Damped free vibrations
with single degree of freedom

II-8- Introduction

Damped free vibrations, are vibrations whose amplitudes decrease overtime until they are completly vinished from the system. This is due to dissipation of energy by frictions in mechanical systems or by joule effect in electrical systems.

In mechanical systems there are two kinds of frictions:

- fluid friction (viscous friction): $\vec{f}_f = -\alpha \vec{v}$
- solid friction $\|\vec{f}_s\| = + mg\,\mathrm{tg}\,\gamma$

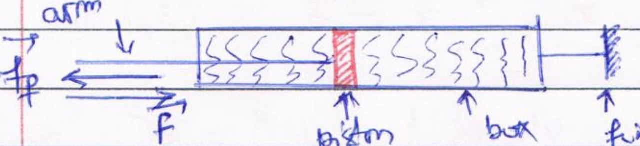
$\alpha =$ linear coefficient of viscous friction

In electrical systems the joule effect is due to ohmic resistance in the circuit and the potential drop is given by:

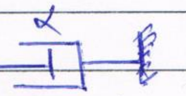
$$U = Ri$$

II-9- Fluid damping of free vibrations in mechanical systems

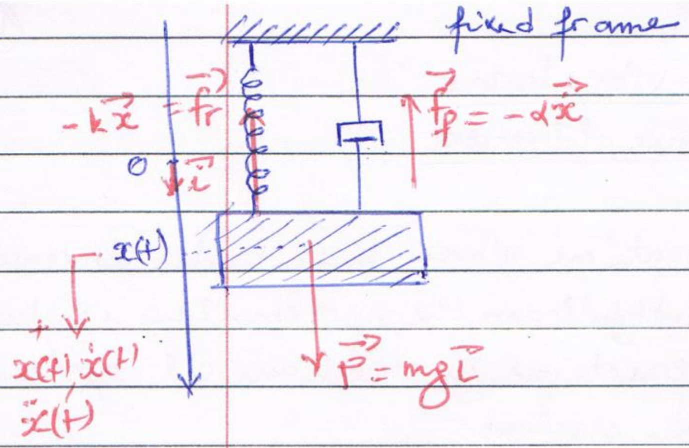
A frictional force can be created by a device consisting of a box, hermetically sealed and filled with a given fluid (liquid or gaz), through which a piston can move along the box and is operated by an arm which leads outside this box. When a force $\vec{F}$ acts on the arm, the device responds with a fluid frictional force $\vec{f}_f$ opposes the applied force (see the figure below).



the device is called damper(مخمد)

This device can be schematically represented by: 

let's have now the k + m (spring + mass), where the mass m is moving through a fluid midium and consequently undergoes the action of a fluid friction force from it. the syst is represented as follows:



9.1_ Extraction of the motion equation by Lagrange's method

$$T = \frac{1}{2} m \dot{x}^2$$
$$U = \frac{1}{2} k x^2$$
$$\Rightarrow \mathcal{L}(x,\dot{x}) = T - U = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2$$

calculate of dissipation function D:

$$D = -\frac{1}{2} \frac{\delta W(\vec{f_p})}{\partial \dot{x}} \qquad \left(D = \frac{1}{2} P_d \right)$$

thus:

$$D = \frac{1}{2} \alpha \dot{x}^2$$

the Lagrange's equation is written as:

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{x}}\right] - \frac{\partial \mathcal{L}}{\partial x} + \frac{\partial D}{\partial \dot{x}} = 0$$

and therefore we find after having deriving $\mathcal{L}$ and D and after having dividing the two sides of the equation by m we find:

$$\boxed{\ddot{x} + \frac{\alpha}{m} \dot{x} + \frac{k}{m} x = 0} \quad \ldots (3)$$

In the equation above we put: $\frac{\alpha}{m} = 2\delta$ and $\frac{k}{m} = \omega_0^2$

δ = damping coefficient معامل التخامد

ω_0 = natural angular frequency

we denote by ξ the ratio of δ over ω_0 i.e:

$$\xi = \frac{\delta}{\omega_0}$$

ξ = is called the damping ratio. and then the equation (3) can be written by two different ways:

