec{F}_0 = constante$$

$$\vec{F} = \vec{F}\,(\dot{\vec{r}}, \vec{r}, t) = \vec{F}_0 = m\ddot{\vec{r}} \quad \Longrightarrow \quad \ddot{\vec{r}} = \frac{\vec{F}_0}{m} = \frac{d}{dt}\left(\frac{d\dot{\vec{r}}}{dt}\right)$$

$$\Longrightarrow \quad d\left(\frac{d\vec{r}}{dt}\right) = \frac{\vec{F}_0}{m}dt \qquad \Longrightarrow \qquad \int_{\dot{\vec{r}}_0}^{\dot{\vec{r}}} d(\dot{\vec{r}}) = \frac{\vec{F}_0}{m}\int_{t_0}^{t} dt$$

$$\Longrightarrow \quad \dot{\vec{r}} - \dot{\vec{r}}_0 = \frac{\vec{F}_0}{m}(t - t_0) \qquad \Longrightarrow \qquad \dot{\vec{r}} = \frac{d\vec{r}}{dt} = \dot{\vec{r}}_0 + \frac{\vec{F}_0}{m}(t - t_0)$$

Finally:

$$\int_{\vec{r}_0}^{\vec{r}} d\vec{r} = \int_{t_0}^{t}\left[\dot{\vec{r}}_0 + \frac{\vec{F}_0}{m}(t - t_0)\right]dt$$

$$\Longrightarrow \qquad \vec{r} = \frac{1}{2}\frac{\vec{F}_0}{m}(t - t_0)^2 + \dot{\vec{r}}_0(t - t_0) + \vec{r}_0$$

It is the law of uniformly varied motion

Example: Free Fall $\qquad \vec{F}_0 = m\vec{g} \Longrightarrow \vec{a} = \vec{g} = \ddot{\vec{r}}$

$$\Longrightarrow \quad \vec{v} = \frac{d\vec{r}}{dt} = \vec{v}_0 + \vec{a}(t - t_0)$$

$$\Longrightarrow \quad \vec{r} = \frac{1}{2}\vec{a}(t - t_0)^2 + \vec{v}_0(t - t_0) + \vec{r}_0$$

Since the motion is done in a straight line

$$\Longrightarrow \quad r = \frac{1}{2}a(t - t_0)^2 + v_0(t - t_0) + h_0$$

5.2- Time-dependent force

$$\vec{F}\,(\dot{\vec{r}}, \vec{r}, t) = \vec{F}(t)$$

$$\ddot{\vec{r}} = \frac{\vec{F}(t)}{m} = \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right) \quad \Longrightarrow \quad \int_{\dot{\vec{r}}_0}^{\dot{\vec{r}}} d\left(\dot{\vec{r}}\right) = \frac{1}{m}\int_{t_0}^{t} \vec{F}(t)dt$$

$$\Longrightarrow \quad \dot{\vec{r}} = \frac{d\vec{r}}{dt} = \dot{\vec{r}}_0 + \frac{1}{m}\int_{t_0}^{t} \vec{F}(t)dt \quad \Longrightarrow \quad \int_{\vec{r}_0}^{\vec{r}} d\vec{r} = \int_{t_0}^{t}\left[\dot{\vec{r}}_0 + \frac{1}{m}\int_{t_0}^{t} \vec{F}(t)dt\right]dt$$

Finally: $\vec{r} = \int_{t_0}^{t}\left[\dot{\vec{r}}_0 + \frac{1}{m}\int_{t_0}^{t} \vec{F}(t)dt\right]dt + \vec{r}_0$

Example: Point Charge **Q** in a Variable Electric Field $E(t) = E_0 sin(\omega t)$.

We know the force of an electric charge is: $F = QE$

$$F = QE_0 sin(\omega t) \Longrightarrow F = ma = QE_0 sin(\omega t)$$

$$\Longrightarrow \quad a = \frac{QE_0 sin(\omega t)}{m}$$

$$r = \int_0^t \left[\dot{r}_0 + \int_0^t \frac{QE_0 sin(\omega t)}{m}\right]dt + r_0 = r_0 + v_0 t + \frac{QE_0}{m\omega^2}(\omega t - sin\omega t)$$

