1) Introduction

In the previous chapter, we discussed the motion of a particle by considering only the changes in its position, velocity, acceleration, and trajectory, without delving into the reasons for these changes. In this chapter, we will address the laws of motion that connect the characteristics of motion (position, velocity, acceleration) on one hand, and the factors affecting motion and what influences them (forces and the mass of the particle) on the other hand.

2) NEWTON'S LAWS OF MOTION

Newton's Laws of Motion are a set of three fundamental principles in classical physics that describe the relationship between the motion of objects and the forces acting on them. These laws provide a framework for understanding the motion of objects and the interactions between them. These laws were formulated by the English physicist Sir Isaac Newton in the 17th century and remain a cornerstone of classical mechanics. Here are the explanations of Newton's Laws of Motion.

Before delving into the explanation of Newton's Three Laws, we will elucidate some terms associated with these laws:

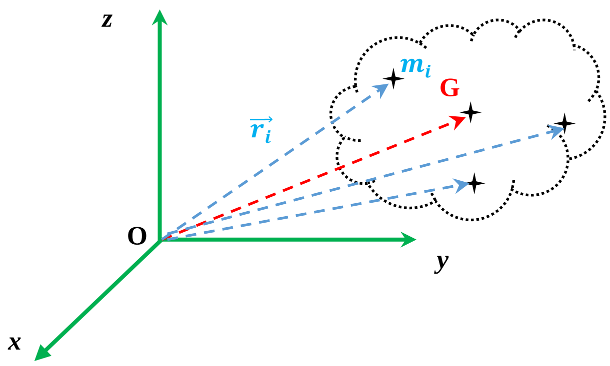
2-1)Mass

***Mass** is a scalar quantity that determines the amount of matter present in a body. From a dynamic standpoint, it represents the inertia of the body, meaning it is a measure of the resistance to any change in the motion of the body.*

2-2) Center of inertia (G) :

Considering a body as a collection of material points (each with mass m_i and position vectors $\vec{r}_i$ relative to a reference point O), the center of mass of the body coincides with the center of mass (the centroid) where it satisfies the following relationship:

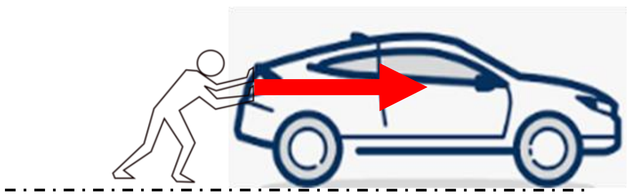
$$\overrightarrow{OG} = \frac{\sum_i m_i \vec{r}_i}{\sum_i m_i}$$



2-3) Forces:

- ***Force*** *is any effect that leads to a change in the motion state of an object (change in the velocity, direction, path, or stopping of the object) or a change in the shape of the object.*
- ***Force*** *is a vector quantity used to describe the interaction between two bodies, represented by a vector* $\vec{F}$*, where this vector represents the characteristics of force, including direction, support, and magnitude.*
- ***Forces****in nature are one of two:*

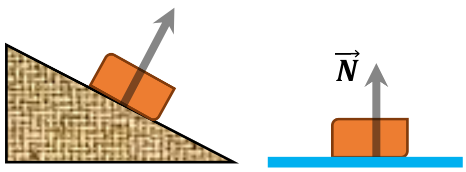
2-3-1)Contact forces *resulting from direct contact between two objects (for example, pulling an object with a rope, pushing a cart)*



We list below the properties of some forces that act at a distance:

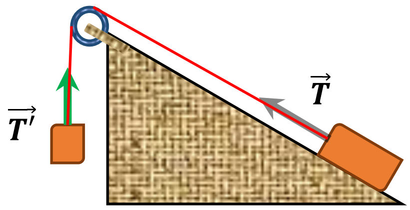
a) Normal Force $(\vec{N})$

this force represents the effect exerted by the supporting surface on the body and its direction is perpendicular to the surfaceof contact.



b) Tension force($\vec{T}$):

The tension force $\vec{T}$ is the force that a cord, rope, or cable exerts on an object attached to it. Thisforce is directed along the rope away from the object at thepoint where the rope is attached.



c) Friction Force:

The frictional force is due to the interaction between the surface atoms of any two bodies in contact. The direction of this force is always parallel to the surface of contact, opposing the motion or the planned motion of one object relative to the other. Hence, the normal and frictional forces are both contact forces and they are always perpendicular to each other.