ξ = نسبة التخامد

(19)

$$\ddot{x} + 2\delta\dot{x} + w_0^2 x = 0 \quad (3a)$$
$$\ddot{x} + 2\xi w_0 \dot{x} + w_0^2 x = 0 \quad (3b)$$

9_2 - Equation's solution

(3a) or (3b) is a linear diff equation of second ordre. we must first write the following caracteristic equation:

$$r^2 + 2\delta r + w_0^2 = 0$$

this algebriac equation admits two roots (real or complexs) in accordance with the sign of the reduced discriminant Δ' which is given by:

$$\Delta' = \delta^2 - w_0^2$$

the two roots are:

$$r_{1,2} = -\delta \mp \sqrt{\Delta'} = -\delta \mp \sqrt{\delta^2 - w_0^2}$$

according to the sign of Δ' we distinguish three cases:

1) first case: weak damping: تخامد ضعيف

if $\Delta' < 0 \Rightarrow \delta^2 - w_0^2 < 0 \Rightarrow 0 < \xi < 1 \quad (\delta < w_0)$

two complex roots are obtained:

$$r_{1,2} = -\delta \mp \sqrt{\Delta'} = -\delta \mp \sqrt{-|\Delta|}$$
$$= -\delta \mp j \underbrace{\sqrt{w_0^2 - \delta^2}}_{w_a} = -\delta \mp j w_a$$

where: $w_a = \sqrt{w_0^2 - \delta^2} = w_0 \sqrt{1 - \xi^2}$

the solution is then given by:

$$x(t) = A_1 e^{r_1 t} + A_2 e^{r_2 t} = A_1 e^{(-\delta + j w_a)t} + A_2 e^{(-\delta - j w_a)t}$$

this expression can be written in the following form:

$$x(t) = C e^{-\delta t} \cos(w_a t + \varphi)$$

In accordance with this expression the motion is called pseudosinusoidal because the amplitude $Ce^{-\delta t}$ is decreasing over time. The motion is then caracterized by a pseudo angular frequency given by:

$$w_a = w_0 \sqrt{1 - \xi^2} \quad : \quad w_a \xrightarrow{\xi \to 0} w_0$$

and the pseudoperiod is:

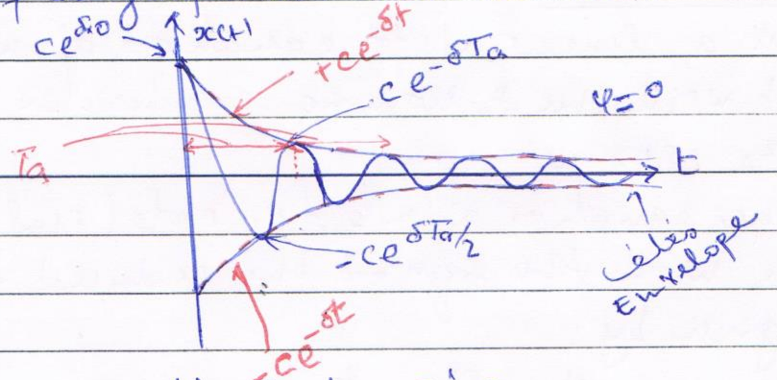
$$T_a = \frac{2\pi}{w_a} = \frac{2\pi}{w_0 \sqrt{1 - \xi^2}} = \frac{T_0}{\sqrt{1 - \xi^2}}$$

(20)

plotting of the graph $x(t)$:

according to the expression of $x(t)$ the graph has Ridges at points $c\,e^{-\delta t}$ which is decreasing exponential function.

the shape of the graph $x(t)$ will be then:



2) Second case: critical damping   تخامد حرج

if $\Delta' = 0 \Rightarrow \delta^2 - \omega_0^2 = 0 \Rightarrow \xi = 1$

knowing that: $\delta = \dfrac{\alpha}{2m} \Rightarrow \alpha_c = 2\sqrt{km}$

$\omega_0 = \sqrt{k/m}$

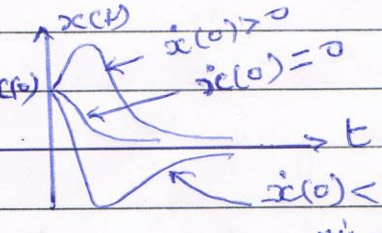
In this case we have a double root $r_1 = r_2 = r = -\delta < 0$

then:

$$x(t) = (A_1 t + A_2)\, e^{-\delta t}$$

the motion is not vibratory the system return to its equilibrium position in a minimal time.

