MOHAMED BOUDIAF UNIVERSITY OF M'SILA

DEPARTMENT OF COMPUTER SCIENCE

The Foundations of Graph Theory

Chapter 2: Representation of Graphs

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2017 - 2018

Methods of graph representation

We distinguish 02 classes of methods:

1. Static methods (use of matrices)

- **Adjacency matrix (vertices – vertices)**
- **Incidence matrix (vertices – arcs)**
- **List of arcs (sorted – unsorted)**
- **List of successors**
- **List of predecessors**
- **Linear list of successors**

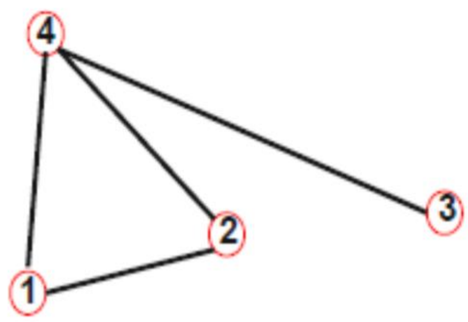
2. Dynamic methods (using linked lists)

- **Dynamic list of vertices**
- **Dynamic list of arcs**

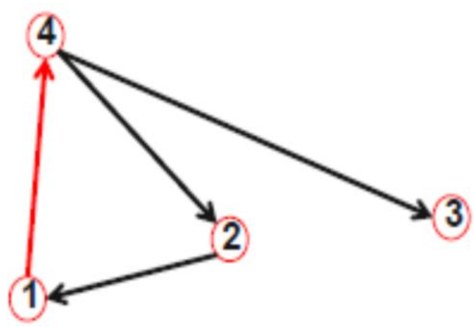
Static Methods

1. Adjacency matrix (vertices – vertices)

Graph G1



Graphe G2



$$\begin{cases} A_{ij} = 1 & \text{if there exist a link between vertices } i \text{ and } j \\ A_{ij} = 0 & \text{if vertices } i \text{ and } j \text{ are not connected} \end{cases}$$

$$A_{ij} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

A(N, N)

A_{14}

$$A_{ij} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

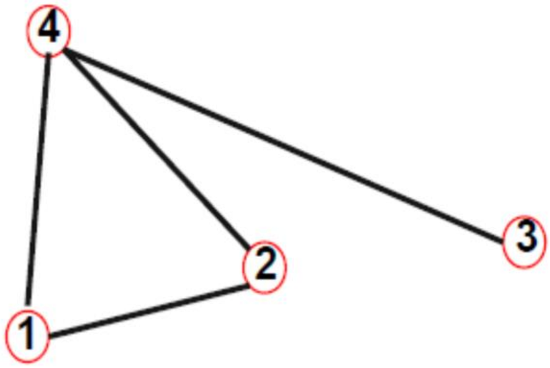
A(N, N)

NB: Note that for a directed graph (right) the matrix is not symmetrical.

Adjacency matrix and vertex degrees

Undirected graph

$$A_{ij} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

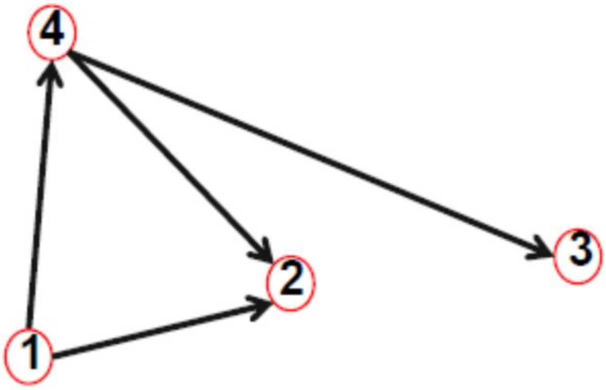


$$\begin{cases} A_{ij} = A_{ji} \\ A_{ii} = 0 \end{cases}$$

$$\mathbf{k_i} = \sum_{j=1}^{N} \mathbf{A_{ij}} = \mathbf{k_j} = \sum_{i}^{n} \mathbf{A_{ij}}$$

$$L = \frac{1}{2}\sum_{i=1}^{N} k_i = \frac{1}{2}\sum_{ij}^{N} A_{ij}$$

Directed graph



$$A_{ij} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

$$\begin{cases} A_{ij} \neq A_{ji} \\ A_{ii} = 0 \end{cases}$$

$$k_i^{ex} = \sum_{j=1}^{N} A_{ij} \quad ; \quad k_j^{in} = \sum_{i=1}^{N} A_{ij}$$

$$L = \sum_{i=1}^{N} k_i^{in} = \sum_{j=1}^{N} k_j^{out} = \sum_{i,j}^{N} A_{ij}$$