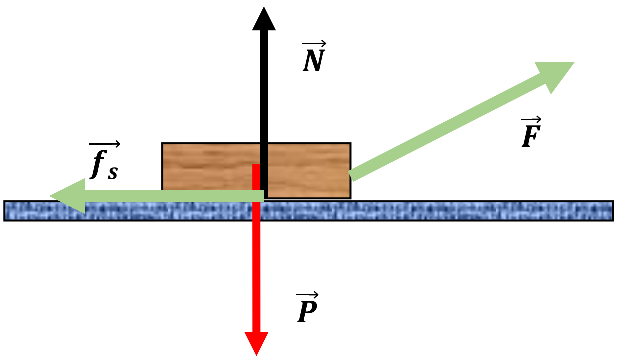
There are two types of frictional forces:

- ✓ ***The statistical frictional force $\vec{f}_s$***

  The name statistical comes from the fact that the body remains stationary. Statical frictional force is directly proportional to the magnitude of the normal force, $\vec{N}$, according to the following relationship

$$\vec{f}_s = \mu_s \vec{N}$$

where μ_s coefficient of static friction.



- ✓ ***The kinetic frictional force $\vec{f}_k$***

When the body moves, the retarding frictional force is then called the kinetic frictional force $\vec{f}_k$ *, The expression for this force is given by the following relation:*

$$\vec{f}_k = \mu_k \vec{N}$$

where μ_k *is the coefficient of kinetic friction.*

The dimensionless coefficients μ_s *and* μ_k *depend on the nature of the surfaces in contact.*

d) The elastic force of a spring

The elastic force of a spring, also known as the spring force or restoring force, is the force generated by a spring when it is displaced from its equilibrium position (compressed or extended). The expression for the elastic force is given by the following relation:

$$\|\vec{F}\| = -k\,\Delta x$$

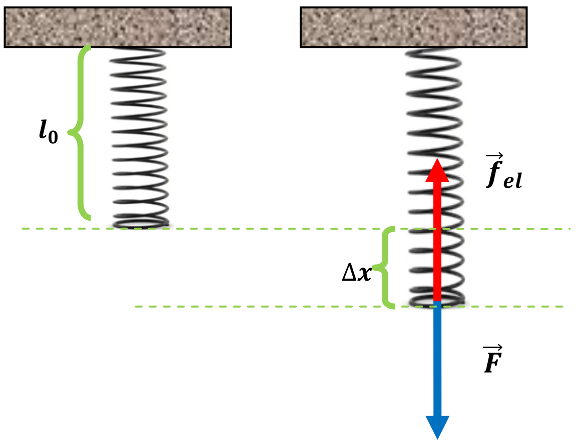
where k *is the spring constant, and* Δx *is the displacement from the equilibrium position.*

The negative sign indicates that the force opposes the direction of displacement.

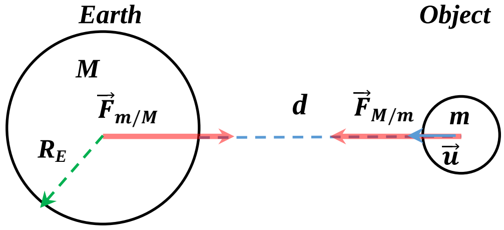
Direction and Orientation of the elastic force:

Direction: The elastic force $\vec{f}_{el}$ *is always in the opposite direction of the displacement, meaning it is a restoring force that works to bring the spring back to its equilibrium position.*

Orientation: If the spring is stretched (positive displacement), the elastic force is directed inward (toward the equilibrium position). If the spring is compressed (negative displacement), the force is directed outward (toward the equilibrium position)



2-3-2) Field forces (Forces acting at a distance): *these forces do not require contact between the two bodies, such as the force of gravity between two objects,* *Attraction and repulsion between two magnets, and the electric force between two electric charges).*



