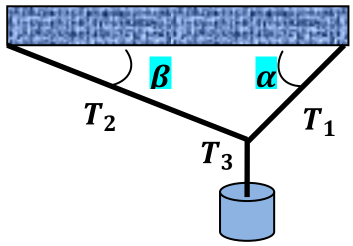
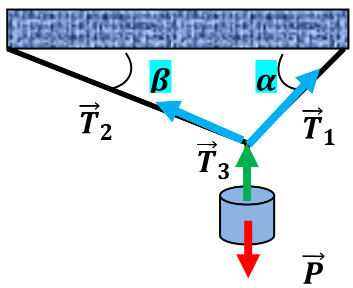
EXERCISE 01

A body of mass m= 3 Kg suspends by three cords, as illustrated in the figure:



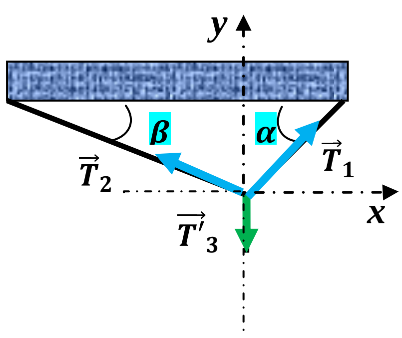
If the angles $\boldsymbol{\alpha}$ *and* $\boldsymbol{\beta}$ *formed by cords 1 and 2 with the horizontal plane are* $\frac{\pi}{6}$ *and* $\frac{\pi}{3}$, *respectively, find the tension force of the three cords.*

Solution



$$\sum \vec{F}_{ext} = \vec{0} \Rightarrow \vec{P} + \vec{T}_3 = \vec{0} \Rightarrow T_3 = m\,g = 3 \times 9.8 = 29.4\ N$$

at the knot



$$\sum \vec{F}_{ext} = \vec{0} \Rightarrow \vec{T}_1 + \vec{T}_2 + \vec{T'}_3 = \vec{0} \quad \text{where } \vec{T'}_3 = -\vec{T}_3$$

$$\begin{cases} \vec{T}_1 = T_1 \cos\alpha \ \vec{\imath} + T_1 \sin\alpha \ \vec{\jmath} \\ \vec{T}_2 = -T_2 \cos\beta \ \vec{\imath} + T_2 \sin\beta \ \vec{\jmath} \\ \overrightarrow{T'}_3 = -T_3 \ \vec{\jmath} \end{cases}$$

$$\vec{T}_1 + \vec{T}_2 + \overrightarrow{T'}_3 = \vec{0} \Rightarrow T_1 \cos\alpha \ \vec{\imath} + T_1 \sin\alpha \ \vec{\jmath} - T_2 \cos\beta \ \vec{\imath} + T_2 \sin\beta \ \vec{\jmath} - T_3 \ \vec{\jmath} = \vec{0}$$

$$(T_1 \cos\alpha - T_2 \cos\beta)\ \vec{\imath} + (T_1 \sin\alpha + T_2 \sin\beta - T_3)\ \vec{\jmath} = \vec{0}$$

$$\begin{cases} T_1 \cos\alpha - T_2 \cos\beta = 0 \\ T_1 \sin\alpha + T_2 \sin\beta - T_3 = 0 \end{cases} \Rightarrow \begin{cases} \frac{T_1\sqrt{3}}{2} - \frac{T_2}{2} = 0 \\ \frac{T_1}{2} + \frac{T_2\sqrt{3}}{2} - T_3 = 0 \end{cases}$$

$$\begin{cases} T_1\sqrt{3} = T_2 \\ \frac{T_1}{2} + \frac{T_1\sqrt{3}\sqrt{3}}{2} = T_3 \end{cases} \Rightarrow \begin{cases} T_1\sqrt{3} = T_2 \\ \frac{T_1}{2} + \frac{3T_1}{2} = T_3 \end{cases} \Rightarrow \begin{cases} T_2 = T_1\sqrt{3} = \frac{T_3\sqrt{3}}{2} = 25.46N \\ T_1 = \frac{T_3}{2} = 14.7\ N \end{cases}$$

EXERCISE 02

A billet with a mass of 5 g is launched from rest and reached a velocity of 200 m/s at the exit of handgun barrel of 8 cm length .

1- Assuming that the acceleration of the bullet is constant inside the barrel, calculate the acceleration of the bullet.

2- Calculate the force applied to the bullet that causes this acceleration.

Solution

1- Since the bullet's motion is straight and uniformly varied with constant acceleration, we can write the motion equations of this bullet:

$$\begin{cases} x = \frac{1}{2}\, a\, t^2 + v_0\, t + x_0 \\ v = at + v_0 \end{cases} \quad \textit{where } v_0 = 0\,\frac{m}{s} \textit{ and } x_0 = 0$$

$$v = at \Rightarrow t = \frac{v}{a} \quad \textit{and } x = \frac{1}{2}\, a\, t^2 = \; x = \frac{1}{2}\, a\, (\frac{v}{a})^2$$

$$x = \frac{v^2}{2a} \;\Rightarrow a = \frac{v^2}{2x}$$

x is the length of the gun = 8cm =0.08m

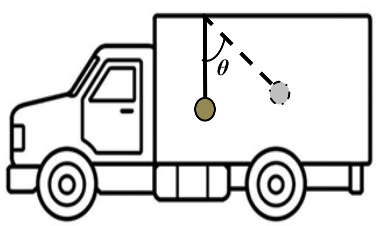
$$a = \frac{(200)^2}{2 \times 0.08} = 2500000 \text{ m/s}^2$$

2- The applied force

$$F = m \times a = 0.005 \times 2500000 = 12500 \text{ N}$$

EXERCISE 03

A small ball of mass 10 g is suspended by a massless thread from the roof of a truck at rest.



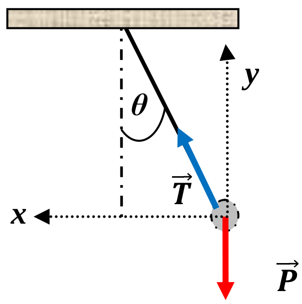
1- Calculate the angle θ made by the thread when the truck is moving with an acceleration of 3 m/s²

, Determine the value of the thread tension.

2- Calculate the angle θ made by the thread when the truck is moving with a constant speed of 50 m/s , Determine the value of the thread tension.