three cases according to initial velocity



3) Third case: heavy damping: تخامد قوي

if $\Delta' > 0 \Rightarrow \delta^2 - \omega_0^2 > 0 \Leftrightarrow \xi > 1$

the caracteristic equation admits two real negative roots

$$r_{1,2} = -\delta \mp \sqrt{\delta^2 - \omega_0^2} < 0$$

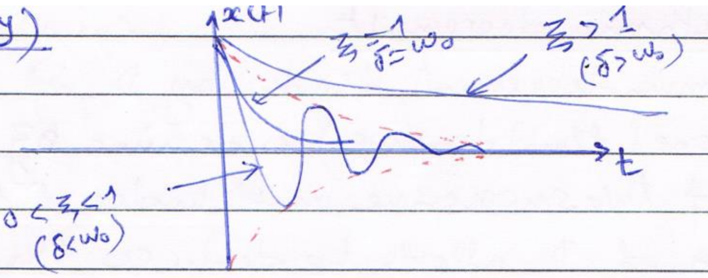
$$\Rightarrow r_{1,2} = -\delta \mp \omega_0\sqrt{\xi^2 - 1}$$

the solution in this case will be:

$$x(t) = e^{-\delta t}\left(A_1 e^{\omega_0\sqrt{\xi^2-1}\,t} + A_2 e^{-\omega_0\sqrt{\xi^2-1}\,t}\right)$$

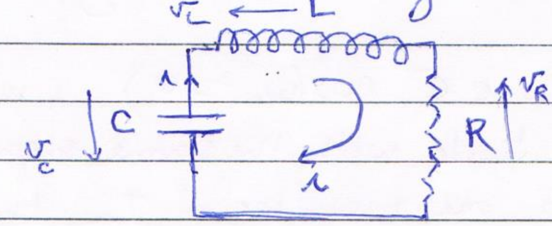
the motion in this case is not vibratory too. The system return to its equilibrium position without vibrations, but it takes a long time for this in comparaison with the previousely seen cases.

Recap (summary) (21)



II-10 - Damping free vibrations in electrical systems RLC

let's have the electrical system RLC in the figure below:



R = play the role of damping in ele systems

we apply Kirchhoff's voltage law:

$$v_L + v_R + v_C = 0 \quad \text{where:} \quad v_L = L\frac{di}{dt} = L\ddot{q}$$
$$v_R = Ri = R\frac{dq}{dt}$$
$$v_C = \frac{1}{C}\int i\,dt = q/C$$

then the differential equation will be:

$$\ddot{q} + \frac{R}{L}\dot{q} + \frac{1}{LC}q = 0$$

we put:

$$\frac{R}{L} = 2\delta \quad \text{and} \quad \frac{1}{LC} = \omega_0^2$$

then the equation become:

$$\ddot{q} + 2\delta\dot{q} + \omega_0^2 q = 0 \quad ; \quad q = \text{charge in } C$$

The solution of this equation is identic to the previous one (mechanical system). Then depending on δ and ω_0 we confirm that the three cases are present in RLC system identically to the mechanical system. Especially for the case of critical damping, which is observed if:

$$\delta = \omega_0 \Rightarrow R_c = 2\sqrt{\frac{L}{C}}$$

Application of critical damping:

moving coil galvanometer (galvanometre à cadre mobile)

II - 11 - Logarithmic decrement التناقص اللوغاريتمي (22)

the logarithmic decrement denoted by D, of weakly damped free oscillations (fluid damping) is defined by the logarithm of the ratio of two successive amplitudes of the same sign. Calculation of D allows to deduce the amount of damping ratio ξ for a given system.