We list below the properties of some forces that act at a distance:

a) Weight ($\vec{w}$)

- *The weight of and object (The gravitational force) is a non-contact force exerted by the Earth on an object. Its magnitude is proportional to the mass of the object and the value of the gravitational constant* $\vec{g}$.
- *The weight force is a consequence of the universal law of gravitation between two bodies (the Earth with a mass* **M** *and the object with a mass* **m***), separated by a distance* **d**.

$$\vec{w} = \vec{F}_{M/m} = -\vec{F}_{\frac{m}{M}} = G\,\frac{M \times m}{(R_E+d)^2}\vec{u}$$

Where$G=6.67\times10^{-11} N\,m^2.\,Kg^{-2}$, $M = 5{,}98\times10^{24}$ *Kg, and* $R_E= 6.37\times10^6 m$

For $d=0 \rightarrow \vec{w} = G\,\frac{M\times m}{(R_E)^2}\vec{u} \rightarrow \vec{w} = m\,\vec{g}$ *where* $\vec{g} = G\,\frac{M}{(R_E)^2}\vec{u} = 9.8\,\vec{u}$

- *The weight force is denoted by a vector* $\vec{w}$ *and itexpression is given by:*

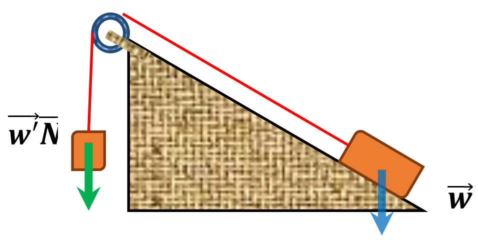
$$\vec{w} = m\,\vec{g}$$

- *The characteristics of the gravitational force vector are:*

Direction*: Vertically towards the center of the Earth.*

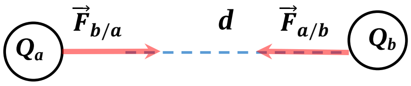
Support*: The line connecting the center of mass of the object and the center of the Earth.*

Magnitude*: Proportional to the mass of the object and the gravitational acceleration of magnitude 9.8m/s².* $\|\vec{w}\| = m\,\|\vec{g}\| = m\,\times 9.8$



b) Electrostatic forces (Coulomb force):

These forces arise between two charged bodies $\boldsymbol{Q_a}$ *and* $\boldsymbol{Q_b}$*separated by a distance* **d.**



$$\vec{F}_{a/b} = -\vec{F}_{\frac{b}{a}} = K\frac{Q_a Q_b}{\|\overrightarrow{AB}\|^3}\overrightarrow{AB}$$

c)The electromagnetic force:

This force arises due to the presence of a charged body **Q** *moving with a velocity* $\vec{v}$ *within an electric field* $\vec{E}$ *and a magnetic field* $\vec{B}$

$$\vec{F} = Q(\vec{E} + \vec{v}\vec{B})$$

3) Linear momentum

Linear momentum $\vec{P}$ *for a moving object is a vector quantity equal to the product of the mass of the object and its velocity.* $\vec{P} = m\,\vec{V}$

If the system consists of a collection of bodies, the total linear momentum of the system is the sum of the linear momentum of its individual bodies

$$\vec{P} = \sum \vec{P}_i = \sum m_i.\vec{V}_i$$

3-1) Conservation of Linear Momentum

The linear momentum of a system is conserved if the derivative of the linear momentum with respect to time is zero. The linear momentum of an isolated system is conserved if its value and direction remain constant throughout the entire time. the law of conservation of momentum, which can be written as:

$$\vec{P}(t_i) = \vec{P}\,(t_f) \Rightarrow \frac{d\vec{P}}{dt} = \vec{0}$$

where the subscripts refer to the total momentum of the system at initial time i *and final time* f

4) Newton's Laws

4-1) Newton's First Law (The Law of Inertia):

Newton's first law states that if the net force on an object is zero, it must stay at rest or move with constant velocity.

$$1^{st}\ \text{law} \quad \Rightarrow \quad \sum \overrightarrow{F_{ext}} = \vec{0} \begin{cases} \vec{v} = \vec{0} \\ \vec{v} = constant \end{cases}$$