Solution

1- Calculating the angle θ made by the thread when the truck is moving with an acceleration of 3 m/s^2, and determining the thread tension value



$$\sum \vec{F}_{ext} = m\,\vec{a} \Rightarrow \vec{P}+\vec{T} = m\,\vec{a} \text{ where } \begin{cases} \vec{P} = -mg\,\vec{j} \\ \vec{T} = T\sin\theta\,\vec{i} + T\cos\theta\,\vec{j} \\ \vec{a} = a\,\vec{i} \end{cases}$$

$$-mg\,\vec{j} + T\sin\theta\,\vec{i} + T\cos\theta\,\vec{j} = m\,a\,\vec{i} \Rightarrow T\sin\theta\,\vec{i} + (T\cos\theta - mg)\,\vec{j} = m\,a\,\vec{i}$$

$$\Rightarrow \begin{cases} T\sin\theta = ma \\ T\cos\theta - mg = 0 \end{cases} \Rightarrow \begin{cases} T\sin\theta = ma \\ T\cos\theta = mg \end{cases} \Rightarrow \frac{T\sin\theta}{T\cos\theta} = \frac{ma}{m\,g} \Rightarrow \tan\theta = \frac{a}{g} = \frac{3}{9.8}$$

$$\theta = 17.0^0$$

$$T\cos\theta = mg \Rightarrow T = \frac{mg}{\cos\theta} = \frac{0.01\times 9.8}{0.95} = 0.103\,N$$

2- Calculating the angle θ made by the thread when the truck is moving with a constant speed of 50 m/s , and determining the thread tension value

in this case the acceleration is zero ($\vec{a} = \vec{0}$)since the truck is moving at a constant velocity

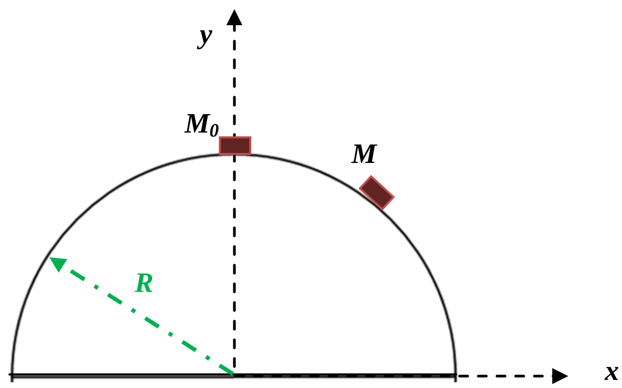
$$\sum \vec{F}_{ext} = \vec{0} \Rightarrow \vec{P}+\vec{T} = \vec{0} \text{ where } \begin{cases} \vec{P} = -mg\,\vec{j} \\ \vec{T} = T\sin\theta\,\vec{i} + T\cos\theta\,\vec{j} \\ \vec{a} = 0\,\vec{i} \end{cases}$$

$$-mg\,\vec{j} + T\sin\theta\,\vec{i} + T\cos\theta\,\vec{j} = 0\,\vec{i} \Rightarrow T\sin\theta\,\vec{i} + (T\cos\theta - mg)\,\vec{j} = 0\,\vec{i}$$

$$\Rightarrow \begin{cases} T\ \sin\theta = 0 \\ T\ \cos\theta\ - mg = 0 \end{cases} \Rightarrow \begin{cases} \sin\theta = 0 \\ T\ \cos\theta = mg \end{cases} \Rightarrow \begin{cases} \theta = 0 \\ T = mg = 0.098\,N \end{cases}$$

EXERCISE 04

*A block of 20 g slides without friction from the highest point of a Surface of a hemisphere of radius **R** (as shown in the figure), starting at position M_0 at time t = 0 s.*

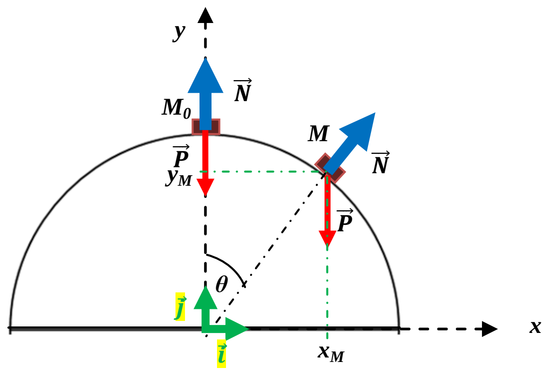


1. *represent the forces acting on the body at positions M_0 and M.*
2. *Using Cartesian coordinates (xy), write the position vector, instantaneous velocity, and acceleration of the body, then calculate their magnitudes.*
3. *Using Cartesian coordinates (xy), write the analytical expression for each force at the point M.*
4. *Write the velocity as a function of g R and cos θ, then deduce the formula for the normal force.*
5. *Calculate the angle at which the body stops touching the surface of the sphere.*
6. *Re-answer all the previous questions using the polar coordinates (r, θ).*

Solution

1- Representing the forces acting on the body at positions M_0 and M.

(the forces acting on the block are: the gravitational force (the weigh force) $\vec{\boldsymbol{P}}$ (vertical towards the centre of the earth), and the normal force $\vec{\boldsymbol{N}}$ (perpendicular to the surface)



2- Writing the position, instantaneous velocity, and instantaneous acceleration vectors of the body using Cartesian coordinates (xy).

- Position vector

$$\overrightarrow{OM} = \vec{r} = x_M\,\vec{\imath} + y_M\vec{\jmath} \Rightarrow \begin{cases} x_M = R\sin\theta \\ y_M = R\cos\theta \end{cases} \Rightarrow \overrightarrow{OM} = \vec{r} = R\sin\theta\ \vec{\imath} + R\cos\theta\,\vec{\jmath}$$

$$\left\|\overrightarrow{OM}\right\| = r = \sqrt{(R\sin\theta)^2 + (R\cos\theta)^2} = \sqrt{(R)^2\,(\,cos^2\theta +\ sin^2\theta)} = R$$

- Velocity vector

$$\vec{v} = v_x\,\vec{\imath} + v_y\vec{\jmath} = \frac{d\vec{r}}{dt} = \frac{dx_M}{dt}\,\vec{\imath} + \frac{dy_M}{dt}\vec{\jmath}$$

$$\Rightarrow \begin{cases} v_x = \dfrac{dx_M}{dt} = \dot{x}_M = \dfrac{d\theta}{dt}R\cos\theta = \dot{\theta}R\cos\theta \\ v_y = \dfrac{dx_M}{dt} = \dot{y}_M = -\dfrac{d\theta}{dt}R\sin\theta = -\ \dot{\theta}R\sin\theta \end{cases}$$

$$\vec{v} = \frac{d\theta}{dt}R\cos\theta\ \vec{\imath} - \frac{d\theta}{dt}R\sin\theta\,\vec{\jmath} = R\dot{\theta}\cos\theta\ \vec{\imath} - R\dot{\theta}\ \sin\theta\,\vec{\jmath}$$

$$\|\vec{v}\| = v = \sqrt{\left(R\dot{\theta}\cos\theta\right)^2 + \left(-R\dot{\theta}\ \sin\theta\right)^2} = \sqrt{(R\dot{\theta})^2\,(\,cos^2\theta +\ sin^2\theta)} = R\dot{\theta}$$