2. Incidence matrix (vertices – arcs)

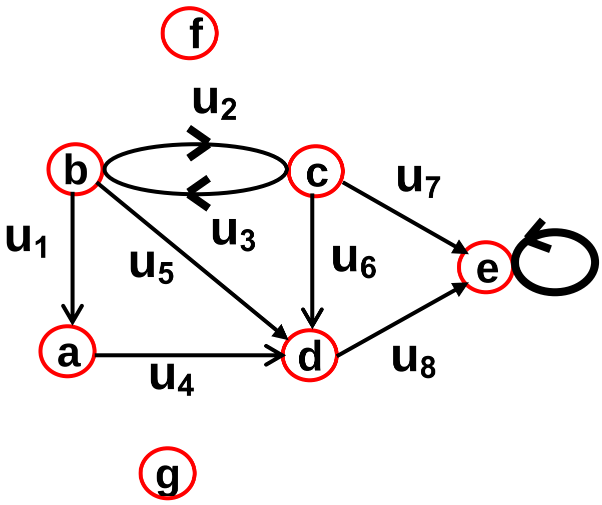
Let the following graph G(X, U):

X = {a, b, c, d, e, f, g}

U = {(a, d), (b, a), (b, c), (b, d), (c, b), (c, d), (c, e), (d, e) , (e, e)}

	u1	u2	u3	u4	u5	u6	u7	u8	u9
a	-1	0	0	+1	0	0	0	0	0
b	+1	+1	-1	0	+1	0	0	0	0
c	0	-1	+1	0	0	+1	+1	0	0
d	0	0	0	-1	-1	-1	0	+1	0
e	0	0	0	0	0	0	-1	-1	2
f	0	0	0	0	0	0	0	0	0
g	0	0	0	0	0	0	0	0	0

A(N, M)



Remarks:

- **Each column in the matrix contains a single (+1) corresponding to the initial endpoint of the arc, and a single (-1) corresponding to its terminal endpoint**
- **The number of (+1) on the line gives the external 1/2 degree of the vertex, although the number of (-1) gives the internal 1/2 degree of the same vertex.**

3. List of sorted arcs (by initial end)

U4	U1	U2	U5	U3	U6	U7	U8	U9
a	b	b	b	c	c	c	d	e
d	a	c	d	b	d	e	e	e

f	g	.....

Isolated vertices

A(2, M)/**A(3, M)**

4. List of unsorted arcs (taken arbitrarily)

U4	U1	U3	U2	U8	U9	U5	U6	U7
a	b	c	b	d	e	b	c	c
d	a	b	c	e	e	d	d	e

f	g	.....

Isolated vertices

A(2, M)/**A(3, M)**

5. List of successors

1	a	d		
2	b	a	c	d
3	c	b	d	e
4	d	e		
5	e	e		
6	f			
7	g			

A(N, d^{+}_{max})