4-2) Newton's Second Law

Newton's Second Law states that the change in the linear momentum of an object is directly proportional to the net external acting forces. The second law of Newton can be formulated mathematically as follows:

$$\sum \vec{F}_{ext} = \frac{d\vec{P}}{dt}$$

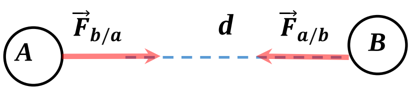
$$\sum \vec{F}_{ext} = \frac{d(m\,\vec{v})}{dt} = m\frac{d(\vec{v})}{dt} + \vec{v}\frac{d(m)}{dt}$$

$$\frac{d(m)}{dt} = 0\ ;\ \frac{d(\vec{v})}{dt} = \vec{a}$$

The acceleration of an object, $\vec{a}$, is related to its mass, m, and the net external acting forces on it $\sum \vec{F}_{ext}$. this second law can be expressed by the relation:

$$\sum \vec{F}_{ext} = m\,\vec{a}$$

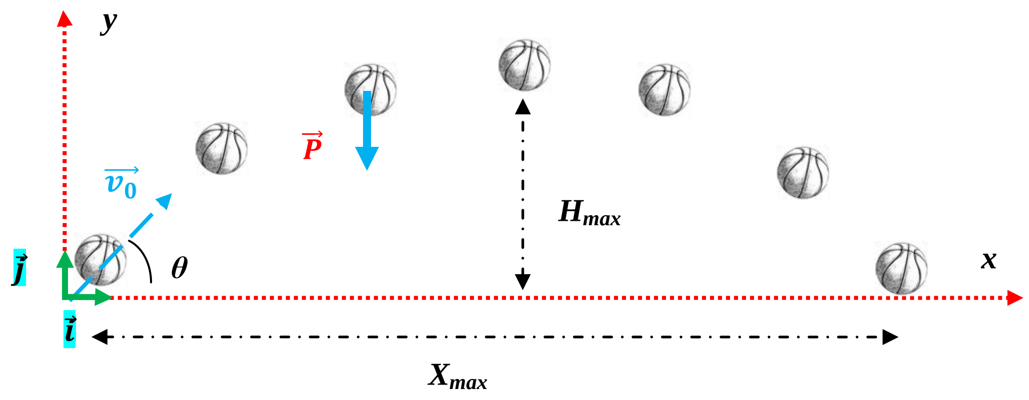
4-3) Newton's Third Law (Action and Reaction)



For every action (force) on an object, there is an equal and opposite reaction (force).The third law of Newton can be formulated mathematically as follows:

$$\vec{F}_{A/B} = -\vec{F}_{B/A}$$

4-4) Application : Projectile Motion



Study of the motion of a projectile subject only to the force of gravity as it is thrown upwards with an initial velocity and at an angle of α . The motion of this projectile is in the plane and is studied in a Cartesian axis (oxy)

Based on the basic principle of motion, we can write:

$$\sum \vec{F}_{ext} = m\,\vec{a} \Rightarrow \vec{P} = m\,\vec{a} \Rightarrow -m\ g\,\vec{j} = m\,\vec{a} \Rightarrow -g\,\vec{j} = \vec{a} = \overrightarrow{cnstant}$$

To find the equations of motion, we follow the steps:

$$\vec{a} = \frac{d\vec{v}}{dt} = -\ g\,\vec{j} \Rightarrow d\vec{v} = -\ g\ dt\ \vec{j} \Rightarrow \int_{\vec{v}_0}^{\vec{v}} d\vec{v} = \int_0^t -\ g\ dt\ \vec{j}$$

$$\vec{v} - \vec{v}_0 = -\ g\ t\ \vec{j} \Rightarrow \vec{v} = -\ g\ t\ \vec{j} + \vec{v}_0$$

$$\vec{v} = \frac{d\vec{r}}{dt} = -gt\,\vec{j} + \vec{v}_0 \Rightarrow d\vec{r} = (-gt\,\vec{j} + \vec{v}_0)dt \Rightarrow \int_{\vec{r}_0}^{\vec{r}} d\vec{r} = \int_0^t (-gt\,\vec{j} + \vec{v}_0)dt$$

$$\vec{r} - \vec{r}_0 = -\frac{1}{2}gt^2\,\vec{j} + \vec{v}_0\,t \Rightarrow \vec{r} = -\frac{1}{2}gt^2\,\vec{j} + \vec{v}_0\,t + \vec{r}_0$$

