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Module: Probability-Statistics

Chapter 3 : COMBINATORIAL ANALYSIS

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1 Introduction

Combinatorial analysis is the science of counting and is used to determine the number of possible outcomes of a given experiment.
Knowledge of these enumeration methods is essential for the elementary calculation of probabilities.

2 Preliminaries

2.1 Definitions :

• A set Ω is finite when it has a finite number of elements.
The number of elements of Ω is called the cardinal of the set and it is noted : $\sharp\Omega$ or $|\Omega|$.

• To count is to determine the number of elements in a finite set, i.e. to determine its cardinal.

Example 1. *If* $\Omega = \{heads, tails\}$*, its cardinal is* $|(\Omega)| = 2.$.
- *If* $\Omega = \{1,\ 2,\ 3,\ 4,\ 5,\ 6\}$, $|(\Omega)| = 6$.
If Ω *is the set of integers,* $|(\Omega)| = +\infty$. $card(\phi) = 0$.
The set Omega of players in a football team is a finite set. . Then $|(\Omega)| = 11$.

Definition 1. *We say that two sets* Ω_1 *and* Ω_2 *are* ***disjoint****, if they have no elements in common, i.e.*

$$\Omega_1 \cap \Omega_2 = \phi$$

2.2 Factor notation

Definition 2. *Let* n $n \in \mathbb{N}$*. The product of the integers from 1 to n is called n factorial, denoted n!*

$$n! = \begin{cases} 1 \quad if\, n = 0 \\ n \times (n-1) \times (n-2) \times ... \times 3 \times 2 \times 1 \quad if\, n > 0 \end{cases}$$

Example 2. $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$. $6! = 6 \times 5! = 720$.
$0! = 1$.

Rremark 1. $\dfrac{n!}{(n-1)!} = n, \qquad n! = n \times (n-1)!$

3 Basic Counting Principles

The basic counting principles are **the product rule** and **the sum rule** will be presented and show how can be used tosolve many different counng problems.

3.1 Addition Principle

Suppose that a set Ω is partitioned into pairwise disjoint parts Ω_1, Ω_2,...,Ω_k , .

The addition principle claims that in this case

$$|\Omega| = |\,(\Omega_1 \cup \Omega_2 \cup ... \cup \Omega_k)\,| = |\,(\Omega_1)\,| + |\,(\Omega_2)\,| + ... + |\,(\Omega_k)\,|$$

Example 3. *Let* $\Omega_1 = \{a, b, c, d\}$ *and* $\Omega_2 = \{g, h, k\}$. *we have* $\Omega_1 \cap \Omega_2 = \phi$, *then*

$$|\,(\Omega_1 \cup \Omega_2)\,| = |\,(\Omega_1)\,| + |\,(\Omega_2)\,| = 4 + 3 = 7.$$

Example 4. *Let* Ω *be the set of students attending the combinatorics lecture. It can be partitioned into parts* Ω_1 *and* Ω_2 *where :*
Ω_1= *set of students that like easy examples.*
Ω_2 = *set of students that don't like easy examples.*
If $|\Omega_1| = 22$ *and* $|\Omega_2| = 8$ *then we can conclude* $|\Omega| = 8$.

3.2 Multiplication Principle

If k experiments that are to be performed are such that :
the first one may result in any of n_1 possible outcomes ;
and if, for each of these n_1 possible outcomes, there are n_2 possible outcomes of the second experiment ;
and if, for each of the possible outcomes of the first two experiments, there are n_3 possible outcomes of the third experiment ;
and if ...,
then there is a total of $n_1 \times n_2 \times ... \times n_k$ possible outcomes of the k experiments.

Example 5. *We want to print a car number plate with, from left to right, 2 distinct letters and 3 digits, the first of which is not zero (a global experiment). How many plates of this type are there ?* ($CH124, DE665$, ..., ...)

Solution 1.

letter	letter	number	number	number
↑	↑	↑	↑	↑
26	25	9	10	10

According to the Multiplication Principle, the possible number of plates of this type is

$$26 \times 25 \times 9 \times 10 \times 10 = 585000$$

Example 6. *Three 6-sided dice are rolled in succession (a global experiment). How many possible outcomes are there ?*

Solution 2. *For D_1, we have (6 distinct numbers)*

For D_2, we have (6 distinct numbers)

For D_3, we have (6 distinct numbers)

According to the Multiplication Principle, the number of possible results is

$$6 \times 6 \times 6 = 216.$$

4 Permutations

4.1 Permutations of n distinct elements

Definition 3. *A permutation of n distinct elements is an ordered arrangement where each element is used once.*

Example 7. *The possible permutations of the 3 letters a, b, c are :*
abc, bca, cab, bac, acb, cba.

