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# Cours de Probabilité-Statistiques

# Chapitre 4: Introduction aux probabilités

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## Contents



## 1 Introduction

The objective of probability calculus is to provide a mathematical treatment of the intuitive notion of chance.
In this chapter we want to discuss the basic concepts of probability that covering the random experiments, the space of elementary events, probability axioms, and rules of probabilities.

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# 2 Definitions

## 2.1 Random Experiment-Sample Space-Events

**Definition 1** *A **random experiment** is an experiment that has the following two properties:*

**i)** *the outcome of the experiment cannot be predicted with certainty.*

**ii)** *the set of possible outcomes can be described BEFORE the experiment.*

**Definition 2** *The set of all possible outcomes of a random experiment is called the **sample space** of the experiment. The sample space is denoted as* $\Omega$.

**Example 1** *When we toss a coin, all the possible outcomes are Heads or Tails. Therefore, the sample space of a coin tossing is* $\Omega = \{heads, tails\}$ *. or as we usually write it,* $\Omega = \{H, T\}$ *.*

**Example 2** *When we toss a die, one of the 6 faces is going to come up. Therefore, the sample space of a die tossing is* $\Omega = \{1, 2, 3, 4, 5, 6\}$.

**Example 3** *We toss a coin 3 times and observe the sequence of heads/tails. The sample space here may be defined as*
$\Omega = \{(H, H, H), (H, H, T), (H, T, H), (T, H, H), (H, T, T), (T, H, T), (T, T, H), (T, T, T)\}$ .

**Example 4** *We toss a coin until the head comes up for the first time. The sample space here may be defined as* $\Omega = \{H, TH, TTH, TTTH, ...\}$ .

An event is a subset A of the sample space S, i.e., . If the outcome of an experiment is an element of A, we say that the event A has occurred. An event consisting of a single element of $\Omega$ is often called a **elementary event**.

**Definition 3** *An **event** A is a subset of the sample space, i.e.,*$A \subset \Omega$.
*An event consisting of a single element of* $\Omega$ *is often called a **elementary event**.*
*The empty subset* $\phi$ *is the **impossible event** and the sample space* $\Omega$ *is the **certain event**.*

**Example 5** *If we throw a 6 face dice which is not truqued.*
*we get* $\Omega = \{1, 2, 3, 4, 5, 6\}$, $|\Omega| =$*6.*
*A is the event where an even number is drawn* $\gg$ *then* $A = \{2; 4; 6\}$
*B is the event an odd number is drawn* $\gg$ *then* $B = \{1; 3; 5\}$
*C is the event* $\ll$ *a number* $\geq 4$ $\gg$ *then* $C = \{4; 5; 6\}$
*D is the elementary event* $\ll$ *the smallest number* $\gg$ *then* $D = \{4; 5; 6\}$.

# 3 Algebra of events

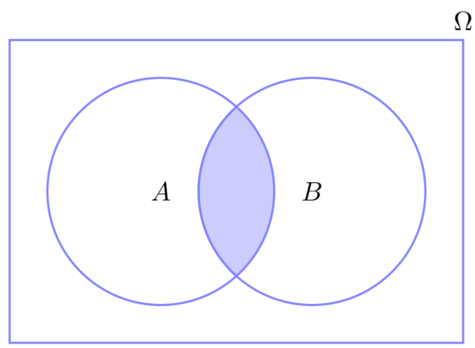
## 3.1 Operations on events

Let $\Omega$ be the sample space of a random experiment, and Let $A, B \in \Omega$.