- Acceleration vector

$$\vec{a} = a_x\,\vec{\imath} + a_y\vec{\jmath} = \frac{d\vec{v}}{dt} = \frac{dv_x}{dt}\,\vec{\imath} + \frac{dv_y}{dt}\vec{\jmath} \Rightarrow \begin{cases} a_x = \dfrac{dv_x}{dt} = \dot{v}_x = \dfrac{d(\dot{\theta}R\cos\theta)}{dt} \\ a_y = \dfrac{dv_y}{dt} = \dot{v}_y = \dfrac{d(-\dot{\theta}R\sin\theta)}{dt} \end{cases}$$

$$\begin{cases} a_x = \dfrac{d(\dot{\theta}R\cos\theta)}{dt} = R\cos\theta\,\dfrac{d(\dot{\theta})}{dt} + \dot{\theta}\,\dfrac{d(R\cos\theta)}{dt} = R\cos\theta\,\ddot{\theta} - R\dot{\theta}^2\sin\theta \\ a_y = \dfrac{d(-\dot{\theta}R\sin\theta)}{dt} = -R\sin\theta\,\dfrac{d(\dot{\theta})}{dt} - \dot{\theta}\,\dfrac{d(R\sin\theta)}{dt} = -R\sin\theta\,\ddot{\theta} - R\dot{\theta}^2\cos\theta \end{cases}$$

$$\vec{a} = \left(R\ddot{\theta}\cos\theta - R\dot{\theta}^2\sin\theta\right)\vec{\imath} + (-R\ddot{\theta}\sin\theta - R\dot{\theta}^2\cos\theta)\vec{\jmath}$$

$$\|\vec{a}\| = a = \sqrt{\left(R\ddot{\theta}\cos\theta - R\dot{\theta}^2\sin\theta\right)^2 + \left(-R\ddot{\theta}\sin\theta - R\dot{\theta}^2\cos\theta\right)^2}$$

$$a = \sqrt{\left(R\ddot{\theta}\right)^2 + \left(R\dot{\theta}^2\right)^2}$$

3- Writing the analytical expression for each force at the point M using Cartesian coordinates (xy).

$$\begin{cases} \vec{P} = -mg\,\vec{\jmath} \\ \vec{N} = N\sin\theta\;\vec{\imath} + N\cos\theta\,\vec{\jmath} \end{cases}$$

4- Writing the velocity as a function of g R and cosθ

$$\sum \vec{F}_{ext} = m\,\vec{a} \Rightarrow \vec{P} + \vec{N} = m\,\vec{a}$$

$$N\sin\theta\;\vec{\imath} + N\cos\theta\,\vec{\jmath} - mg\,\vec{\jmath} = m\left(R\ddot{\theta}\cos\theta - R\dot{\theta}^2\sin\theta\right)\vec{\imath} + m(-R\ddot{\theta}\sin\theta - R\dot{\theta}^2\cos\theta)\vec{\jmath}$$

$$\begin{cases} N\sin\theta = m\left(R\ddot{\theta}\cos\theta - R\dot{\theta}^2\sin\theta\right) \\ N\cos\theta = m(-R\ddot{\theta}\sin\theta - R\dot{\theta}^2\cos\theta) - mg \end{cases}$$

By multiplying both sides of the 1st equations by $\cos\theta$ *, and both sides of the 2nd equation by* $\sin\theta$ *, we get:*

$$\begin{cases} \cos\theta\, N \sin\theta = \cos\theta\, mR\ddot{\theta}\cos\theta - \cos\theta\; mR\dot{\theta}^2 \sin\theta \\ \sin\theta\; N\cos\theta = \sin\theta\; mg - \sin\theta\; mR\ddot{\theta}\sin\theta - \sin\theta\; mR\dot{\theta}^2\cos\theta \end{cases}$$

$$\cos\theta\, mR\ddot{\theta}\cos\theta - \cos\theta\; mR\dot{\theta}^2\sin\theta = \sin\theta\; mg - \sin\theta\; mR\ddot{\theta}\sin\theta - \sin\theta\; mR\dot{\theta}^2\cos\theta$$

$$\sin\theta\; mR\ddot{\theta}\sin\theta + \cos\theta\, mR\ddot{\theta}\cos\theta = \sin\theta\; mg - \sin\theta\; mR\dot{\theta}^2\cos\theta + \cos\theta\; mR\dot{\theta}^2\sin\theta$$

$$mR\ddot{\theta}(\sin^2\theta + \cos^2\theta) = \sin\theta\; mg + mR\dot{\theta}^2(-\sin\theta\cos\theta + \cos\theta\;\sin\theta)$$

$$\sin^2\theta + \cos^2\theta = 1$$

$$R\ddot{\theta} = \sin\theta\; g \quad \Rightarrow R\frac{d\dot{\theta}}{dt} = \sin\theta\; g$$

$$Note\ that\ \frac{d\theta}{dt} = \dot{\theta} = \frac{v}{R} \Rightarrow \frac{R}{R}\frac{dv}{dt} = \sin\theta\; g \Rightarrow \frac{dv}{dt} = \sin\theta\; g$$

By multiplying both sides of last equation by $d\theta$

$$d\theta\,\frac{dv}{dt} = g\sin\theta\; d\theta \Rightarrow dv\,\frac{d\theta}{dt} = g\sin\theta\; d\theta \Rightarrow v\,dv = Rg\sin\theta\; d\theta$$

$$\int_0^v v\,dv = \int_0^\theta Rg\sin\theta\; d\theta \Rightarrow \frac{1}{2}\,v^2 = [-Rg\cos\theta]_0^\theta = Rg\,(1-\cos\theta)$$

$$\boldsymbol{v = \sqrt{2Rg\,(1-\cos\theta)}}$$

** Deducing the formula for the normal force.*

$$N\sin\theta = mR\ddot{\theta}\cos\theta - mR\dot{\theta}^2\sin\theta$$

$$note\ that\quad R\ddot{\theta} = \sin\theta\; g \Rightarrow N\sin\theta = mg\sin\theta\;\cos\theta - mR\dot{\theta}^2\sin\theta$$

$$N = mg\;\cos\theta - mR\dot{\theta}^2 \quad Note\ that\ \frac{d\theta}{dt} = \dot{\theta} = \frac{v}{R} \Rightarrow N = mg\;\cos\theta - mR(\frac{v}{R})^2$$

$$N = mg\ \cos\theta - \frac{m}{R}v^2 \quad note\ that\ \boldsymbol{v} = \sqrt{\boldsymbol{2Rg\ (1-\cos\theta)}}$$

$$N = mg\ \cos\theta - \frac{m}{R}\boldsymbol{2Rg\ (1-\cos\theta)} \Rightarrow N = mg\ \cos\theta - 2m\ \boldsymbol{g\ (1-\cos\theta)}$$

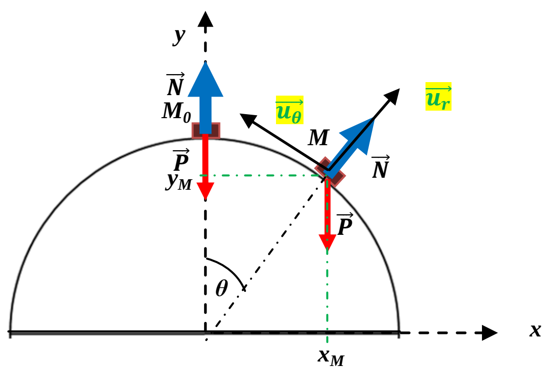
$$\boldsymbol{N = mg\ (\cos\theta - 2 + 2\cos\theta) = m\ g\ (-2 + 3\cos\theta\ )}$$

5- Calculate the angle at which the body stops touching the surface of the sphere.

$$N = 0 = m\ g\ (-2 + 3\ \cos\theta\ ) \Rightarrow -2 + 3\ \cos\theta = 0 \Rightarrow 3\ \cos\theta = 2 \Rightarrow \cos\theta = \frac{2}{3} \Rightarrow \theta = 48^0$$

6- Re-answer all the previous questions using ***the polar coordinates (r, θ).***

a- Writing the position, instantaneous velocity, and instantaneous acceleration vectors of the body, using Cartesian coordinates (r, θ).