Proposition 1. *The number of permutations with n distinct elements is*

$$P(n, n) = n!$$

Rremark 2. ***Use permutations*** *: if a problem calls for the number of arrangements of objects and different orders are to be counted.*

Example 8. *In the example 7, on a* $n = 3$, *donc* $P_3 = 3! = 6$.

4.2 Permutations with Similar Elements

Definition 4. *If we classify in a particular order n elements of which*
n_1 *are identical of type 1,*
n_2 *are identical of type 2,*
.......,
n_k *are identical of type k,*
we form a permutation with repetitions of these $n = n_1 + n_2 + ... + n_k$ *elements.*

Proposition 2. *The number of permutations with repetitions is*

$$\bar{P}_n\left(n_1, n_2, ..., n_k\right) = \frac{n!}{n_1!n_2!...n_k!}$$

Example 9. *How many different words can you make with all the letters in the word ≪ ERRER ≫*

We have 5 letters $(n = 5)$ *some of which are similar.*

the letter 'E' appears twice so $n_1 = 2$, *the letter 'R' appears three times so* $n_2 = 3$, *then*

$$\bar{P}_5\left(2; 3\right) = \frac{5!}{2!3!} = 10$$

RRREE, RRERE, RREER, RERRE, RERER, REERR, ERRRE, ERRER; ERERR; EERRR.

Example 10. *A chess tournament has 10 competitors, of which 4 are Russian, 3 are from theUnited States, 2 are from Great Britain, and 1 is from Brazil. If the tournamentresult lists just the nationalities of the players in the order in which they placed,how many outcomes are possible ?*

Solution *There are* $\frac{10!}{2!3!1!4!} = 12600$ *possible outcomes*

Rremark 3. *a)* $\bar{P}_n\,(n_1, n_2, ..., n_k) < P_n$.

b) The bar above the P stands for 'with repetition'.

4.3 Circular Permutations

Definition 5. *Circular permutation is the total number of ways in which n distinct objects can be arranged around a fixed circle.*

Proposition 3. *The number of circular permutations of n objects is equal to*

$$(n-1)!$$

Example 11. *How many ways can you arrange 4 people :*
a) On a line ?
b) Around a round table ?

Solution 3. *a) 5 !=120.*
b)(5-1) !=4 !=24.

4.4 r-Permutation (arrangement)

Definition 6. *Let r be a positive integer* $(r \leq n)$*. By an r-permutation of a set* Ω *of n elements, we understand an ordered arrangement of r of the n elements.*

Example 12. *If* $\Omega = \{a, b, c\}$ *then the three 1-permutations of* Ω *are a , b, c, the six 2-permutations of S are ab ac ba be ca cb,*
and the six 3-permutations of S are abc acb bac bca cab cba.

Proposition 4. *The number of r-permutations froma set containing n elements is given by*

$$\begin{aligned} P(n,r) &= \frac{n!}{(n-r)!} \\ &= n\,(n-1)\,...\,(n-r+1) \end{aligned}$$

Rremark 4. *If* $r = n$*, we have* $P(n,r) = \dfrac{n!}{(n-n)!} = \dfrac{n!}{0!} = n!,$

Example 13. *With the letters of the word ≪ RELATION ≫, how many can you form, with or without meaning, different 5-letter words ?*

Answer *:* $n = 8$ *et* $r = 5$ *,there are* $A_8^5 = \dfrac{8!}{(8-5)!} = 8 \times 7 \times 6 = 336$ *possible words*

Proposition 5. *The number of permutations of n objects, taken r at a time, when repetition of objects is allowed, is*

$$n^r$$

Example 14. *How many 3 digit numbers can be formed with the digits 1 and 2 ?*

Answer *:* $n = 2$, *there are* $2^3 = 2^3 = 8$. *numbers*
111, 112, 121, 211, 122, 222, 221,212.

Example 15. *How many two-digit numbers can you make using the digits 5, 6, 7, 8, 9 ?*

Answer *:* $n = 5$, $r = 2$, $5^2 = 25$.

5 Combinations

5.1 Combination

Definition 7. *A combination of a set of elements is an arrangement where each element is used once, and order is not important.*

Example 16. *Four people* $\{1; 2; 3; 4\}$ *want to play doubles table tennis. How many different teams can they form ?*

Answer : It's a combination : $\{1; 2\}$,$\{3; \}$,$\{1; 3\}$,$\{2; 4\}$,$\{1; 4\}$,$\{2 : 3\}$ *We can form 6 teams.*

Proposition 6. *The Number of Combinations of n Objects Taken r at a Time*

$$\begin{aligned} \begin{pmatrix} n \\ r \end{pmatrix} &= \frac{n!}{r!\,(n-r)!} \\ &= \frac{P(n,r)}{r!} \end{aligned}$$

Rremark 5. *1)We also note* $C(n, r)$

2) Use combinations if a problem calls for the number of ways of selecting objects and the order of selection is not to be counted.

