

Solution du TD n° 2

Exercice 1 :

x_i	$x_0 = 0$	$x_1 = 1$	$x_2 = 3$
$f(x_i) = y_i$	$y_0 = 1$	$y_1 = 0.5$	$y_2 = 0.25$

1. Polynôme de Lagrange passant par les 3 points :

On a 3 point donc le degré du polynôme est $n \leq 2$.

$$P_2(x) = \sum_{K=0}^2 \frac{L_K(x)}{L_K(x_K)} y_K = \frac{L_0(x)}{L_0(x_0)} y_0 + \frac{L_1(x)}{L_1(x_1)} y_1 + \frac{L_2(x)}{L_2(x_2)} y_2$$

Pour $K = 0$:

$$\frac{L_0(x)}{L_0(x_0)} = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} = \frac{(x - 1)(x - 3)}{(0 - 1)(0 - 3)} = \frac{1}{3}(x - 1)(x - 3)$$

Pour $K = 1$:

$$\frac{L_1(x)}{L_1(x_1)} = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} = \frac{(x - 0)(x - 3)}{(1 - 0)(1 - 3)} = -\frac{1}{2}x(x - 3)$$

Pour $K = 2$:

$$\frac{L_2(x)}{L_2(x_2)} = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} = \frac{(x - 0)(x - 1)}{(3 - 0)(3 - 1)} = \frac{1}{6}x(x - 1)$$

Donc,

$$P_2(x) = \frac{1}{3}(x - 1)(x - 3)(1) - \frac{1}{2}x(x - 3)(0.5) + \frac{1}{6}x(x - 1)(0.25)$$

$$P_2(x) = 0.125x^2 - 0.625x + 1$$

2. Approximation de (1.5) :

En utilisant le polynôme de Lagrange $P_2(x)$, on trouve :

$$y(x) = P_2(1.5) = 0.125(1.5)^2 - 0.625(1.5) + 1$$

$$\Rightarrow y(1.5) \simeq 0.344$$

Exercice 2 :

Polynôme de Lagrange

a. $P(x)$

x_i	$x_0 = 0$	$x_1 = 1$	$x_2 = 2$
$P(x_i) = y_i$	$y_0 = -6$	$y_1 = 3$	$y_2 = 21$

On a 3 point donc le degré du polynôme est $n \leq 2$.

$$P_2(x) = \sum_{K=0}^2 \frac{L_K(x)}{L_K(x_K)} y_K = \frac{L_0(x)}{L_0(x_0)} y_0 + \frac{L_1(x)}{L_1(x_1)} y_1 + \frac{L_2(x)}{L_2(x_2)} y_2$$

Pour $K = 0$:

$$\frac{L_0(x)}{L_0(x_0)} = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} = \frac{(x - 1)(x - 2)}{(0 - 1)(0 - 2)} = \frac{1}{2}(x - 1)(x - 2)$$

Pour $K = 1$:

$$\frac{L_1(x)}{L_1(x_1)} = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} = \frac{(x - 0)(x - 2)}{(1 - 0)(1 - 2)} = -x(x - 2)$$

Pour $K = 2$:

$$\frac{L_2(x)}{L_2(x_2)} = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} = \frac{(x - 0)(x - 1)}{(2 - 0)(2 - 1)} = \frac{1}{2}x(x - 1)$$

Donc,

$$P_2(x) = \frac{1}{2}(x - 1)(x - 2)(-6) - x(x - 2)(3) + \frac{1}{2}x(x - 1)(21)$$

$$P_2(x) = \frac{9}{2}x^2 + \frac{9}{2}x - 6$$

b. $Q(x)$

x_i	$x_0 = 0$	$x_1 = 1$	$x_2 = 2$
$Q(x_i) = y_i$	$y_0 = 10$	$y_1 = 15$	$y_2 = 40$

On a 3 point donc le degré du polynôme est $n \leq 2$.

$$Q_2(x) = \sum_{K=0}^2 \frac{L_K(x)}{L_K(x_K)} y_K = \frac{L_0(x)}{L_0(x_0)} y_0 + \frac{L_1(x)}{L_1(x_1)} y_1 + \frac{L_2(x)}{L_2(x_2)} y_2$$

Pour $K = 0$:

$$\frac{L_0(x)}{L_0(x_0)} = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} = \frac{(x - 1)(x - 2)}{(0 - 1)(0 - 2)} = \frac{1}{2}(x - 1)(x - 2)$$

Pour $K = 1$:

$$\frac{L_1(x)}{L_1(x_1)} = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} = \frac{(x - 0)(x - 2)}{(1 - 0)(1 - 2)} = -x(x - 2)$$