- *Position vector*

$$\overrightarrow{OM} = R\,\vec{u}_r$$

- *Velocity vector*

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dR}{dt}\vec{u}_r + \frac{d\vec{u}_r}{dt}R \Rightarrow \begin{cases} \dfrac{dR}{dt} = 0 \\ \dfrac{d\vec{u}_r}{dt} = \dfrac{d\theta}{dt}\vec{u}_\theta \end{cases} \Rightarrow \vec{v} = \frac{d\vec{r}}{dt} = R\frac{d\theta}{dt}\vec{u}_\theta = R\dot{\theta}\vec{u}_\theta$$

- Acceleration vector

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d(R\frac{d\theta}{dt}\vec{u}_\theta)}{dt} = \frac{d\theta}{dt}\vec{u}_\theta \frac{d(R)}{dt} + R\frac{d\theta}{dt}\frac{d(\vec{u}_\theta)}{dt} + R\vec{u}_\theta \frac{d(\frac{d\theta}{dt})}{dt}$$

$$\Rightarrow \begin{cases} \dfrac{dR}{dt} = 0 \\ \dfrac{d\vec{u}_\theta}{dt} = -\dfrac{d\theta}{dt}\vec{u}_r = -\dot{\theta}\,\dot{\vec{u}}_r \\ d\left(\dfrac{d\theta}{dt}\right) = \dfrac{d^2\theta}{d^2t} = \ddot{\theta} \end{cases}$$

$$\vec{a} = -R\dot{\theta}\,\dot{\theta}\,\vec{u}_r + R\vec{u}_\theta\ddot{\theta} = -R\dot{\theta}^2\,\vec{u}_r + R\ddot{\theta}\,\vec{u}_\theta$$

b- Writing the analytical expression for each force at the point M using Cartesian coordinates (r, θ).

$$\begin{cases} \vec{N} = N\,\vec{u}_r \\ \vec{p} = -P\cos\theta\ \vec{u}_r - P\sin\theta\,\vec{u}_\theta \end{cases}$$

c- Writing the velocity as a function of g R and cosθ

$$\sum \vec{F}_{ext} = m\,\overrightarrow{a} \Rightarrow \vec{P} + \vec{N} = m\,\overrightarrow{a}$$

$$N\,\vec{u}_r - P\cos\theta\ \vec{u}_r - P\sin\theta\,\vec{u}_\theta = -m\,R\dot{\theta}^2\,\vec{u}_r + mR\ddot{\theta}\,\vec{u}_\theta$$

$$(N - P\cos\theta)\,\vec{u}_r - P\sin\theta\,\vec{u}_\theta = -m\,R\dot{\theta}^2\,\vec{u}_r + mR\ddot{\theta}\,\vec{u}_\theta$$

$$\Rightarrow \begin{cases} N - P\cos\theta = -m\,R\dot{\theta}^2 \\ -P\sin\theta = mR\ddot{\theta} \end{cases}$$

$$N = P\cos\theta - m\,R\dot{\theta}^2$$

$$-P\sin\theta = mR\ddot{\theta} \Rightarrow R\ddot{\theta} = \sin\theta\ g \quad \Rightarrow R\frac{d\dot{\theta}}{dt} = \sin\theta\ g$$

$$Note\ that\ \frac{d\theta}{dt} = \dot{\theta} = \frac{v}{R} \Rightarrow \frac{R}{R}\frac{dv}{dt} = \sin\theta\ g \Rightarrow \frac{dv}{dt} = \sin\theta\ g$$

By multiplying both sides of last equation by $d\theta$

$$d\theta\ \frac{dv}{dt} = g\sin\theta\ d\theta \Rightarrow dv\ \frac{d\theta}{dt} = g\sin\theta\ d\theta \Rightarrow v\ dv = Rg\sin\theta\ d\theta$$

$$\int_0^v v\ dv = \int_0^\theta Rg\sin\theta\ d\theta \Rightarrow \frac{1}{2}\ v^2 = [-Rg\cos\theta]_0^\theta = Rg\ (1-\cos\theta)$$

$$v = \sqrt{2Rg\ (1-\cos\theta)}$$

** Deducing the formula for the normal force*

$$N - P\cos\theta = -m\,R\dot{\theta}^2 \Rightarrow N = P\cos\theta - m\,R\dot{\theta}^2$$

$$Note\ that\ \frac{d\theta}{dt} = \dot{\theta} = \frac{v}{R} \Rightarrow N = mg\ \cos\theta - mR(\frac{v}{R})^2$$

$$N = mg\ \cos\theta - \frac{m}{R}v^2 \quad note\ that\ \boldsymbol{v} = \sqrt{\boldsymbol{2Rg\ (1-\cos\theta)}}$$

$$N = mg\ \cos\theta - \frac{m}{R}\boldsymbol{2Rg\ (1-\cos\theta)} \Rightarrow N = mg\ \cos\theta - 2m\ \boldsymbol{g\ (1-\cos\theta)}$$

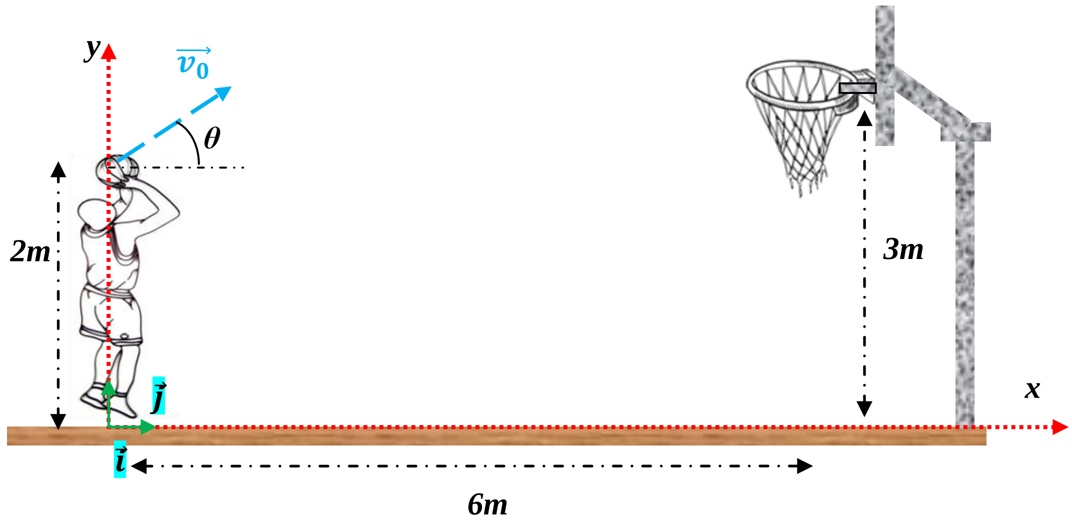
$$N = mg\ (\cos\theta - 2 + 2\cos\theta) = m\ g\ (-2 + 3\cos\theta)$$