the expression of $x(t)$ for weakly damping free oscillations is of the form:

$$x(t) = C e^{-\delta t} \cos(\omega_a t + \varphi) \quad ; \text{ where } \omega_a = \sqrt{\omega_0^2 - \delta^2}$$

two successive amplitude with the same sign are separated by a time equal to one pseudperiod T_a, then the phase difference will be : ($t_2 - t_1 = T_a$)

$$\Delta\varphi = \omega_a (t_2 - t_1) = \omega_a T_a = 2\pi$$

where t_1 and t_2 are two instants for which we have two ridges of the same sign $x_1(t_1)$ and $x_2(t_2)$

so we can write:

$$D = \ln\left(\frac{x_1}{x_2}\right) = \ln\left[\frac{C e^{-\delta t_1} \cos(\omega_a t_1 + \varphi)}{C e^{-\delta t_2} \cos(\omega_a t_2 + \varphi)}\right]$$

$$= \ln\left[\frac{e^{-\delta t_1} \cos(\omega_a t_1 + \varphi)}{e^{-\delta t_2} \cos(\omega_a t_1 + \omega_a T_a + \varphi)}\right]$$

$$= \ln\left[e^{\delta(t_2 - t_1)} \frac{\cancel{\cos(\omega_a t_1 + \varphi)}}{\cancel{\cos(\omega_a t_1 + \varphi + 2\pi)}}\right] = \delta(t_2 - t_1) = \delta T_a$$

Knowing that: $\delta = \omega_0 \xi$ and $T_a = \dfrac{T_0}{\sqrt{1 - \xi^2}}$ then:

$$D = \omega_0 \xi \frac{T_0}{\sqrt{1 - \xi^2}} = \boxed{\frac{2\pi \xi}{\sqrt{1 - \xi^2}}}$$

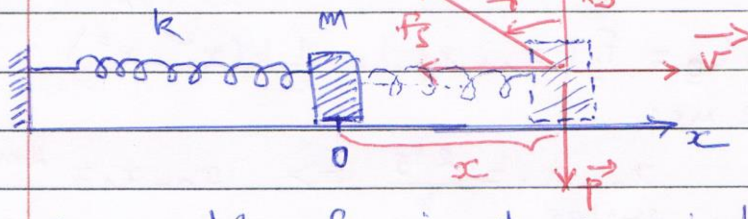
<u>note</u> : If we take two instants t_1 an t_2 separated by n pseudoperiod T_a i.e: $t_2 - t_1 = n T_a$ then:

$$\boxed{\ln\left(\frac{x_1}{x_2}\right) = n D}$$

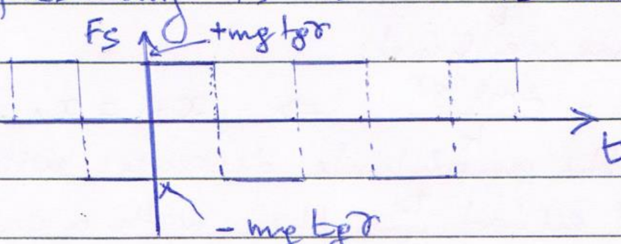
<u>Conclusion</u>: for weakly damped free oscillations the amplitude decreases exponentially over time

II - 12 - Solid damping of free vibrations (23)

Let have the vibratory system represented on the figure below. the mass m is moving horizontally on horizontal plane (solid) for which we associate an axis ox of motion. During its deplacement on ox the mass m undergoes the action of the force of solid friction created by the contact between m and the plane. this force denoted $\|\vec{F_s}\| = mg\,tg\gamma$ changes its direction every half period $\frac{T_0}{2}$, because m has a vibratory motion of period T_0. then f_s will have a shape of square function.



the curve representing f_s is shown in the graph below



so f_s is a periodic function and can be developped as a fourier's series as follows:

$$F_s = \frac{4}{\pi} mg\,tg\gamma \sum_{n=0}^{\infty} \frac{\sin(2n+1)w_0 t}{(2n+1)} \quad ; \quad w_0 = \frac{2\pi}{T_0}$$

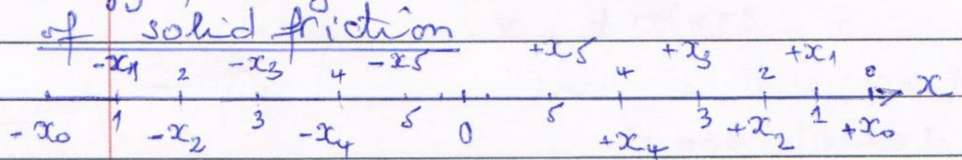
the differential equation is then given as:

$$m\ddot{x} + kx = \frac{4}{\pi} mg\,tg\gamma \sum_{n=0}^{\infty} \frac{\sin(2n+1)w_0 t}{(2n+1)}$$

whose solution $x(t)$ will be of the form of fourier's series, which can be find through inverse fourier's transformation.