6. List of predecessors

1	a	b		
2	b	c		
3	c	b		
4	d	a	b	c
5	e	e		
6	f			
7	g			

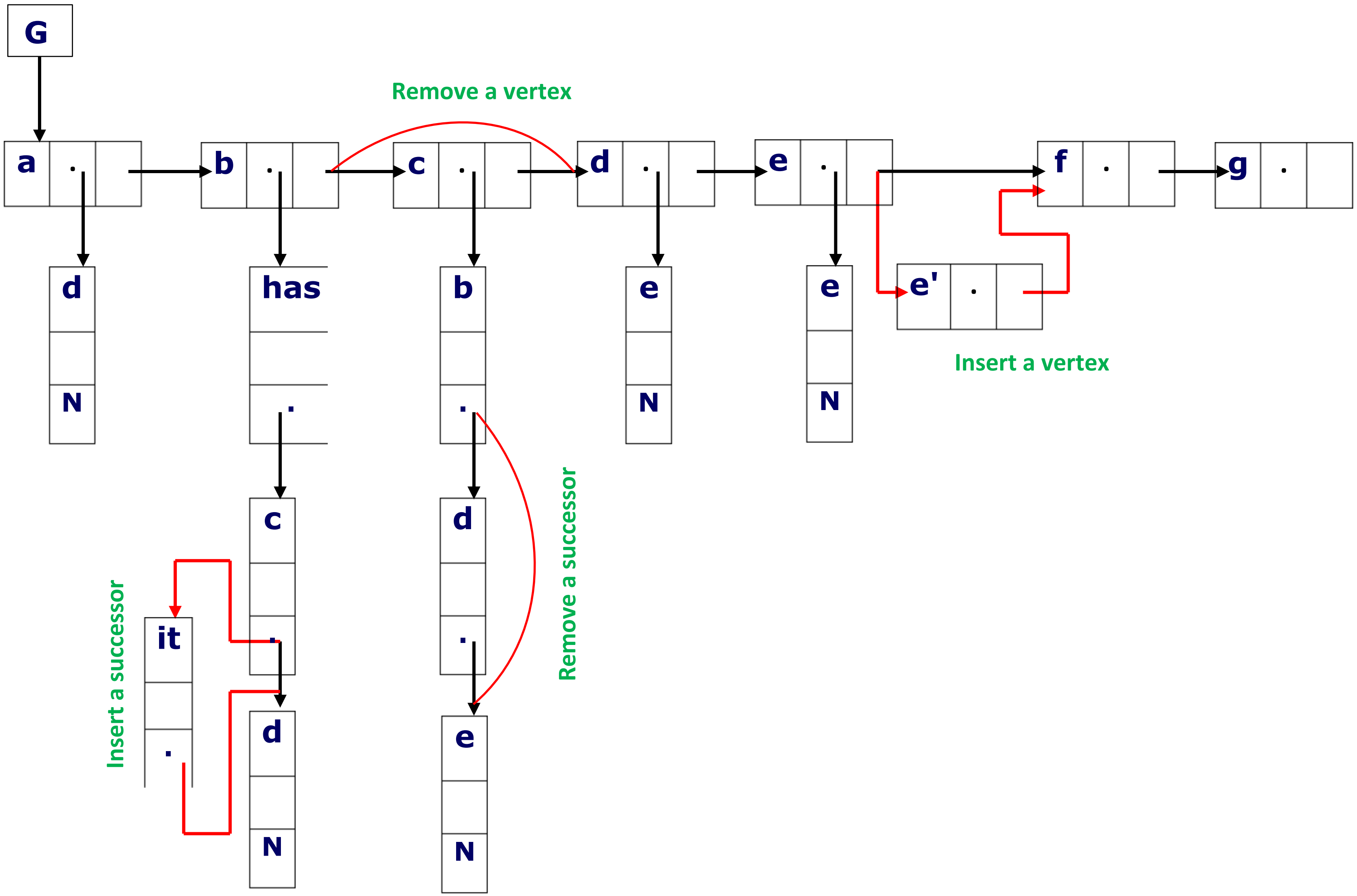
A(N, d^{-}_{max})