5- Calculate the angle at which the body stops touching the surface of the sphere.

$$N = 0 = m\ g\ (-2 + 3\cos\theta) \Rightarrow -2 + 3\cos\theta = 0 \Rightarrow 3\cos\theta = 2 \Rightarrow \cos\theta = \frac{2}{3} \Rightarrow \theta = 48^0$$

EXERCISE 05

A basketball player, standing 2 meters tall, throws a ball at an angle of 30^0, as shown in the figure. The basket is 3 meters above the ground and 6 meters away from the player.



1. Study the motion of this ball and write the time-dependent equations that describe its motion.

2. Find the velocity required for the ball to reach the basket? And how long does it take to reach the basket?

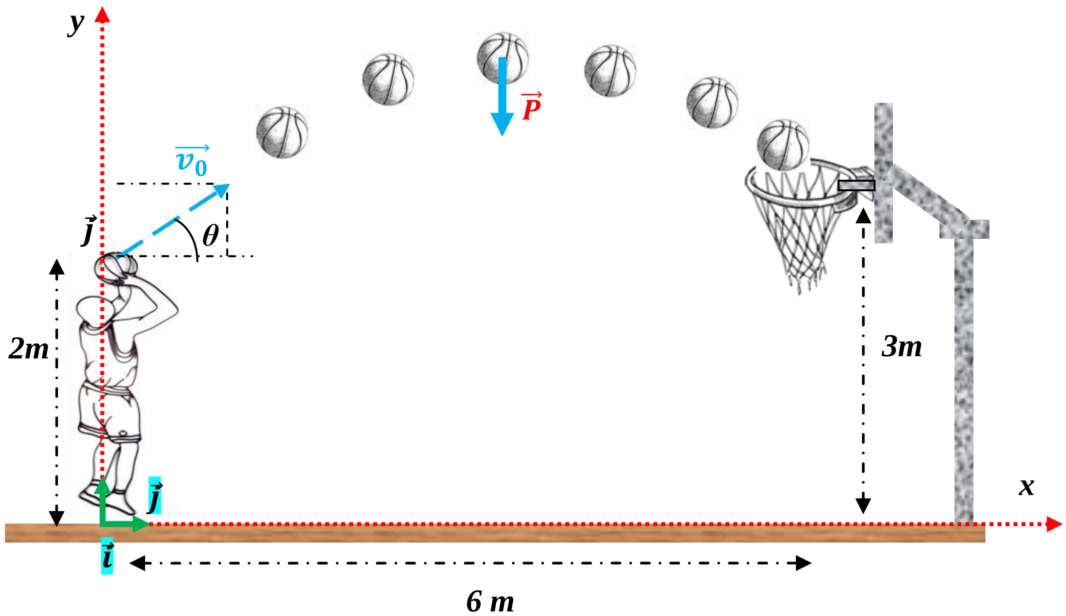
3. Write the equation of a basketball's trajectory, then find the maximum height the ball can reach.

4. If the player throws the ball at initial velocity = 10 m/s at an angle of 45^0, find the distance at which he must be positioned for the ball to go into the basket.

Solution

1- *Writing the time-dependent equations that describe its motion.*

$$\sum \vec{F}_{ext} = m\,\vec{a} \Rightarrow \vec{P} = m\,\vec{a} \Rightarrow -m\ g\,\vec{j} = m\,\vec{a} = m\,a_x\,\vec{i} + m\,a_y\,\vec{j}$$



To find the equations of motion, we follow the steps:

$$\vec{a} = -\,g\,\vec{j}$$

$$\vec{a} = a_x\,\vec{i} + a_y\,\vec{j} = \frac{d\vec{v}}{dt} = \frac{dv_x}{dt}\,\vec{i} + \frac{dv_y}{dt}\vec{j} = -\,g\,\vec{j} \Rightarrow \begin{cases} \dfrac{dv_x}{dt} = 0 \\ \dfrac{dv_y}{dt} = -g \end{cases}$$

$$\Rightarrow \begin{cases} dv_x = 0\,dt \\ dv_y = -g\,dt \end{cases} \Rightarrow \begin{cases} \displaystyle\int_{v_{x0}}^{v_x} dv_x = \int_0^t 0\,dt \\ \displaystyle\int_{v_{y0}}^{v_y} dv_y = \int_0^t -g\,dt \end{cases} \Rightarrow \begin{cases} v_x - v_{x_0} = 0 \\ v_y - v_{y_0} = -gt \end{cases}$$

$$\begin{cases} v_x = v_{x_0} = v_0\cos\theta = \dfrac{v_0\sqrt{3}}{2} \\ v_y = -gt + v_{y_0} = -\,g\,t + v_0\sin\theta = -gt + \dfrac{v_0}{2} \end{cases}$$

$$\vec{v} = \frac{v_0\sqrt{3}}{2}\,\vec{i} + \left(-gt + \frac{v_0}{2}\right)\vec{j}$$

$$\vec{v} = v_x\,\vec{i} + v_y\,\vec{j} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\,\vec{i} + \frac{dy}{dt}\vec{j} = \frac{v_0\sqrt{3}}{2}\,\vec{i} + \left(-gt + \frac{v_0}{2}\right)\vec{j}$$

$$\Rightarrow \begin{cases} \dfrac{dx}{dt} = \dfrac{v_0\sqrt{3}}{2} \\ \dfrac{dy}{dt} = -gt + \dfrac{v_0}{2} \end{cases} \Rightarrow \begin{cases} dx = \dfrac{v_0\sqrt{3}}{2}\ dt \\ dy = \left(-gt + \dfrac{v_0}{2}\right) dt \end{cases} \Rightarrow \begin{cases} \displaystyle\int_{x_0}^{x} dx = \int_0^t \frac{v_0\sqrt{3}}{2}\ dt \\ \displaystyle\int_{y_0}^{y} dy = \int_0^t \left(-gt + \frac{v_0}{2}\right) dt \end{cases}$$

$$\Rightarrow \begin{cases} x - x_0 = \dfrac{v_0\sqrt{3}}{2} t \\ y - y_0 = -\dfrac{1}{2}gt^2 + \dfrac{v_0}{2} t \end{cases} \Rightarrow \begin{cases} x = \dfrac{v_0\sqrt{3}}{2} t \ + x_0 \\ y = -\dfrac{1}{2}gt^2 + \dfrac{v_0}{2} t + y_0 \end{cases}$$