According to initial data, at the moment $\vec{r}_0 = \vec{0};\ \vec{v}_0 = v_0\cos\alpha\,\vec{i} + v_0\sin\alpha\,\vec{j}$

So the equations of motion for the basketball are as follows:

$$\begin{cases} \vec{r} = x\,\vec{i} + y\,\vec{j} = -\frac{1}{2}gt^2\,\vec{j} + (v_0\cos\alpha\,\vec{i} + v_0\sin\alpha\,\vec{j})\,t \\ \vec{v} = v_x\,\vec{i} + v_y\,\vec{j} = -\ g\ t\ \vec{j} + v_0\cos\alpha\,\vec{i} + v_0\sin\alpha\,\vec{j} \\ \vec{a} = a_x\,\vec{i} + a_y\,\vec{j} = -\ g\,\vec{j} \end{cases}$$

$$\begin{cases} \vec{r} = x\vec{\imath} + y\vec{\jmath} = (-\frac{1}{2}gt^2 + tv_0 \sin\alpha)\,\vec{\jmath} + t\ v_0 \cos\alpha\,\vec{\imath} \\ \vec{v} = v_x\vec{\imath} + v_y\vec{\jmath} = (-\ g\,t + v_0 \sin\alpha\,\vec{\jmath})\ \vec{\jmath} + v_0 \cos\alpha\,\vec{\imath} \\ \vec{a} = a_x\vec{\imath} + a_y\vec{\jmath} = -\ g\,\vec{\jmath} \end{cases}$$

Therefore, the projectile's motion is straight and uniform with respect to the (Ox) axis; and straight and uniformly variable with respect to the (Oy) axis. Its motion equations with respect to each axis are given as follows:

$$\Rightarrow along\ (Ox) \begin{cases} a_x = 0 \\ v_x = v_0 \cos\alpha \\ x = t\ v_0 \cos\alpha \end{cases} \quad and\ along\ (Oy) \begin{cases} a_y = -g \\ v_y = -gt + v_0 \cos\alpha \\ y = -\frac{1}{2}gt^2 + tv_0 \sin\alpha \end{cases}$$

At the maximum height the projectile reaches, the velocity v_y *becomes null*

$$-gt + v_0 \cos\alpha = 0 \Rightarrow t = \frac{v_0 \cos\alpha}{g}$$

The projectile reaches its maximum height at the moment $t = \frac{v_0 \cos\alpha}{g}$

$$H_{max} = -\frac{v^2{}_0}{2g}\cos^2\alpha + \frac{\cos\alpha}{g}v^2{}_0 \sin\alpha$$

The distance at which a projectile falls:

$$-\frac{1}{2}gt^2 + tv_0 \sin\alpha = 0 \Rightarrow \frac{1}{2}gt^2 = tv_0 \sin\alpha \Rightarrow \frac{1}{2}gt = v_0 \sin\alpha \Rightarrow t = \frac{2v_0}{g}\sin\alpha$$

The moment at which the projectile touches the ground's surface is $t = \frac{2v_0}{g}\sin\alpha$

And it touches the ground surface after covering a distance of

$$x = t\ v_0 \cos\alpha = \frac{2v^2{}_0}{g}\sin\alpha\cos\alpha$$

5) The torque

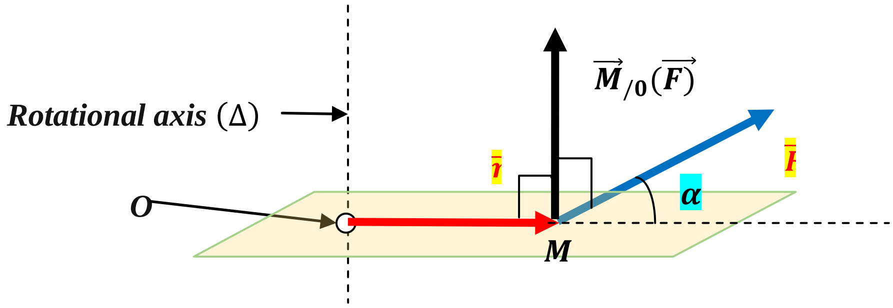
Torque is a vector quantity that represents the ability (tendency) of a force to induce rotational motion of an object around an axis (Δ)*passing through point O .*

