CHAPTER 1

THE FINITE EXPANSIONS

1.1 Londau Notation

Definition 1.1.1 *let $x_0 \in \mathbb{R}$ et f, g tow continuous functions defines in the neighborhood to x_0. We say that f is negligible compared to g around to x_0 if $\lim\limits_{x \to x_0} \dfrac{f(x)}{g(x)} = 0$.*

We note $f << g$ (notation de Hardy) or $f = o(g)$ (notation de Londau)

Remark 1.1.2

1. *$f = o(1) \Rightarrow \lim\limits_{x \to x_0} f = 0$.*
2. *$f \mathfrak{R} g \Leftrightarrow f = o(g)$ is not equivalent relation (because it is transitive, but not reflexive and not symmetric).*
3. *Si $\alpha < \beta \Rightarrow x^\alpha = o(x^\beta)$.*

Theorem 1.1.3 *Let x_0, $\lambda \in \mathbb{R}$ and f, g and h three functions defined around to x_0 then:*

1. *$f = o(h)$ and $g = o(h) \Rightarrow f + \lambda g = o(h)$.*
2. *$f = o(h) \Rightarrow f.g = o(hg)$.*
3. *f is bounded and g tends to infinity, then $f = o(g)$.*

Definition 1.1.4 *The finite expansions essentially to find a polynomial approximation to a more complicated function in the neighborhood of a some point. They have numerous applications in other sciences, but also in mathematics itself, particularly in numerical analysis.*

Definition 1.1.5 *We said that $f : I \to \mathbb{R}$, is represented by the polynomial approximation of degree n, for x near to $x_0 \in I$ if and only if there exist a polynomial $P \in \mathbb{R}_n[X]$, such that*

$$\forall x \in I : f(x) = P(x - x_0) + o(x - x_0)^n.$$

*We call $P(x - x_0)$ is **the mean part** of the finie expansions and $o(x - x_0)^n$ is **the remainder part (or, rest)** of degree n.*

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1.2 Taylor formula (or:Taylor polynomial of degree n)

The approximations of the function f by the Taylor polynomial of degree n, denoted by $P(x-x_0)$ for x near x_0 using more derivatives $f'(x), f''(x), ..., f^n(x)$ is given by

$$f(x)=f(x_0)+\frac{f'(x_0)}{1!}(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+...+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+O(x-x_0)^n$$

Particular case. if $x=0$.

$$f(x)=f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+...+\frac{f^{(n)}(0)}{n!}x^n+O(x^n)$$

with $\epsilon(x)\longrightarrow 0$ quand $x\longmapsto x_0$. This formula is called the formula of **Mac-Laurin**

Example 1.2.1 *We will write the Mac-Laurin formula of the function* $\cos: x\mapsto \cos x$ *near to* 0. *of degree* n

$$\begin{aligned}
\cos' x &= -\sin x,\\
\cos'' x &= -\cos x,\\
\cos^{(3)} x &= \sin x,\\
\cos^{(4)} x &= \cos x,\\
\cos^{(5)} x &= -\sin x,\\
\cos^{(5)} x &= -\cos x,
\end{aligned}$$

It's easy to find that,

$$\cos^{(n)}(x)=\cos(x+\frac{n\pi}{2})$$

Then

$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\frac{x^8}{8!}+...+(-1)^n\frac{x^{2n}}{(2n)!}+O(x^n)$$

The polynomial approximation of the function $\cos$ *of degree* 4 *is*

$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}+O(x^4),$$

The polynomial approximation of degree 5 *is given by:*

$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}+O(x^5),$$

The first statement informs us that there are terms of order x^4 in the expansion. The second statement is stronger as it informs us that there are no terms of order x^5.

Definition 1.2.2 ***(Finite expansions at zero)*** *Let f be a real valued function. We said that the function f is represented by a finite expansion at zero if there exist real numbers $a_0, a_1, ...a_n$ and a real valued function ϵ such that*

$$f(x)=a_0+a_1x+a_2x^2+...+a_nx^n+x^n\epsilon(x),\qquad \lim_{x\to 0}\epsilon(x)=0.$$

Then the function f is represented by the polynomial approximation of degree n, denoted by $P_n(x)$ for x near zero, which is called the main part of finite expansions at zero, such that: $P_n(x)=a_0+a_1x+a_2x^2+...+a_nx^n$

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Remark 1.2.3 *Note that:* $x^n\epsilon(x) = O(x^n)$

Using the euclidean division by increasing power order, one has the finite expansion at zero of

$$f(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + ... + x^n + \frac{x^{n+1}}{1-x} = 1 + x + x^2 + x^3 + ... + x^n + x^n(\frac{x}{1-x})$$

in this case $\epsilon(x) = \dfrac{x}{1-x}$. We generally do not try to determine the function $\epsilon(x)$.