According to the given data, at the moment t = 0s $x_0 = 0\ m$ *and* $y_0 = 2\ m$

$$\begin{cases} x = \dfrac{v_0\sqrt{3}}{2} t \\ y = -\dfrac{1}{2}gt^2 + \dfrac{v_0}{2} t + 2 \end{cases}$$

So the equations of motion for the basketball are as follows:

$$\begin{cases} \vec{r} = x\,\vec{i} + y\,\vec{j} = \dfrac{v_0\sqrt{3}}{2} t\,\vec{i} + \left(-\dfrac{1}{2}gt^2 + \dfrac{v_0}{2} t + 2\right)\vec{j} \\ \vec{v} = v_x\,\vec{i} + v_y\,\vec{j} = \dfrac{v_0\sqrt{3}}{2}\,\vec{i} + \left(-gt + \dfrac{v_0}{2}\right)\vec{j} \\ \vec{a} = a_x\,\vec{i} + a_y\,\vec{j} = -\ g\,\vec{j} \end{cases}$$

2- Finding the velocity required for the ball to reach the basket

To reach the basket which has a height of 3 meters, the ball must cover a distance of 6 meters.

$$6 = \frac{v_0\sqrt{3}}{2} t \ \Rightarrow t = \frac{12}{v_0\sqrt{3}}$$

$$3 = -\frac{1}{2}g\left(\frac{12}{v_0\sqrt{3}}\right)^2 + \frac{v_0}{2}\frac{12}{v_0\sqrt{3}} + 2 \ \Rightarrow g\frac{72}{v^2{}_0} = 6\sqrt{3} - 3$$

$$v^2{}_0 = 9.8\frac{72}{6\sqrt{3} - 3} = 9.7\ \mathrm{m/s}$$

* *Finding the time taken by the basketball to reach the basket*

To reach the basket which has a height of 3 meters, the ball must cover a distance of 6 meters.

$$t = \frac{12}{v_0\sqrt{3}} = \frac{12}{9.7\sqrt{3}} = 0.7s$$

3- Writing the equation of a basketball's trajectory,

$$t = \frac{2x}{v_0\sqrt{3}}$$

$$y = -\frac{1}{2}g\left(\frac{2x}{v_0\sqrt{3}}\right)^2 + \frac{v_0}{2}\frac{2x}{v_0\sqrt{3}} + 2 \quad \Rightarrow y = -\frac{2g}{3\,v^2{}_0}\,x^2 + \frac{x}{\sqrt{3}} + 2$$

$$y = -\frac{2g}{3\,v^2{}_0}\,x^2 + \frac{x}{\sqrt{3}} + 2$$

**Finding the maximum height the ball can reach.*

$$v_y = 0 = -gt + \frac{v_0}{2} \Rightarrow gt = \frac{v_0}{2} \Rightarrow t = \frac{v_0}{2g}$$

$$y = -\frac{1}{2}g\left(\frac{v_0}{2g}\right)^2 + \frac{v_0}{2}\left(\frac{v_0}{2g}\right) + 2 \Rightarrow y = -\frac{v^2{}_0}{8g} + \frac{v^2{}_0}{4g} + 2 \Rightarrow 3.2\,m$$

4- Finding the distance at which he must be positioned for the ball to go into the basket, in case when the player If the player throws the ball at initial velocity $v_0 = 6\,m/s$

$$\vec{v} = v_x\,\vec{\imath} + v_y\,\vec{\jmath} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\,\vec{\imath} + \frac{dy}{dt}\vec{\jmath} \Rightarrow \begin{cases} v_x = \dfrac{dx}{dt} = v_0\cos\theta \\ v_y = \dfrac{dy}{dt} = -gt + v_{y_0} = -g\,t + v_0\sin\theta \end{cases}$$

$$\begin{cases} \dfrac{dx}{dt} = v_0\cos\theta \\ \dfrac{dy}{dt} = -g\,t + v_0\sin\theta \end{cases} \Rightarrow \begin{cases} dx = v_0\cos\theta\,dt \\ dy = (-g\,t + v_0\sin\theta)dt \end{cases}$$

$$\Rightarrow \begin{cases} \displaystyle\int_{x_0}^{x} dx = \int_0^t v_0\cos\theta\,dt \\ \displaystyle\int_{y_0}^{y} dy = \int_0^t (-g\,t + v_0\sin\theta)dt \end{cases}$$

$$\Rightarrow \begin{cases} x - x_0 = v_0\cos\theta\,t \\ y - y_0 = -\dfrac{1}{2}gt^2 + v_0\sin\theta\,t \end{cases} \Rightarrow \begin{cases} x = v_0\cos\theta\,t + x_0 \\ y = -\dfrac{1}{2}gt^2 + v_0\sin\theta\,t + y_0 \end{cases}$$

According to the given data, at the moment $t = 0s$ $x_0 = 0\ m$ and $y_0 = 2\ m$

$$\begin{cases} x = v_0 \cos\theta\, t \\ y = -\frac{1}{2}gt^2 + v_0 \sin\theta\, t + 2 \end{cases}$$

To reach the basket, the ball must arrive at the coordinates of the basket (x, 3). Therefore we can write:

$$\begin{cases} x = 10\, t \cos(\frac{\pi}{4}) \\ 3 = -\frac{1}{2}gt^2 + 8\, t\ \sin(\frac{\pi}{4}) + 2 \end{cases} \Rightarrow \begin{cases} x = 10\, t\frac{\sqrt{2}}{2} = 5\sqrt{2}\ t \\ 1 = -\frac{1}{2}gt^2 + 5\sqrt{2}\, t \end{cases}$$

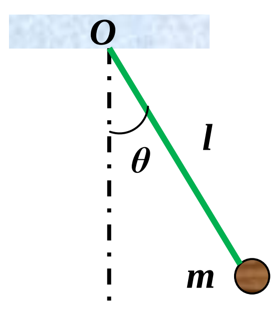
$$-\frac{1}{2}gt^2 + 5\sqrt{2}\, t - 1 = 0$$

$$\Delta = 30.36 \Rightarrow \sqrt{\Delta} = 5.51\ , t = 1.28s$$

$$x = 5\sqrt{2}\ t = 9.07\ m$$

EXERCISE 06

A small ball of mass 10 g is suspended by a massless thread from the roof as shown in the following figure:



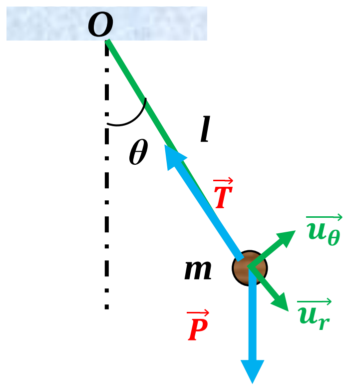
1- Using both methods, demonstrate that the differential equation for the motion of a simple pendulum is as follows:

$$\ddot{\theta} + \frac{g}{l}\theta = 0$$

a- The first method involves using polar coordinates and Newton's second law.
b- The second method employs Cartesian coordinates and the angular momentum theorem .