Example 17. *A committee of 3 is to be formed from a group of 24 people. How many different committees are possible ?*

Solution
There are

$$\begin{pmatrix} 24 \\ 3 \end{pmatrix} = \frac{24!}{3!21!} = \frac{24 \times 23 \times 22}{6} = 2024$$

possible committees

Properties

1) $\begin{pmatrix} n \\ n \end{pmatrix} = \begin{pmatrix} n \\ 0 \end{pmatrix}$, $\begin{pmatrix} n \\ r \end{pmatrix} = \begin{pmatrix} n \\ n-r \end{pmatrix}$ (symmetry formula)
2) $\begin{pmatrix} n \\ 1 \end{pmatrix} = \begin{pmatrix} n \\ n-1 \end{pmatrix} = n.$
3) $\begin{pmatrix} n+1 \\ r+1 \end{pmatrix} = \begin{pmatrix} n \\ r \end{pmatrix} + \begin{pmatrix} n \\ r+1 \end{pmatrix}$ (Pascal's triangle).
4) $\sum_{i=0}^{n} \begin{pmatrix} n \\ i \end{pmatrix} = 2^n$

Démonstration. exercice. □

Example 18. $\begin{pmatrix} 3 \\ 0 \end{pmatrix} = \dfrac{3!}{0!\,(3-0)!} = 1$; $\begin{pmatrix} 3 \\ 3 \end{pmatrix} = 1$; $\begin{pmatrix} 3 \\ 2 \end{pmatrix}) = \begin{pmatrix} 3 \\ 1 \end{pmatrix} = 3$

5.2 Combinations with Repetition

Definition 8. *A combination with repetition of r objects from n is a way of selecting r objects from a list of n. The selection rules are :*
1) the order of selection does not matter
2) each object can be selected more than once.

Example 19. *Let $\Omega = a, b, c$. We want to obtain the list of combinations of k=4 elements taken from the set E, i.e. generate the list of 4-combinations with repetition :*
(a, a, a, a), (a, a, a, b), (a, a, a, c), (a, a, b, b), (a, a, b, c), , (a, a, c, c) , (a, b, b, b) , (a, b, b, c) , (a, b, c, c) , (a, c, c, c) ,(b, b, b, b) , (b, b, b, c) , (b, b, c, c) , (b, c, c, c) , (c, c, c, c)

Proposition 7.

Number of combinations with repetition of r objects from n is

$$\begin{pmatrix} n+r-1 \\ r \end{pmatrix} = \frac{(n+r-1)!}{r!(n-1)!}$$

Example 20. *In the previous example , we have* $\begin{pmatrix} 3+4-1 \\ 4 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \end{pmatrix} = 15$

5.3 Pascal's Trianlge

The relationship $\begin{pmatrix} n+1 \\ r+1 \end{pmatrix} = \begin{pmatrix} n \\ r \end{pmatrix} + \begin{pmatrix} n \\ r+1 \end{pmatrix}$

allows practical determination of the various coefficients $\begin{pmatrix} n \\ r \end{pmatrix}$ using Pascal's triangle.

If $r > n$, , we put $\begin{pmatrix} n+r-1 \\ r \end{pmatrix} = 0$

$n \backslash r$	0	1	2	3	4	5	6
0	1						
1	1	1					
2	1	2	1				
3	1	3	3	1			
4	1	4	6	4	1		
5	1	5	10	10	5	1	
6	1	6	15	20	15	6	1

5.4 Newton's Binomial Theorem

Let n be a positive integer. Then, for all x and y,

$$(a+b)^n = \sum_{i=0}^{n} \begin{pmatrix} n \\ i \end{pmatrix})a^{n-i}b^i$$

Example 21. $(a+b)^2 = \begin{pmatrix} 2 \\ 0 \end{pmatrix})a^2 + \begin{pmatrix} 2 \\ 1 \end{pmatrix})ab + \begin{pmatrix} 2 \\ 2 \end{pmatrix})a^0b^2 = 1a^2+2ab+1b^2$

$(a+b)^3 = \begin{pmatrix} 3 \\ 0 \end{pmatrix})a^3 + \begin{pmatrix} 3 \\ 1 \end{pmatrix})a^2b\begin{pmatrix} 3 \\ 2 \end{pmatrix})a^1b^2 + \begin{pmatrix} 3 \\ 3 \end{pmatrix})a^0b^2$

$= 1a^3 + 3a^2b + 3a^1b^2 + 1b^3$.

5.5 Summary of the combinatorial analysis

order ╲ repetition	repetitions are allowed	repetitions are not allowed
order does matter	n^r	$\frac{n!}{(n-r)!}$
does not matter	$\begin{pmatrix} n+r-1 \\ r \end{pmatrix}$	$\begin{pmatrix} n \\ r \end{pmatrix}$