Dynamic methods

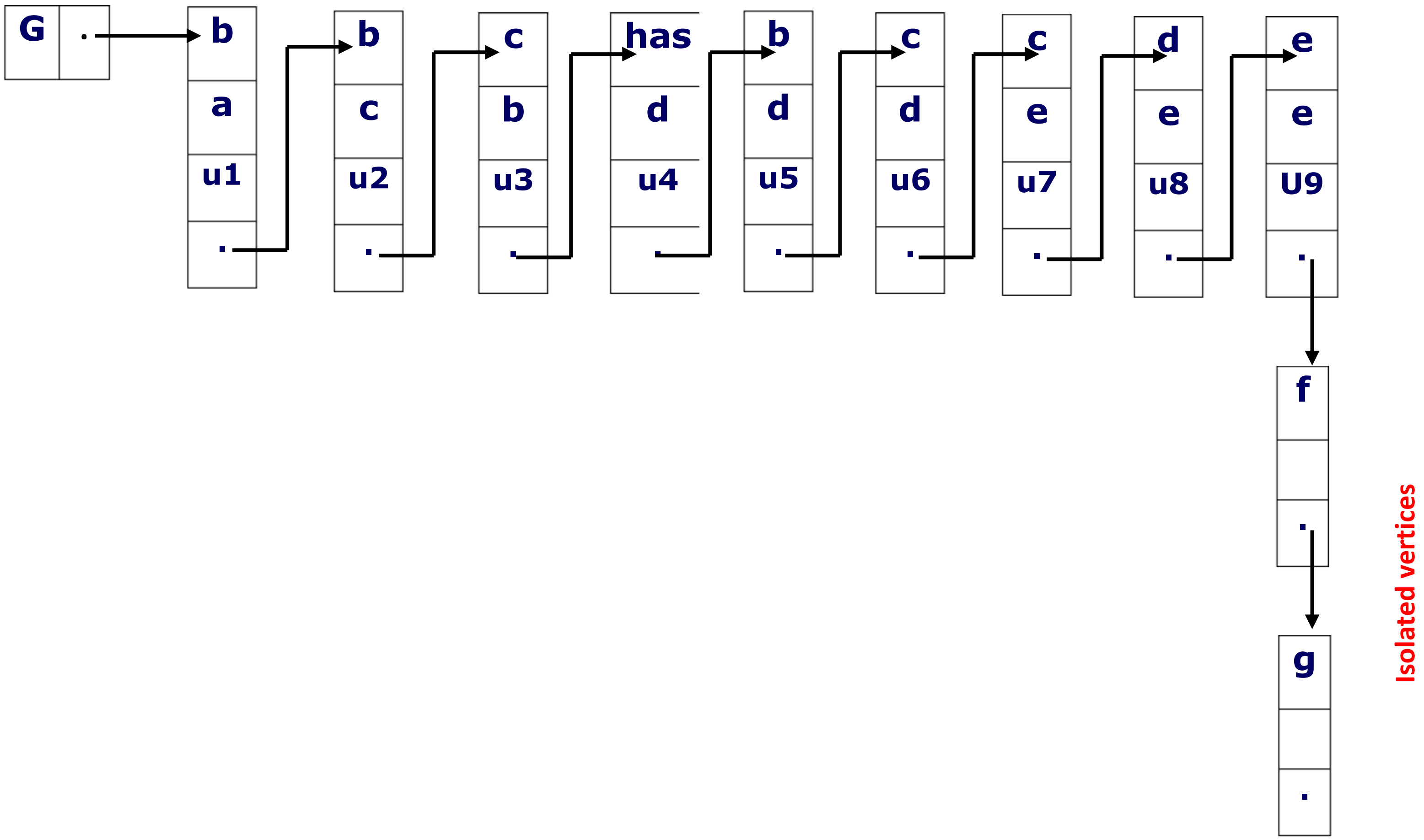
For each vertex we associate a sequence of boxes (boxes/nodes), each box contains elementary information, such as:

- **Name of the vertex (a, b, c, 1, 2, ...)**
- **Address of the 1st successor**
- **Address of the last successor.**
- **Number of successors**
- **Value of the arc.**
- **Others**