Torque is denoted by $\overrightarrow{\mathcal{M}}_{/0}(\vec{F})$, *where* $\vec{F}$ *is the applied force* $\boldsymbol{O}$ *is the point through which the axis of rotation passes, and* $\vec{r} = \overrightarrow{OM}$ *is the vector extending from point* $\boldsymbol{O}$ *(the point on the axis of rotation) to the* $\boldsymbol{M}$ *point where the force is applied. The expression for torque is given by the cross product between* $\vec{r}$ *and* $\vec{F}$ *as follow :*

$$\overrightarrow{\mathcal{M}}_{/0}(\vec{F}) = \vec{r} \wedge \vec{F}$$

The direction of $\overrightarrow{\mathcal{M}}_{/0}(\vec{F})$ *is perpendicular to the plane formed by* $\vec{r}$ *and* $\vec{F}$ *and its sense is given by the right-hand rule or of advance of a right-handed screw rotating from* $\vec{r}$ *to* $\vec{F}$.

the magnitude of the torque : $\|\overrightarrow{\mathcal{M}}_{/0}(\vec{F})\| = \|\vec{r}\| \, \|\vec{F}\| \sin\alpha$ *, where* α *is the smaller angle between* $\vec{r}$ *and* $\vec{F}$.

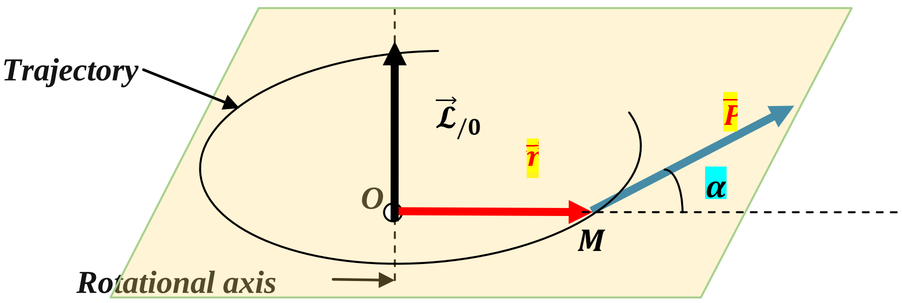


6) Angular momentum

The angular momentum of the point M in rotation around a fixed point O is vector quantity denoted $\vec{\mathcal{L}}_{/0}$ *, its expression is given by is given by the cross product between the vector* $\vec{r}$ *and the linear momentum* $\vec{P}$ *as follow :*

$$\vec{\mathcal{L}}_{/0} = \vec{r} \wedge \vec{P}$$

- ✓ *The angular momentum is a vector perpendicular to the plane containing the vectors* $\vec{r}$ *and* $\vec{P}$.
- ✓ *The magnitude of the angular momentum:* $\|\vec{\mathcal{L}}_{/0}\| = \|\vec{r}\| \, \|\vec{P}\| \sin\alpha$ *, where* α *is the smaller angle between* $\vec{r}$ *and* $\vec{P}$.



The angular momentum expression can be written in Cartesian coordinates as follows:

$$\begin{cases} \vec{r} = x\,\vec{\imath} + y\,\vec{\jmath} + z\,\vec{k} \\ \vec{P} = P_x\,\vec{\imath} + P_y\,\vec{\jmath} + P_z\,\vec{k} \end{cases}$$

$$\vec{\mathcal{L}}_{/0} = \vec{r} \wedge \vec{P} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ x & y & z \\ P_x & P_y & P_z \end{vmatrix} = \begin{vmatrix} y & z \\ P_y & P_z \end{vmatrix}\vec{\imath} - \begin{vmatrix} x & z \\ P_x & P_z \end{vmatrix}\vec{\jmath} + \begin{vmatrix} x & y \\ P_x & P_y \end{vmatrix}\vec{k}$$

$$\vec{\mathcal{L}}_{/0} = \underbrace{(yP_z - P_y z)}_{\mathcal{L}_x}\vec{\imath} - \underbrace{(xP_z - P_x z)}_{\mathcal{L}_y}\vec{\jmath} + \underbrace{(xP_y - P_x y)}_{\mathcal{L}_z}\vec{k}$$