**Definition 4** *$A \cap B$ is the event "both $A$ and $B$." $A \cap B$ is called the intersection of $A$ and $B$.*

$$A \cap B = \{x \in A \ et \ x \in B\}$$

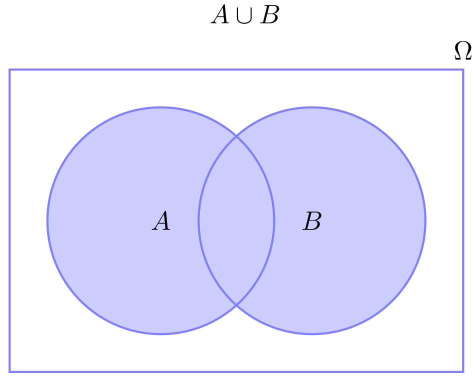
$A \cap B$




**Definition 5** *$A \cup B$ is the event "either $A$ or $B$ or both. $A \cup B$ is called the union of $A$ and $B$*

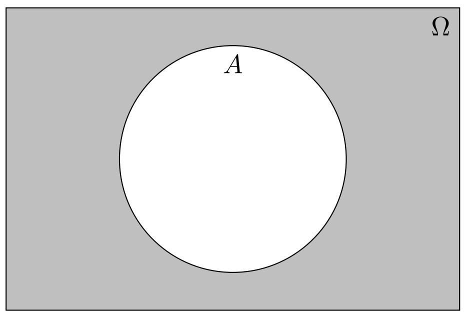
$$A \cup B = \{x \in A \ ou \ x \in B\}$$

$A \cup B$



**Definition 6** *$\bar{A}$ is the event "not A." A $\bar{A}$ is called the complement of A.*

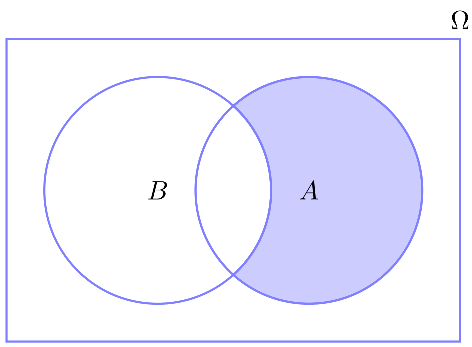
$$\bar{A} = C_{\Omega}^{A} = \{x \in \Omega \ et \ x \notin A\}$$




**Definition 7** *$A - B = A \cap \bar{B}$ is the event "A but not B."*
*In particular, $\bar{A} = \Omega - A$.*

$$A - B = \{x \in A \ et \ x \notin B\}$$

$A - B$




**Remark 1** *If the sets corresponding to events A and B are disjoint, i.e.,$A \cap B = \phi$, we often say that the events are* ***mutually exclusive****.*

**Example 6** *Referring to the experiment of tossing a coin twice, let A be the event "at least one head occurs" and B the event "the second toss results in a tail. Then $A = \{HH, HT, TH\}$ , $B = \{HT, TT\}$,*

$A \cup B = \{HH, HT, TH, TT\} = \Omega,$

$A \cap B = \{HT\}$,

$\bar{A} = \{TT\} =$, $A - B = \{TH, HH\}$

# 4 The Concept of Probability

**Definition 8** *Let $\Omega$ be the sample space of a random experiment.*

*The probability of an event $A$, let $P(A)$, is defined by the ratio:*

$$P(A) = \frac{|A|}{|\Omega|} = \frac{\textit{number of favourable cases}}{\textit{number of possible cases}}$$

**Remark 2** *1) This definition is only valid if all the draws have the same chance of succeeding. We will then say that the outcomes are equiprobable.*

*2)* $0 \leq P(A) \leq 1$.

**Example 7** *What is the probability of obtaining an even number by throwing a 6 face dice?*

***Answer****: Possible cases : 6 Favourable cases : 3 ,, then* $P(A) = \frac{3}{6} = 0.5$.

**Example 8** *A committee of 3 people is chosen from 4 men and 6 women.*

*What is the probability that the 3 people chosen are 2 men and 1 woman ?*

***Answer****: Possible cases:* $\begin{pmatrix} 10 \\ 3 \end{pmatrix} = \dfrac{10!}{7!3!} = 120$
*Favourable cases :*$\begin{pmatrix} 4 \\ 2 \end{pmatrix}\begin{pmatrix} 6 \\ 1 \end{pmatrix} = 36$, *then*

$$P(A) = \tfrac{36}{120} = 0.3.$$

## 4.1 Axioms of probability

**Definition 9** *Let $\Omega$ be a sample space. A probability $P$ is an application of the set of events $\Omega$ in the interval* $[0; 1]$.