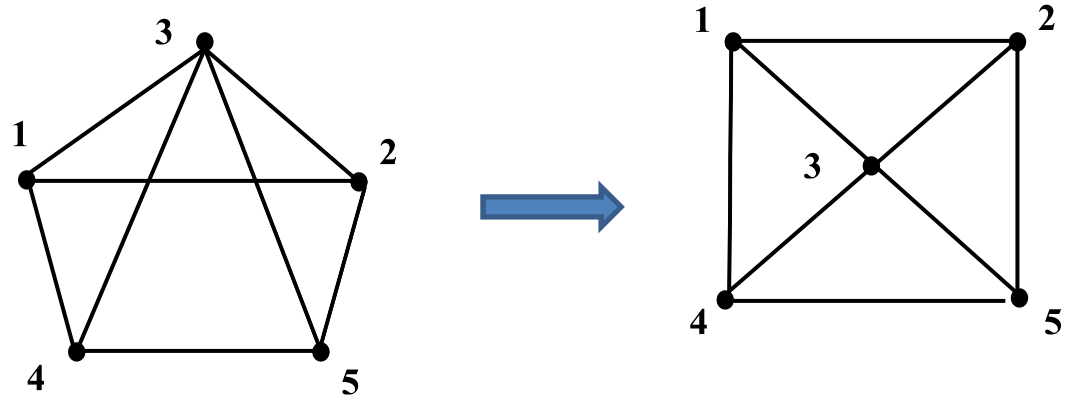
1. Dynamic list of vertices



2. Dynamic list of arcs



Planar graphs



Eulerian graphs and Hamiltonian graphs

Theorem 1:

- **A connected undirected graph has an Eulerian chain if only if the numberof vertices of odd degree is equal to 0 or 2.**
- **It admits an Eulerian cycle if only if all its vertices have a even degree.**

  Necessary condition:

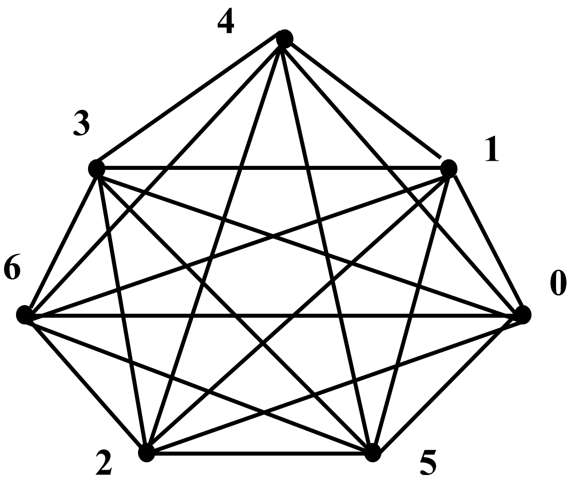
  - **For each vertex x, d-(x) = d+(x) ==> d(x) is even**

Eulerian paths and circuits

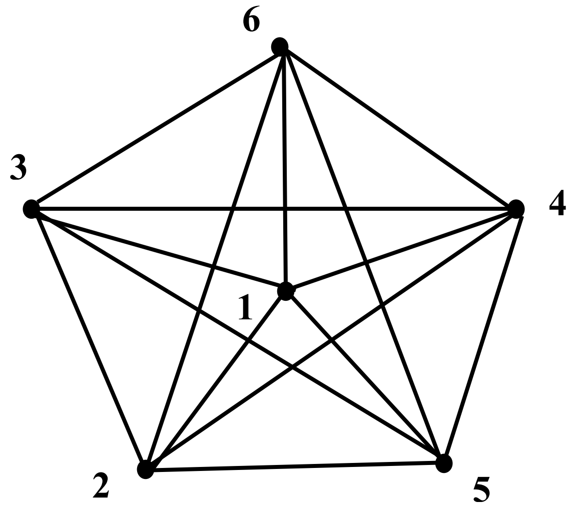
Theorem 2:

- **G admits an Eulerian circuit if only if ∀x∈G $d^{-}(x) = d^{+}(x)$**
- **A simple connected graph, G(X, U) is Eulerian if only if all its vertices are of even degree (∀x∈X, d(x) is even)**

