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Department of Civil Engineering-Department of Electrical Engineering
Module: Probability-Statistics

Chapter 5: Conditional probability

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Contents

1 Introduction

Sometimes, partial information is available about a random event. The aim of conditional probabilities is to take this partial information into account when calculating the probability.

2 Conditional Probability

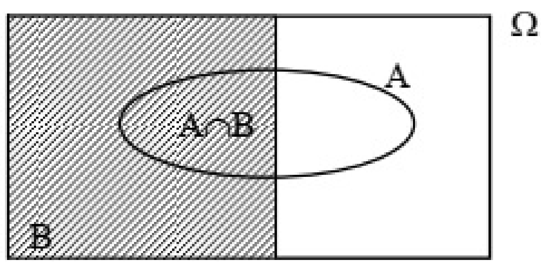
Definition 1. *Let A and B be two events in ω such that $P(B) > 0$. The **conditional probability** of event A, given that event B has occurred is*

*

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

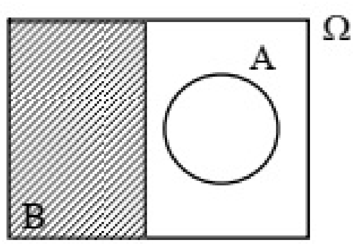
2.1 Remarks

1) $P(A/B)$ can be interpreted as the fact that Ω is restricted to B and the results of A are restricted to $A \cap B$.



2) If $A \cap B = \Omega$ (A and B are mutually exclusive), A cannot occur if B has already occurred, and therefore

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{P(\phi)}{P(B)} = 0.$$



3) In general $P(B/A) \neq P(A/B)$.

4) $P(\Omega/B) = \dfrac{P(B \cap \Omega)}{P(B)} = 1.$

Example 1. *Let's throw a dice and consider the events* $A = \{5\}$ *and* $B = \{1, 3, 5\}$*:*

$$P(A) = \frac{1}{6} \quad , P(B) = \frac{3}{6} \quad et\ P(A \cap B) = \frac{1}{6}$$

- *The probability of '5 coming out' knowing that it is 'an odd number' is :*

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{6}}{\frac{3}{6}} = \frac{1}{3}$$

- *The probability of an 'even number' coming out knowing that it is '5' is*

$$P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{1}{6}}{\frac{3}{6}} = 1$$

Note that $P(B/A) \neq P(A/B)$.

- *The probability that '5 does not come out' knowing that it is 'an odd number' is*

$$P(\bar{A}/B) = 1 - P(A/B) = 1 - \frac{1}{3} = \frac{2}{3}$$

2.2 Theorems on Conditional Probability

Definition 2. *Let n be an integer equal to or greater than 2. To say that n events $B_1, B_2, \ldots, B_n$ form a partition of the Ω universe means that:*

- *For all $i \in \{1; 2; \ldots; n\}, B_i \neq \emptyset$.*
- $\forall i \neq j : B_i \cap B_j = \emptyset$..
- $B_1 \cup B_2 \cup \ldots \cup B_k = \Omega$.

Theorem 1. ***(the total probability formula)***
Let n events $B_1, B_2, \ldots, B_n$ have non-zero probabilities and form a partition of Ω.
For all event A of Ω we have

$$\begin{aligned} P(A) &= P\left(A \cap B_1\right) + P\left(A \cap B_2\right) + \ldots + P\left(A \cap B_n\right) \\ &= P\left(B_1\right) P\left(A/B_1\right) + P\left(B_2\right) P\left(A/B_2\right) + \ldots + P\left(B_n\right) P\left(A/B_n\right) \end{aligned}$$

Remark . *Since ($B \cup \bar{B} = \Omega$ and $B \cup \bar{B} = \phi$), we can apply the total probability formula:*

$$P(A) = P(B) \times P(A/B) + P(\bar{B}) \times P(A/\bar{B})$$

2.3 Formula of compound probabilities

theorem 2. Let A, B, C, D be events in a universe $Omega$. a) $P(A \cap B) = P(A)P(B/A)$.
b) $P(A \cap B \cap C) = P(A) \cdot P(B/A)P(C/(B \cap A))$
c) $P(A \cap B \cap C \cap D) = P(A) \cdot P(B/A)P(C/(B \cap A)) \cdot P(D/(A \cap B \cap C))$.

Demonstration. a) According to the definition

$$P(B/A) = \frac{P(A \cap B)}{P(A)} \Rightarrow P(A \cap B) = P(A)P(B/A)$$

b) According to the definition

$$P(C/(B \cap A)) = \frac{P((A \cap B) \cap C)}{P(A \cap B)}$$
$$\Longrightarrow P(A \cap B \cap C) = P((A \cap B))P(C/(B \cap A)) = P(A) \cdot P(B/A)P(C/(B \cap A))$$

Example 3. Consider an urn containing 4 white balls and 6 red balls. 3 balls are drawn one after the other. What is the probability of obtaining a draw consisting of 3 white balls?

Answer Let B_i be the event: the i-th ball drawn is white. Clearly

$$P(B_1) = \frac{4}{10}$$

$$P(B_2/B_1) = \frac{3}{9}$$
$$P(B_3/(B_2 \cap B_1)) = \frac{2}{8}.$$

By compound probabilities, we obtain

$$P(B_1 \cap B_2 \cap B_3) = \frac{4}{10} \times \frac{3}{9} \times \frac{2}{8} = \frac{1}{30}$$

3 Bayes' formula