A probability must satisfy the following three axioms:

**Axiom 1** $0 \leq P(A) \leq 1$ *for any event* $A$.

**Axiom 2** $P(\Omega) = 1$ *(The probability of the event certain* $\Omega$ *is equal to 1 = 100%)*

**Axiom 3** *If* $A \cap B = \phi$, *then* $P(A \cup B) = P(A) + P(B)$.

# 5 Some Important Theorems on Probability

**Theorem 1** $P(\bar{A}) = 1 - P(A)$.

**Proof 1** *Wehave:* $(A \cup \bar{A}) = \Omega$ *and* $(A \cap \bar{A}) = \phi$, *so :*

$$\begin{aligned} 1 &= P(\Omega) \qquad \textit{(by Axiom 2 )} \\ &= P\left(A \cup \bar{A}\right) \\ &= P(\bar{A}) + P(A) \qquad \textit{(by Axiom 3 )} \\ &\Longrightarrow P(\bar{A}) = 1 - P(A). \end{aligned}$$

**Example 9** *What is the probability of getting tails at least once by tossing 3 coins ?*

***Answer:***

$$\begin{aligned} P(0 \textit{ fois pile}) + P(1 \textit{ fois pile})) + P(2 \textit{ fois pile})) + P(3 \textit{ fois pile})) &= 1 \Leftrightarrow \\ P(1 \textit{ fois pile})) + P(2 \textit{ fois pile})) + P(3 \textit{ fois pile})) &= 1 - P(0 \textit{ fois pile}) \Leftrightarrow \\ P(\textit{ au moins une fois pile}) &= 1 - P(0 \textit{ fois pile}) \\ &= 1 - \frac{1}{8} \\ &= \frac{7}{8} = 0.875\% \end{aligned}$$

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**Theorem 2** $P\left(B \cap \bar{A}\right) = P(B) - P(A \cap B)$

**Proof 2** $\left(B\cap\bar{A}\right)\cup\left(B\cap A\right)=B$ *and* $\left(B\cap\bar{A}\right)\cap\left(B\cap A\right)=\phi$, *so*

$$\begin{aligned} P\left(B\right) &= P\left(\left(B\cap\bar{A}\right)\cup\left(B\cap A\right)\right)\\ &= P\left(B\cap\bar{A}\right)+P\left(B\cap A\right)\ \ (\textit{ axiom 3})\\ &\Rightarrow P\left(B\cap\bar{A}\right)=P\left(B\right)-P\left(B\cap A\right)\end{aligned}$$

**Remark 3** *If* $A\subseteq B$ *then* $B\cap A=A$ , *so* $P\left(B-A\right)=P\left(B\right)-P\left(A\right)$

**Theorem 3** *If* $A\subseteq B$ *then* $P\left(A\right)\leq P\left(B\right).$

**Proof 3** $\left(B\cap\bar{A}\right)\cup A=B$ *et* $\left(B\cap\bar{A}\right)\cap A=\phi$, *so*

$$\begin{aligned} P\left(B\right) &= P\left(\left(B\cap\bar{A}\right)\cup A\right)\\ &= P\left(B\cap\bar{A}\right)+P\left(A\right)>P\left(A\right)\ \ (\textit{ axiom 3})\\ \textit{car } P\left(B\cap\bar{A}\right) &\geq 0\ (\textit{ axiom 1})\end{aligned}$$

**Remark 4** $\phi\subseteq A\subseteq\Omega$ *alors* $0\leq P(A)\leq 1$

**Theorem 4** $P\left(A\cup B\right)=P\left(A\right)+P\left(B\right)-P\left(B\cap A\right).$

**Proof 4** *a) If* $P\left(B\cap A\right)=\phi$ , *so* $P\left(A\cup B\right)=P\left(A\right)+P\left(B\right)-\underbrace{P\left(\phi\right)}_{}=0$

*b) If* $P\left(B\cap A\right)\neq\phi$ , *we have*

$A\cup B=\left(B-A\right)\cup A$ *et* $\left(B-A\right)\cap A=\phi$, *so,*

$$\begin{aligned} P\left(A\cup B\right) &= P\left(\left(B-A\right)\cup A\right)\\ &= P\left(B-A\right)+P\left(A\right)\\ &= P\left(B\right)-P\left(A\cap B\right)+P\left(A\right)\\ &= P\left(A\right)+P\left(B\right)-P\left(B\cap A\right)\end{aligned}$$