Solution

a- The first method involves using polar coordinates and Newton's second law.



$$\overrightarrow{OM} = l\,\overrightarrow{u_r} \quad \Rightarrow \vec{v} = \frac{d\left(\overrightarrow{OM}\right)}{dt} = l\frac{d\overrightarrow{u_r}}{dt} = l\frac{d\theta}{dt}\,\overrightarrow{u_\theta} \quad \Rightarrow \vec{a} = \frac{d(\vec{v})}{dt} = l\frac{d^2\theta}{dt^2}\,\overrightarrow{u_\theta} + l\frac{d\theta}{dt}\frac{d\overrightarrow{u_\theta}}{dt}$$

$$\frac{d\overrightarrow{u_\theta}}{dt} = -\frac{d\theta}{dt}\,\overrightarrow{u_r}$$

$$\vec{a} = l\ddot{\theta}\,\overrightarrow{u_\theta} - l\dot{\theta}^2\overrightarrow{u_r}$$

$$\sum \vec{F}_{ext} = m\,\vec{a} \Rightarrow \quad \vec{P} + \vec{T} = m\,\vec{a} \;\; where \begin{cases} \vec{P} = P\cos\theta\,\overrightarrow{u_r} - P\sin\theta\,\overrightarrow{u_\theta} \\ \vec{T} = T\,\overrightarrow{u_r} \end{cases}$$

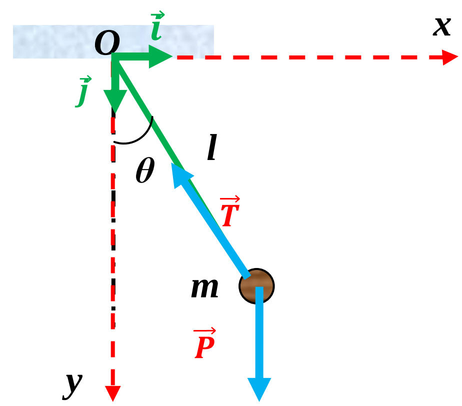
$$P\cos\theta\,\overrightarrow{u_r} - P\sin\theta\,\overrightarrow{u_\theta} + T\,\overrightarrow{u_r} = m\,l\ddot{\theta}\,\overrightarrow{u_\theta} - ml\dot{\theta}^2\overrightarrow{u_r}$$

$$\begin{cases} m\,l\ddot{\theta} = -P\sin\theta \\ -ml\dot{\theta}^2 = T + P\cos\theta \end{cases}$$

or small values of angle $\theta \Rightarrow \sin\theta \approx \theta$

$$m\,l\ddot{\theta} + P\sin\theta = 0 \quad \Rightarrow \boxed{\ddot{\theta} + \frac{g}{l}\theta = 0}$$

b- The second method employs Cartesian coordinates and the angular momentum theorem



By Applying the angular momentum theorem:

$$\frac{d\vec{\mathcal{L}}_{/0}}{dt} = \sum_i^N \overrightarrow{\boldsymbol{\mathcal{M}}_{\mathrm{i}}}_{/0}(\vec{F}) = \overrightarrow{\boldsymbol{\mathcal{M}}_{\mathrm{i}}}_{/0}(\vec{P}) + \overrightarrow{\boldsymbol{\mathcal{M}}_{\mathrm{i}}}_{/0}(\vec{T})$$

the vector $\overrightarrow{OM} = x\,\vec{\imath} + y\,\vec{\jmath} = l\sin\theta\,\vec{\imath} + l\,\cos\theta\,\vec{\jmath}$

$$\vec{v} = \frac{d\overrightarrow{OM}}{dt} = \frac{d\theta}{dt}\,l\cos\theta\,\vec{\imath} - \frac{d\theta}{dt}\,l\,\sin\theta\,\vec{\jmath}$$

the forces applied to the mass m are the weight $\vec{P}$ *and the tension* $\vec{T}$, *their expressions in the polar coordinates are given as :*

$$\begin{cases} \vec{P} = mg\,\vec{\jmath} \\ \vec{T} = -T\,\sin\theta\,\vec{\imath} - T\,\cos\theta\,\vec{\jmath} \end{cases}$$

By Applying the angular momentum theorem:

$$\frac{d\vec{\mathcal{L}}_{/0}}{dt} = \sum_i^N \overrightarrow{\boldsymbol{\mathcal{M}}_{\mathrm{i}}}_{/0}(\vec{F}) = \overrightarrow{\boldsymbol{\mathcal{M}}_{\mathrm{i}}}_{/0}(\vec{P}) + \overrightarrow{\boldsymbol{\mathcal{M}}_{\mathrm{i}}}_{/0}(\vec{T})$$

$$\begin{cases} \overrightarrow{\boldsymbol{\mathcal{M}}_{\mathrm{i}}}_{/0}(\vec{P}) = \overrightarrow{OM} \wedge \vec{P} \\ \overrightarrow{\boldsymbol{\mathcal{M}}_{\mathrm{i}}}_{/0}(\vec{T}) = \overrightarrow{OM} \wedge \vec{T} \end{cases}$$

$$\overrightarrow{\boldsymbol{\mathcal{M}}_{i}}_{/0}(\vec{P}) = \overrightarrow{OM} \wedge \vec{P} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ l\sin\theta & l\cos\theta & 0 \\ 0 & mg & 0 \end{vmatrix}$$

$$= \begin{vmatrix} l\cos\theta & 0 \\ mg & 0 \end{vmatrix}\vec{\imath} - \begin{vmatrix} l\sin\theta & 0 \\ 0 & 0 \end{vmatrix}\vec{\jmath} + \begin{vmatrix} l\sin\theta & l\cos\theta \\ 0 & mg \end{vmatrix}\vec{k} = l\,mg\,\sin\theta\,\vec{k}$$

$$\overrightarrow{\boldsymbol{\mathcal{M}}_{i}}_{/0}(\vec{T}) = \overrightarrow{OM} \wedge \vec{T} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ l\sin\theta & l\cos\theta & 0 \\ -T\sin\theta & -T\cos\theta & 0 \end{vmatrix}$$

$$= \begin{vmatrix} l\cos\theta & 0 \\ -T\cos\theta & 0 \end{vmatrix}\vec{\imath} - \begin{vmatrix} l\sin\theta & 0 \\ -T\sin\theta & 0 \end{vmatrix}\vec{\jmath} + \begin{vmatrix} l\sin\theta & l\cos\theta \\ -T\sin\theta & -T\cos\theta \end{vmatrix}\vec{k} = 0\ \vec{k}$$