Pour $K = 2$:

$$\frac{L_2(x)}{L_2(x_2)} = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} = \frac{(x - 0)(x - 1)}{(2 - 0)(2 - 1)} = \frac{1}{2}x(x - 1)$$

Donc,

$$Q_2(x) = \frac{1}{2}(x - 1)(x - 2)(10) - x(x - 2)(15) + \frac{1}{2}x(x - 1)(40)$$

$$Q_2(x) = 10x^2 - 5x + 10$$

Les points d'intersection

Aux points d'intersection $P_2(x) = Q_2(x)$

$$\frac{9}{2}x^2 + \frac{9}{2}x - 6 = 10x^2 - 5x + 10$$

$$\frac{11}{2}x^2 - \frac{19}{2}x + 16 = 0 \Rightarrow 11x^2 - 19x + 32 = 0$$

$\Delta = -1047 < 0$ donc on a aucun point d'intersection.

Exercice 3 :

Approximation de $f(4.5)$:

Puisque le polynôme est de degré 2, nous avons besoin uniquement de 3 points. On prend des points dont les abscisses sont proches de 4.5. Ces points sont présentés dans le tableau suivant :

x_i	$x_0 = 3$	$x_1 = 5$	$x_2 = 7$
$f(x_i) = y_i$	$y_0 = 1.2528$	$y_1 = 1.6094$	$y_2 = 1.9459$

On a 3 points donc le degré du polynôme est $n = 2$. Le polynôme de Newton qui passe par ces 3 points est :

$$P_2(x) = y_0 + \frac{\Delta y_0}{1! h^1}(x - x_0) + \frac{\Delta^2 y_0}{2! h^2}(x - x_0)(x - x_1)$$

$$\text{où : } h = x_1 - x_0 = x_2 - x_1 = 2$$

$$\left. \begin{array}{l} x_0 = 3 \longrightarrow y_0 = 1.2528 \\ x_1 = 5 \longrightarrow y_1 = 1.6094 \\ x_2 = 7 \longrightarrow y_2 = 1.9459 \end{array} \right\} \begin{array}{l} \Delta y_0 = y_1 - y_0 = 0.3566 \\ \Delta y_1 = y_2 - y_1 = 0.3365 \end{array} \left. \right\} \Delta^2 y_0 = \Delta y_1 - \Delta y_0 = -0.0201$$

Donc :

$$P_2(x) = 1.2528 + \frac{0.3566}{2}(x - 3) + \frac{-0.0201}{2! 2^2}(x - 3)(x - 5)$$

$$P_2(x) = 1.2528 + 0.1783(x - 3) - 0.0025(x - 3)(x - 5)$$

L'approximation de $f(4.5)$ est donc :

$$f(4.5) \simeq P_2(4.5) = 1.2528 + 0.1783(4.5 - 3) - 0.0025(4.5 - 3)(4.5 - 5)$$

$$P_2(4.5) = 1.5221.$$

Exercice 4 :

1. Calcul de l'approximation de $\sqrt{1.6}$

x_i	$x_0 = 1$	$x_1 = 2$	$x_2 = 3$
$f(x_i) = y_i$	$y_0 = 1$	$y_1 = 1.4142$	$y_2 = 1.7320$

Le polynôme de Newton qui passe par ces 3 points est :

$$P_2(x) = y_0 + \frac{\Delta y_0}{1! h^1} (x - x_0) + \frac{\Delta^2 y_0}{2! h^2} (x - x_0)(x - x_1)$$

où $h = x_1 - x_0 = x_2 - x_1 = 1$

$$\left. \begin{array}{l} x_0 = 1 \longrightarrow y_0 = 1 \\ x_1 = 2 \longrightarrow y_1 = 1.4142 \\ x_2 = 3 \longrightarrow y_2 = 1.7320 \end{array} \right\} \begin{array}{l} \Delta y_0 = y_1 - y_0 = 0.4142 \\ \Delta y_1 = y_2 - y_1 = 0.3178 \end{array} \left. \vphantom{\begin{array}{l} x_0 = 1 \\ x_1 = 2 \\ x_2 = 3 \end{array}} \right\} \Delta^2 y_0 = \Delta y_1 - \Delta y_0 = -0.0964$$