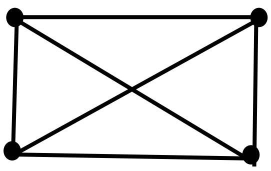
Examples



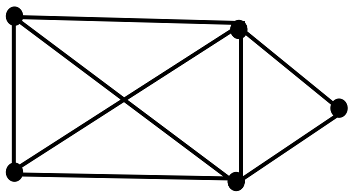
∀x∈X, d(x) = 6 even
G is Eulerian



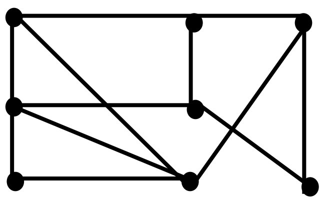
∀x∈X, d(x) = 5 odd
G is not Eulerian



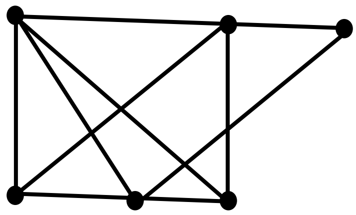
G is not Eulerian



G is Eulerian

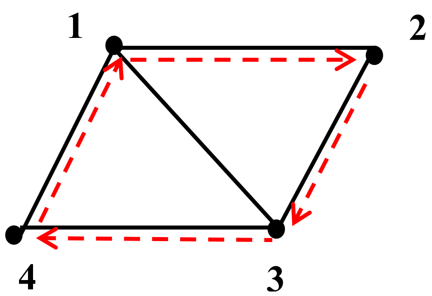


G is not Eulerian



G is Eulerian

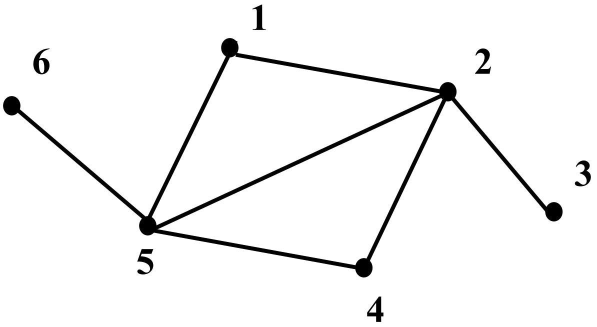
Hamiltonian cycles



∃ a Hamiltonian cycle

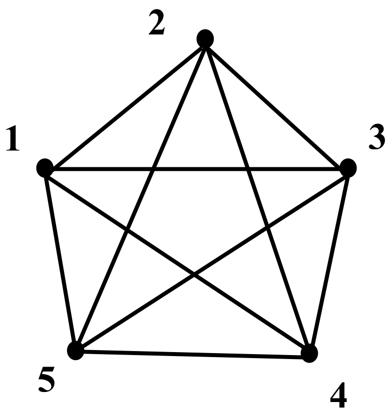
Observation

- **A graph with a vertex of degree 1 cannot be Hamiltonian**

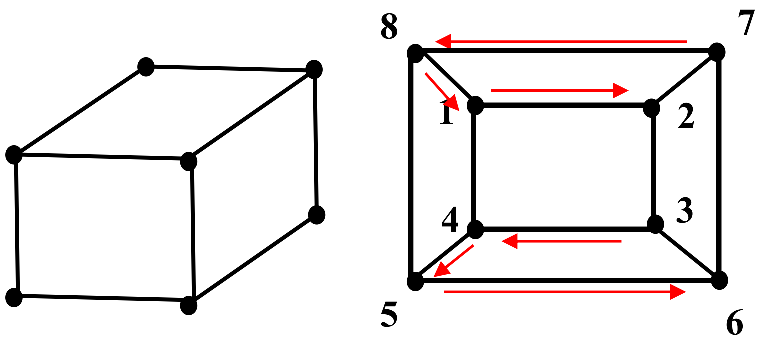


G is not Hamiltonian

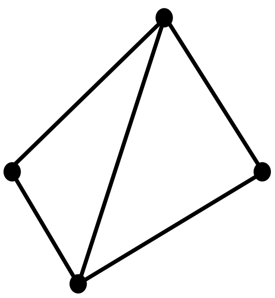
- **The complete graphs k_n are Hamiltonian if n > 3**



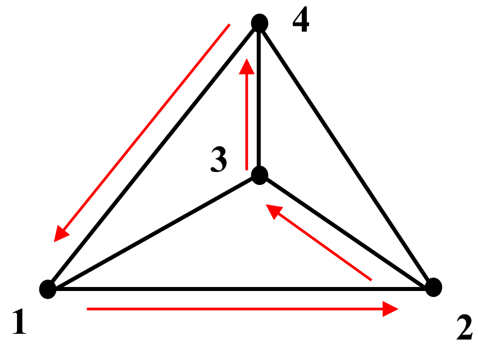
G is Hamiltonian (complete, k4)



G contains a Hamiltonian cycle



k3 type graph (pyramid)



G is Hamiltonian (complete, k3)