$$\vec{\mathcal{L}}_{/0} = \overrightarrow{OM} \wedge \vec{P} = \overrightarrow{OM} \wedge \boldsymbol{m}\,\vec{v}$$

$$\vec{\mathcal{L}}_{/0} = \overrightarrow{OM} \wedge \vec{P} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ l\sin\theta & l\cos\theta & 0 \\ \frac{d\theta}{dt}\,ml\cos\theta & -\frac{d\theta}{dt}\,m\,l\,\sin\theta & 0 \end{vmatrix}$$

$$= \begin{vmatrix} l\cos\theta & 0 \\ -\frac{d\theta}{dt}\,m\,l\,\sin\theta & 0 \end{vmatrix}\vec{\imath} - \begin{vmatrix} l\sin\theta & 0 \\ \frac{d\theta}{dt}\,ml\cos\theta & 0 \end{vmatrix}\vec{\jmath} + \begin{vmatrix} l\sin\theta & l\cos\theta \\ \frac{d\theta}{dt}\,ml\cos\theta & -\frac{d\theta}{dt}\,m\,l\,\sin\theta \end{vmatrix}\vec{k}$$

$$\vec{\mathcal{L}}_{/0} = \left(-\frac{d\theta}{dt}\,m\,l^2\,sin^2\,\theta + \frac{d\theta}{dt}\,ml^2\cos^2\theta\right)\vec{k} = -\frac{d\theta}{dt}\,m\,l^2\ \vec{k}$$

$$\frac{d\vec{\mathcal{L}}_{/0}}{dt} = \frac{d(-\frac{d\theta}{dt}\,m\,l^2\ \vec{k})}{dt} = -\,m\,l^2\,\frac{d^2\theta}{d^2t}\,\vec{k} = -m\,l^2\ddot{\theta}\,\vec{k}$$

$$-m\ l^2\ddot{\theta}\,\vec{k} = l\,mg\,sin\,\theta\,\vec{k} \Rightarrow -ml^2\ddot{\theta} = l\,mg\,sin\,\theta$$

For small values of angle $\theta \Rightarrow sin\,\theta \approx \theta$

$$\ddot{\theta} + l\frac{mg}{ml^2}\theta = 0 \quad \Rightarrow \ddot{\theta} + \frac{g}{l}\theta = 0$$

EXERCISE 07

The position vector of a body weighing 2 kg is given as follows.

$$\overrightarrow{OM} = (t^2 + 4t + 3)\vec{\imath} + (2t^2)\vec{\jmath} + (t+3)\vec{k}$$

1- Find the mathematical expressions for:

- *The velocity*
- *The acceleration*
- *The linear momentum*
- *The applied force*
- *The torque of this force*
- *The angular momentum*

2- Calculate the derivative of the linear momentum with respect to time, and state your conclusions.

3- Calculate the derivative of the angular momentum with respect to time, and state your conclusions from these calculations.

Solution

♣ *The velocity*

$$\vec{v} = \frac{d\overrightarrow{OM}}{dt} = \frac{d((t^2+4t+3)\vec{\imath} + (2t^2)\vec{\jmath} + (t+3)\overrightarrow{k\,})}{dt} = (2t+4)\vec{\imath} + 4t\,\vec{\jmath} + \overrightarrow{k}$$

♣ *The acceleration*

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d((2t+4)\vec{\imath} + 4t\,\vec{\jmath} + \overrightarrow{k})}{dt} = 2\,\vec{\imath} + 4\,\vec{\jmath}$$

♣ *The linear momentum*

$$\vec{P} = m\,\vec{v} = 2\left((2t+4)\vec{\imath} + 4t\,\vec{\jmath} + \overrightarrow{k}\right) = (4t+8)\vec{\imath} + 8t\,\vec{\jmath} + 2\,\overrightarrow{k}$$

♣ *The applied force*

$$\vec{F} = m\,\vec{a} = 2(2\,\vec{\imath} + 4\,\vec{\jmath}) = 4\,\vec{\imath} + 8\,\vec{\jmath}$$

♣ *The torque of this force*

$$\overrightarrow{\boldsymbol{\mathcal{M}}_{\boldsymbol{i}}}_{/0}(\vec{F}) = \overrightarrow{OM} \wedge \vec{F} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ t^2+4t+3 & 2t^2 & t+3 \\ 4 & 8 & 0 \end{vmatrix}$$

$$\overrightarrow{\boldsymbol{\mathcal{M}}_{\boldsymbol{i}}}_{/0}(\vec{F}) = \begin{vmatrix} 2t^2 & t+3 \\ 8 & 0 \end{vmatrix} \vec{\imath} - \begin{vmatrix} t^2+4t+3 & t+3 \\ 4 & 0 \end{vmatrix} \vec{\jmath} + \begin{vmatrix} t^2+4t+3 & 2t^2 \\ 4 & 8 \end{vmatrix} \vec{k}$$

$$= (-8t-24)\,\vec{\imath} + (-4t-12)\vec{\jmath} + (32t+24)\,\vec{k}$$

♣ *The angular momentum*

2. $$\vec{\mathcal{L}}_{/0} = \overrightarrow{OM} \wedge \vec{P} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ t^2+4t+3 & 2t^2 & t+3 \\ 4t+8 & 8t & 2 \end{vmatrix}$$

$$= \begin{vmatrix} 2t^2 & t+3 \\ 8t & 2 \end{vmatrix} \vec{\imath} - \begin{vmatrix} t^2+4t+3 & t+3 \\ 4t+8 & 2 \end{vmatrix} \vec{\jmath} + \begin{vmatrix} t^2+4t+3 & 2t^2 \\ 4t+8 & 8t \end{vmatrix} \vec{k}$$

$$= (-4t^2-24\,t)\,\vec{\imath} + (-2t^2-12t-18)\vec{\jmath} + (16t^2+24t)\,\vec{k}$$

2- Calculate the derivative of the linear momentum with respect to time, and state your conclusions.

$$\frac{d\vec{P}}{dt} = \frac{d((4t+8)\vec{\imath} + 8t\,\vec{\jmath} + 2\,\vec{k})}{dt} = 4\,\vec{\imath} + 8\,\vec{\jmath}$$

we conclude that:

$$\frac{d\vec{P}}{dt} = m\,\vec{a} = \vec{F}$$

3- Calculate the derivative of the angular momentum with respect to time, and state your conclusions from these calculations.

$$\frac{d\vec{\mathcal{L}}_{/0}}{dt} = \frac{d((-4t^2-24\,t)\,\vec{\imath} + (-2t^2-12t-18)\vec{\jmath} + (16t^2+24t)\,\vec{k})}{dt}$$

$$= (-8t - 24)\,\vec{i} + (-4t - 12)\vec{j} + (32t + 24)\,\vec{k}$$

we conclude that:

$$\frac{d\vec{\mathcal{L}}_{/0}}{dt} = \overrightarrow{\boldsymbol{\mathcal{M}}_{i}}_{/0}(\vec{F})$$