Donc :

$$P_2(x) = 1 + \frac{0.4142}{1! 1^1} (x - 1) + \frac{-0.0964}{2! 1^2} (x - 1)(x - 2)$$

$$P_2(x) = 1 + 0.4142(x - 1) - 0.0482(x - 1)(x - 2)$$

L'approximation de $\sqrt{1.6}$ est donc :

$$P_2(1.6) = 1 + 0.4142(1.6 - 1) - 0.0482(1.6 - 1)(1.6 - 2)$$

$$P_2(1.6) = 1.2601.$$

2. L'erreur commise :

L'erreur commise en approxinant $\sqrt{1.6}$ par $P_2(1.6)$ est :

$$E_{exacte} = |f(1.6) - P_2(1.6)| = |\sqrt{1.6} - 1.2601| = 0.0048.$$

Exercice 5 (supplémentaire) :

x_i	$x_0 = -1$	$x_1 = -0.5$	$x_2 = 0$	$x_3 = 0.5$	$x_4 = 1$
$f(x_i) = y_i$	$y_0 = 1$	$y_1 = 0.5$	$y_2 = 0$	$y_3 = 0.5$	$y_4 = 1$

1. Polynôme de Lagrange de la fonction $f(x)$ passant par les points ci-dessus :

On a 5 point donc le degré du polynôme est $n \leq 4$.

$$P_4(x) = \sum_{K=0}^4 \frac{L_K(x)}{L_K(x_K)} y_K = \frac{L_0(x)}{L_0(x_0)} y_0 + \frac{L_1(x)}{L_1(x_1)} y_1 + \frac{L_2(x)}{L_2(x_2)} y_2 + \frac{L_3(x)}{L_3(x_3)} y_3 + \frac{L_4(x)}{L_4(x_4)} y_4$$

Pour $K = 0$:

$$\begin{aligned} \frac{L_0(x)}{L_0(x_0)} &= \frac{(x - x_1)(x - x_2)(x - x_3)(x - x_4)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)(x_0 - x_4)} \\ &= \frac{(x + 0.5)(x - 0)(x - 0.5)(x - 1)}{(-1 + 0.5)(-1 - 0)(-1 - 0.5)(-1 - 1)} = \frac{2}{3}(x + 0.5)x(x - 0.5)(x - 1) \end{aligned}$$

Pour $K = 1$:

$$\begin{aligned} \frac{L_1(x)}{L_1(x_1)} &= \frac{(x - x_0)(x - x_2)(x - x_3)(x - x_4)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)(x_1 - x_4)} \\ &= \frac{(x + 1)(x - 0)(x - 0.5)(x - 1)}{(-0.5 + 1)(0.5 - 0)(-0.5 - 0.5)(-0.5 - 1)} \\ &= -\frac{8}{3}(x + 1)x(x - 0.5)(x - 1) \end{aligned}$$

Pour $K = 2$:

$$\begin{aligned} \frac{L_2(x)}{L_2(x_2)} &= \frac{(x - x_0)(x - x_1)(x - x_3)(x - x_4)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)(x_2 - x_4)} = \frac{(x + 1)(x + 0.5)(x - 0.5)(x - 1)}{(0 + 1)(0 + 0.5)(0 - 0.5)(0 - 1)} \\ &= 4(x + 1)(x + 0.5)(x - 0.5)(x - 1) \end{aligned}$$

Pour $K = 3$:

$$\begin{aligned} \frac{L_3(x)}{L_3(x_3)} &= \frac{(x - x_0)(x - x_1)(x - x_2)(x - x_4)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)(x_3 - x_4)} = \frac{(x + 1)(x + 0.5)(x - 0)(x - 1)}{(0.5 + 1)(0.5 + 0.5)(0.5 - 0)(0.5 - 1)} \\ &= -\frac{8}{3}(x + 1)(x + 0.5)x(x - 1) \end{aligned}$$

Pour $K = 4$:

$$\begin{aligned} \frac{L_4(x)}{L_4(x_4)} &= \frac{(x - x_0)(x - x_1)(x - x_2)(x - x_3)}{(x_4 - x_0)(x_4 - x_1)(x_4 - x_2)(x_4 - x_3)} = \frac{(x + 1)(x + 0.5)(x - 0)(x - 0.5)}{(1 + 1)(1 + 0.5)(1 - 0)(1 - 0.5)} \\ &= \frac{2}{3}(x + 1)(x + 0.5)x(x - 0.5) \end{aligned}$$

Donc :

$$P_4(x) = \frac{2}{3}(x + 0.5)x(x - 0.5)(x - 1)(1) - \frac{8}{3}(x + 1)x(x - 0.5)(x - 1)(0.5) + \\ 4(x + 1)(x + 0.5)(x - 0.5)(x - 1)(0) - \frac{8}{3}(x + 1)(x + 0.5)x(x - 1)(0.5) + \frac{2}{3}(x + 1)(x + \\ 0.5)x(x - 0.5)(1)$$

$$P_4(x) = -\frac{4}{3}x^4 + \frac{7}{3}x^2.$$