If we take the following initial conditions: $t_0 = 0$; $r_0 = 0$; $v_0 = 0$

$$r = \frac{QE_0}{m\omega^2}(\omega t - sin\omega t)$$

5.3- Velocity-dependent force

$$\vec{F}\left(\dot{\vec{r}}, \vec{r}, t\right) = \vec{F}\left(\dot{\vec{r}}\right) = \vec{F}(\vec{v})$$

$$\Longrightarrow \quad \ddot{\vec{r}} = \frac{\vec{F}(\vec{v})}{m} = \frac{d}{dt}(\vec{v})$$

$$\Longrightarrow \quad dt = m\frac{dv}{F(v)} \quad \Longrightarrow \quad t - t_0 = \int_{v_0}^{v} m\frac{dv}{F(v)}$$

$$\Longrightarrow \quad t = t_0 + f(v; v_0)$$

But:

$$ma = \frac{mdv}{dt} = m\frac{dv}{dr}.\frac{dr}{dt} = mv\frac{dv}{dr} = F(v)$$

$$\Longrightarrow \quad dr = m\frac{vdv}{F(v)}$$

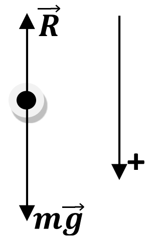
$$\Longrightarrow \quad \int_{r_0}^{r} dr = m\int_{v_0}^{v} \frac{vdv}{F(v)} \quad \Longrightarrow \quad r = r_0 + m\int_{v_0}^{v} \frac{vdv}{F(v)}$$

Example: frictional force (air resistance) acting on a body in free fall: $\vec{R} = -k\vec{v}$

$$\sum \vec{F}^{ex} = m\vec{g} + \vec{R} \quad \Longrightarrow \quad mg - kv = \frac{mdv}{dt}$$

$$\Longrightarrow \quad \frac{dv}{\left(g - \frac{k}{m}v\right)} = dt \quad \Longrightarrow \quad \int \frac{dv}{\left(g - \frac{k}{m}v\right)} = \int dt$$

If we take



$$g - \frac{k}{m}v = u \Longrightarrow -\frac{k}{m}dv = du$$

So

$$\int \frac{du}{u} = -\frac{m}{k}\int dt \quad \Longrightarrow \quad Ln(u) = -\frac{m}{k}t$$

If at $t_0 = 0, \quad v_0 = 0 \quad \Longrightarrow \quad v = \alpha(1 - e^{-\beta t})\beta = \frac{m}{k}$

Where $\alpha = \frac{mg}{k}$

5.4- Position-dependent force

$$\vec{F}\left(\dot{\vec{r}}, \vec{r}, t\right) = \vec{F}(\vec{r})$$

Generally, these types of forces are <u>conservative</u>, so they derive from a potential.

$$F = -\frac{dV}{dr}$$

Where V : is a potential function (potential energy)

$$F = -\frac{dV}{dr} = ma = m\ddot{r} \quad \Longrightarrow \quad \vec{F} \circ \dot{\vec{r}} = m\ddot{\vec{r}} \circ \dot{\vec{r}}$$

$$\vec{F} \circ \frac{d\vec{r}}{dt} = \frac{1}{2}\frac{d(m\dot{r}^2)}{dt}$$

$$\Longrightarrow \int_{r_0}^{r} \vec{F} \circ d\vec{r} = \int_{\dot{r}_0}^{\dot{r}} d\left(\frac{1}{2}m\dot{r}^2\right) = -\int_{V_0}^{V} dV$$

$$\Longrightarrow \quad \frac{1}{2}\left(m\dot{r}^2 - m{\dot{r}_0}^2\right) = V(r_0) - V(r)$$

$$\Longrightarrow \quad \frac{1}{2}m\dot{r}^2 + V(r) = \frac{1}{2}m{\dot{r}_0}^2 + V(r_0) = Constant = E$$

E: total energy (mechanical Energy)

We have:

$$\frac{dr}{dt} = \dot{r} = \mp\sqrt{\frac{2}{m}}\sqrt{E - V(r)} \qquad \Longrightarrow \qquad dt = \mp\sqrt{\frac{m}{2}} \cdot \frac{dr}{\sqrt{E - V(r)}}$$

$$\Longrightarrow \qquad t - t_0 = \mp\sqrt{\frac{m}{2}} \cdot \int_{r_0}^{r} \frac{dr}{\sqrt{E-V(r)}}$$

$$\Longrightarrow \qquad t = t_0 \mp \sqrt{\frac{m}{2}} \cdot \int_{r_0}^{r} \frac{dr}{\sqrt{E-V(r)}} = T(r)$$