If the system consists of a number of particles, the angular momentum of the system is the sum of the angular momentum of these particles $\vec{\mathcal{L}}_{/0} = \sum_i \overrightarrow{\mathcal{L}_i}_{/0}$

6-1) The angular momentum theorem

The angular momentum theorem states that the derivative of the total angular momentum of a system with respect to time is equal to the sum of the external torques applied to the system. The law of the angular momentum theorem can be derived as follows:

$$\vec{\mathcal{L}}_{/0} = \vec{r} \wedge \vec{P}$$

$$\frac{d\vec{\mathcal{L}}_{/0}}{dt} = \frac{d}{dt}(\vec{r} \wedge \vec{P}) = \frac{d\vec{r}}{dt} \wedge \vec{P} + \vec{r} \wedge \frac{d\vec{P}}{dt} = \vec{v} \wedge \boldsymbol{m}\vec{v} + \vec{r} \wedge \frac{d\vec{P}}{dt}$$

$$\vec{v} \wedge \boldsymbol{m}\vec{v} = \vec{0}$$

$$\frac{d\vec{P}}{dt} = \vec{F}$$

$$\frac{d\vec{\mathcal{L}}_{/0}}{dt} = \vec{r} \wedge \vec{F} = \overrightarrow{\boldsymbol{\mathcal{M}}}_{/0}(\vec{\boldsymbol{F}})$$

If the system consists of n particles, then the derivative of the angular momentum of the system is equal to the sum of the torques of all external forces acting on these particles.

$$\frac{d\vec{\mathcal{L}}_{/0}}{dt} = \sum_i^N \overrightarrow{\mathcal{M}_i}_{/0}(\vec{F})$$

6-2) Conservation of Angular Momentum

The angular momentum of a system is conserved if the derivative of the angular momentum with respect to time is zero. Therefore, we can write:

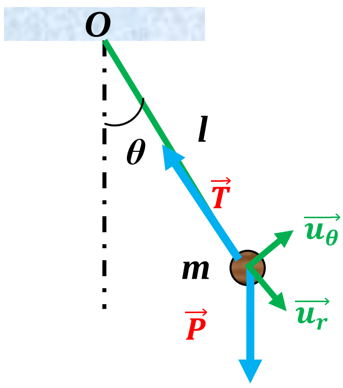
$$\vec{\mathcal{L}}_{/0} = \vec{r} \wedge \vec{P}$$

$$\frac{d\vec{\mathcal{L}}_{/0}}{dt} = \vec{0} \quad if \quad \overrightarrow{\mathcal{M}}_{/0}(\vec{F}) = \vec{0}$$

If the net external torque acting on a system is zero (i.e. an isolated system), the total angular momentum of the system remains constant in both magnitude and direction.

6-3) Application : simple pendulum

A simple pendulum consists of a small mass suspended by a thread of length L of massless and non-stretchable fixed at the other end. If the mass is pulled to the right or left from its equilibrium position and then released, the pendulum will swing in a vertical plane about an axis passing through O.



the forces applied to the mass m are the weight $\vec{P}$ *and the tension* $\vec{T}$*, their expressions in the polar coordinates are given as :*

$$\begin{cases} \vec{P} = P\cos\theta\,\vec{u_r} - P\sin\theta\,\vec{u_\theta} \\ \vec{T} = T\,\vec{u_r} \end{cases}$$