1.2.1 Properties

- If the function f can be expanded at zero, then this expansion is unique.
- If the function f can be expanded at zero, then $\lim_{x\to 0} f(x) = a_0$. This criterion is generally used to demonstrate that a function does not admit an expansion.
- If the function f can be expanded at zero, and if f is even (resp. odd) the polynomial approximation $P_n(x)$ is even (resp. odd)

  In fact:
  f represented by finite expansion at x_0, $\quad f(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n + x^n\epsilon(x)$
  so $f(-x) = a_0 - a_1x + a_2x^2 + ... + (-1)^n a_nx^n + x^n\epsilon(x)$

  ⋆ If f is odd: $f(-x) = -f(x) = -a_0 - a_1x - a_2x^2 - ... - a_nx^n - x^n\epsilon(x)$, then $a_0 = a_2 = a_4 = ... = a_{2k} = 0, \quad 2k \leq n$.

  ⋆ If f even $f(-x) = f(x)$ then $a_1 = a_3 = a_5 = ... = a_{2k+1} = 0, \quad 2k+1 \leq n$.

1.3 Finite expansions of elementary functions

The function defined by: $f(x) = a^x$, $a > 0$. We have $a^x = e^{x\ln a}$ and the n-th derivative of f is $(a^x)^{(n)} = (\ln a)^n e^{x\ln a}$. So the finite expansions of this function is given by

$$a^x = 1 + (\ln a)x + \frac{(\ln a)^2}{2!}x^2 + ... + \frac{(\ln a)^n}{n!}x^n + x^n\epsilon(x)$$

For $a = e$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^n}{n!} + x^n\epsilon(x)$$

$$Ch(x) = Ch(0) + \frac{Ch'(0)}{1!}x + \frac{Ch''(0)}{2!}x^2 + ... + \frac{Ch^{(n)}(0)}{n!}x^n + x^n\epsilon(x)$$

$$Ch(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + ... + \frac{x^{2n}}{(2n)!} + x^{2n+1}\epsilon(x)$$

$$Sh(x) = Sh(0) + \frac{Sh'(0)}{1!}x + \frac{Sh''(0)}{2!}x^2 + ... + \frac{Sh^{(n)}(0)}{n!}x^n + x^n\epsilon(x)$$

$$Sh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + ... + \frac{x^{2n+1}}{(2n+1)!} + x^{2n+2}\epsilon(x)$$

$$\cos(x) = \cos(0) + \frac{\cos'(0)}{1!}x + \frac{\cos''(0)}{2!}x^2 + ... + \frac{\cos^{(n)}(0)}{n!}x^n + x^n\epsilon(x)$$

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$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + ... + (-1)^n \frac{x^{2n}}{(2n)!} + x^{2n+1}\epsilon(x)$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + ... + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + x^{2n+2}\epsilon(x)$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + ... + (-1)^n x^n + x^n\epsilon(x)$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ... + (-1)^{n-1}\frac{x^n}{n} + x^n\epsilon(x)$$

For $\alpha \in \mathbb{R}$ et $x \neq -1$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + ... + \frac{\alpha(\alpha-1)...(\alpha-n+1)}{n!}x^n + x^n\epsilon(x)$$

For $\alpha = \frac{1}{2}$

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + ... + (-1)^{n-1}\frac{1.1.3.5...(2n-3)}{2^n n!}x^n + x^n\epsilon(x)$$

For $\alpha = -1$ we fall back on the finite expansion of $\frac{1}{1+x}$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + ... + x^n + x^n\epsilon(x)$$

1.4 Algebraic combinations of finite expansions

If f and g can both be expanded at zero and λ is any constant, then each of the following functions is also can be expanded at zero: The sum $f+g$, the difference $f-g$, the constant multiple $\lambda \times f$, the product $f \times g$, the quotient $f \div g$, if $g(0) \neq 0$:

Consider the finite expansions at zero of f and g

$$f(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n + x^n\epsilon_1(x)$$

and

$$f(x) = b_0 + b_1 x + b_2 x^2 + ... + b_n x^n + x^n\epsilon_2(x)$$

such that $\lim\limits_{x\to 0}\epsilon_1(x) = 0, \quad \lim\limits_{x\to 0}\epsilon_2(x) = 0$ then the finite expansions of

- **the sum** $f+g$ is

  $(f+g)(x) = f(x)+g(x) = (a_0+a_1x+a_2x^2+...+a_nx^n)+(b_0+b_1x+b_2x^2+...+b_nx^n)+x^n(\epsilon_1(x)+\epsilon_2(x))$