But it's more easy to search $x(t)$ by analysing how energy of the system decreases over time.

12 - 1 - Energy of the system in the case of solid friction



if we assume that the system begins to move from x_0 without initial velocity then:

its total energy is:

$$E_0 = \frac{1}{2} k x_0^2$$

the mass m will move towards the position $-x_1$ and will stop to move instantly in this position $-x_1$. the total energy at $-x_1$ is:

$$E_1 = \frac{1}{2} k(-x_1)^2 = \frac{1}{2} k x_1^2$$

the energy E_1 is less than E_0 because of dissipation of energy due to friction during the motion.
Dissipated energy is transformed into heat by the work of friction force f_s

therefor:

$$E_0 - E_1 = F_s(x_0 + x_1) = \frac{1}{2} k(x_0^2 - x_1^2)$$

$(x_0 - x_1)(x_0 + x_1)$

which will give us:

$$x_0 - x_1 = \frac{2F_s}{k} \Rightarrow x_0 - x_1 = \frac{2mg\,tg\,r}{k}$$

$$\Rightarrow x_1 = x_0 - \frac{2mg\,tg\,r}{k}$$

by the same maner we find:

$$E_1 - E_2 = \frac{2mg\,tg\,r}{k} \Rightarrow x_2 = x_1 - \frac{2mg\,tg\,r}{k}$$

which means that the amplitude decreases with a cadence of $\frac{2mg\,tg\,r}{k}$ every half period $\frac{T_0}{2}$, then with a cadence of $\frac{4mg\,tg\,r}{k}$ every full period.

After n $\frac{\text{periods}}{2}$ of motion $(n\frac{T_0}{2})$ the amplitude will become:

$$x_n = x_0 - n\left(\frac{2mg\,tg\,r}{k}\right)$$

thus it's easy to show that amplitudes in the negative side form together an arithmetic series of a common difference of: $\frac{2mg\,tg\,r}{k}$. the same thing happens for the positive side.

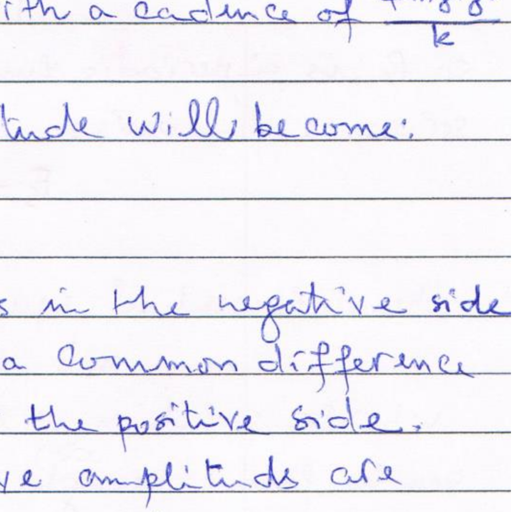
As a conclusion we can say that positive amplitudes are located on the same straight line of equation

$$x = -\frac{8\pi g\,tg\,r}{\omega_0} t + x_0$$

the negative amplitudes are on the straight line of equation

$$x = \frac{8\pi g\,tg\,r}{\omega_0} t - x_0$$

the shape of $x(t)$ will be then:



the condition of complete stop of the system is: $F_r \leq f_s \Rightarrow k x_n \leq mg\,tg\,r$

$$\Rightarrow x_n \leq \frac{mg\,tg\,r}{k} \Rightarrow x_0 - n\left(\frac{2mg\,tg\,r}{k}\right) \leq \frac{mg\,tg\,r}{k}$$

$$\Rightarrow x_0 - n(a) \leq \frac{1}{2} a$$