By Applying the angular momentum theorem:

$$\frac{d\vec{\mathcal{L}}_{/0}}{dt} = \sum_i^N \overrightarrow{\mathcal{M}_i}_{/0}(\vec{F}) = \overrightarrow{\mathcal{M}_i}_{/0}(\vec{P}) + \overrightarrow{\mathcal{M}_i}_{/0}(\vec{T})$$

the vector $\overrightarrow{OM} = l\,\overrightarrow{u_r}$

$$\begin{cases} \overrightarrow{\mathcal{M}_i}_{/0}(\vec{P}) = \overrightarrow{OM} \wedge \vec{P} \\ \overrightarrow{\mathcal{M}_i}_{/0}(\vec{T}) = \overrightarrow{OM} \wedge \vec{T} \end{cases}$$

$$\overrightarrow{\mathcal{M}_i}_{/0}(\vec{P}) = \overrightarrow{OM} \wedge \vec{P} = \begin{vmatrix} \overrightarrow{u_r} & \overrightarrow{u_\theta} & \vec{k} \\ l & 0 & 0 \\ P\cos\theta & -P\sin\theta & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 \\ -P\sin\theta & 0 \end{vmatrix}\overrightarrow{u_r} - \begin{vmatrix} l & 0 \\ P\cos\theta & 0 \end{vmatrix}\overrightarrow{u_\theta} + \begin{vmatrix} l & 0 \\ P\cos\theta & -P\sin\theta \end{vmatrix}\vec{k} = -l\,P\sin\theta\,\vec{k}$$

$$\overrightarrow{\mathcal{M}_i}_{/0}(\vec{T}) = \overrightarrow{OM} \wedge \vec{T} = \begin{vmatrix} \overrightarrow{u_r} & \overrightarrow{u_\theta} & \vec{k} \\ l & 0 & 0 \\ T & 0 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix}\overrightarrow{u_r} - \begin{vmatrix} l & 0 \\ T & 0 \end{vmatrix}\overrightarrow{u_\theta} + \begin{vmatrix} l & 0 \\ T & 0 \end{vmatrix}\vec{k} = 0\,\vec{k}$$

$$\vec{\mathcal{L}}_{/0} = \overrightarrow{OM} \wedge \vec{P} \;=\; \overrightarrow{OM} \wedge \boldsymbol{m}\,\vec{v}$$

$$\vec{v} = \frac{d(\overrightarrow{OM})}{dt} = l\frac{d\overrightarrow{u_r}}{dt} = l\frac{d\theta}{dt}\,\overrightarrow{u_\theta} \;\Rightarrow\; \vec{P} \;=\; \boldsymbol{m}\,\vec{v} = m\,l\frac{d\theta}{dt}\,\overrightarrow{u_\theta}$$

$$\vec{\mathcal{L}}_{/0} = \overrightarrow{OM} \wedge \vec{P} \;= \begin{vmatrix} \overrightarrow{u_r} & \overrightarrow{u_\theta} & \vec{k} \\ l & 0 & 0 \\ 0 & m\,l\frac{d\theta}{dt} & 0 \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ m\,l\frac{d\theta}{dt} & 0 \end{vmatrix}\overrightarrow{u_r} - \begin{vmatrix} l & 0 \\ 0 & 0 \end{vmatrix}\overrightarrow{u_\theta} + \begin{vmatrix} l & 0 \\ 0 & m\,l\frac{d\theta}{dt} \end{vmatrix}\vec{k}$$

$$\vec{\mathcal{L}}_{/0} = m\,l^2\frac{d\theta}{dt}\,\vec{k}$$

$$\frac{d\vec{\mathcal{L}}_{/0}}{dt} = \frac{d(m\,l^2\frac{d\theta}{dt}\,\vec{k})}{dt} = \;m\,l^2\frac{d^2\theta}{d^2t}\,\vec{k} = \;m\,l^2\ddot{\theta}\,\vec{k}$$

$$m\;\,l^2\ddot{\theta}\,\vec{k} = -l\,P\sin\theta\,\vec{k} \Rightarrow\; ml^2\ddot{\theta} \;= -l\,P\sin\theta$$

For small values of angle $\theta \Rightarrow \sin\theta \approx \theta$

$$\ddot{\theta} + l\frac{mg}{ml^2}\theta = 0 \quad\Rightarrow \ddot{\theta} + \frac{g}{l}\theta = 0$$