Time is a function of "r", conversely, we can determine the function that describes the position of the mobile "$r = R(t)$"

6- Angular momentum

A particle of mass "m" and velocity "$\vec{v}$", has momentum "$\vec{P}$" and is subject to forces given by Newton's second law.

$$\vec{F} = \sum_i \vec{F}_i^{ex} = \frac{d\vec{P}}{dt}$$

$$\Longrightarrow \quad \vec{r} \wedge \vec{F} = \sum_i \vec{r} \wedge \vec{F}_i^{ex} = \sum_i \vec{\mathcal{M}}_i(\vec{F}_i^{ex})_{/o} = \vec{r} \wedge \frac{d\vec{P}}{dt}$$

If we add the quantity " $\frac{d\vec{r}}{dt} \wedge \vec{P} = 0$" that does not modify the previous expression in any way, we will have:

$$\sum_i \vec{\mathcal{M}}_{i_{/o}} = \vec{r} \wedge \frac{d\vec{P}}{dt} + \frac{d\vec{r}}{dt} \wedge \vec{P} = \frac{d(\vec{r} \wedge \vec{P})}{dt}$$

Quantity "$\vec{r} \wedge \vec{P}$" plays an important role in rotational motion than momentum in translation. This amount is called angular momentum.

6.1- Definition

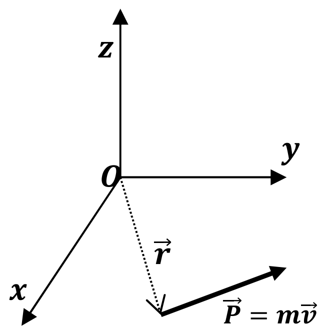
The angular momentum with respect to a point "**O**", denoted "$\vec{L}_O$", of a particle of mass "m" and velocity "$\vec{v}$", is the rotation that results from the effect of its momentum.

$$\vec{L}_O = \vec{\mathcal{M}}(\vec{P})_{/o} = \overrightarrow{OM} \wedge \vec{P} = \vec{r} \wedge \vec{P}.$$

6.2- Relation between angular momentum and resultant forces (Newton's 2nd Law)

Newton's second law for a rotational motion of a body can be written as follows:

$$\vec{\mathcal{M}}(\vec{F})_{/O} = \sum_i \vec{\mathcal{M}}_{i/O} = \frac{d(\vec{r} \wedge \vec{P})}{dt} = \frac{d\vec{L}_O}{dt}$$

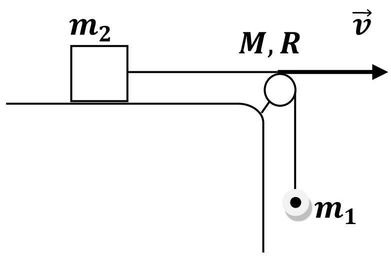


Example:

The mass m_2, slides without friction, on a table, driven by the sphere m_1, with the help of a non stretched wire passing through the groove of a pulley of radius R and mass M distributed on its rim.

Calculate

1. The angular momentum with respect to an axis passing through the center of the pulley.
2. The acceleration of the masses m_1 and m_2



- The angular momentum of m_2 :
  $L_2 = |\vec{r}_2 \wedge m_2\vec{v}_2| = m_2 vR$
- The angular momentum of m_1 :
  $L_1 = |\vec{r}_1 \wedge m_1\vec{v}_1| = m_1 vR$
- The angular momentum of M :

  $L_3 = |\vec{R} \wedge M\vec{v}| = MvR$

Pulley mass distributed over the rim (periphery), so the angular momentum is:

$$L_{/\Delta} = L_1 + L_2 + L_3$$

$$\sum_i \mathcal{M}(\vec{F})_{/\Delta} = \frac{dL_{/\Delta}}{dt} = \frac{d(L_1 + L_2 + L_3)}{dt} = \frac{d(m_1 vR + m_2 vR + MvR)}{dt}$$

$$\sum_i \mathcal{M}(\vec{F})_{/\Delta} = \mathcal{M}\left(m_1\vec{g}\right)_{/\Delta} = m_1 gR = (m_1 + m_2 + M)Ra$$

$$\Rightarrow \qquad a = \frac{m_1 g}{(m_1 + m_2 + M)}$$