  Therefore

  $(f+g)(x) = (a_0+b_0) + (a_1+b_1)x + (a_2+b_2)x^2 + ... + (a_n+b_n)x^n + x^n(\epsilon_1(x)+\epsilon_2(x))$

  and $\lim\limits_{x\to 0}(\epsilon_1(x)+\epsilon_2(x)) = 0$

  For example: If $f(x) = e^x$ and $g(x) = e^{-x}$ then

  $$e^x + e^{-x} = 2 + 2\frac{x^2}{2!} + 2\frac{x^4}{4!} + ... + 2\frac{x^{2k}}{(2k)!} + x^{2k}\epsilon(x) \quad 2k \leq n$$

  Hence

  $$\frac{e^x+e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + ... + \frac{x^{2k}}{(2k)!} + x^{2k}\epsilon(x) = Ch(x) \quad 2k \leq n$$

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- **the product** The finite expansion at zero of the $f.g$ is obtained by the product $(a_0 + a_1x + a_2x^2 + ... + a_nx^n)(b_0 + b_1x + b_2x^2 + ... + b_nx^n)$ and keeping only the monomials of degree less than n in the product

$$(f.g)(x) = f(x).g(x) = (A(x) + x^n\epsilon_1(x)) + (B(x) + x^n\epsilon_2(x))$$

such that $A(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$ and $B(x) = b_0 + b_1x + b_2x^2 + ... + b_nx^n$ Then

$$(f.g)(x) = A(x).B(x) + x^n\epsilon_1(x).B(x) + x^n\epsilon_2(x).B(x) + x^{2n}\epsilon_1(x).\epsilon_2(x)$$

<u>**For example**</u>: Let the function $f(x) = e^x.\sin(x)$. We find the finite expansions of the function f of degree 6 near to zero.

$$f(x) = e^x.\sin(x) = (1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...).(x - \frac{x^3}{3!} + \frac{x^5}{5!} + ...)$$

$$f(x) = x + x^2 + \frac{x^3}{3} - \frac{x^5}{33} + \frac{x^6}{18} + R_6$$

Exercise: Find the finite expansion of the function f defined by: $f(x) = \dfrac{\ln(1+x)}{1+x}$ of degree 6 at zero.

- **the division** The finite expansion at zero of the quotient f/g is obtained by the euclidean division of $(a_0 + a_1x + a_2x^2 + ... + a_nx^n)$ by $(b_0 + b_1x + b_2x^2 + ... + b_nx^n)$ by increasing power order.

$$A(x) = B(x).Q(x) + x^{n+1}R(x), \text{ avec } deg(Q) \leq n.$$

Such that Q is mean part of the finite expansions n de $\dfrac{f}{g}$

<u>**For example**</u>: The finite expansion of degree 2 of $\dfrac{2+x+x^3}{1+x^2}$. We put $A(x) = 2 + x + x^3$ et $B(x) = 1 + x^2$, alors $Q(x) = 2 + x - 2x^2$, $R(x) = 1 + 2x$ therefore

$$A(x) = (1 + x^2)(2 + x - 2x^2) + x^3(1 + 2x)$$

Example 1.4.1 *The finite expansions of degree 5 of the function $th : x \mapsto th(x)$*

$$th(x) = \frac{sh(x)}{ch(x)} = \frac{x + \frac{x^3}{3!} + \frac{x^5}{5!} + x^5\epsilon(x)}{1 + \frac{x^2}{2!} + \frac{x^4}{4!} + x^5\epsilon(x)} = x - \frac{x^3}{3} + \frac{2x^5}{15} + \frac{x^6.(\frac{x}{180} + \frac{x^3}{180})}{1 + \frac{x^2}{2!} + \frac{x^4}{4!}}$$

1.4.1 Composite of finite expansions

Proposition 1.4.2 *If g can be expanded at zero of degree n and if f can be expanded at $g(0)$ of degree n such that $g(0) = 0$. Then the composite function $(f \circ g)$ can be expanded at zero of degree n by replacing the finite expansion of g in the finite expansion of f and by keeping only the monomials of degree less or equal n.*

<u>**Par example**</u> the finite expansion of degree 2 at 0 of the function $x \mapsto e^{\sin x}$. we have

$$\sin x = x + x^2\epsilon(x) \quad \text{and } e^x = 1 + x + \frac{x^2}{2} + \epsilon_2(x)$$

then $e^{\sin x} = 1 + x + \dfrac{x^2}{2} + \epsilon(x)$ avec $\lim\limits_{x\to 0} \epsilon(x) = 0$.

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1.4.2 The finite expansions at a point

We said that the function $f : x \mapsto f(x)$ can be represented by a finite expansion at point x_0 if the function $t \mapsto f(t + x_0)$. can be represented by finite expansion at zero.

Often we therefore reduce the problem to 0 by changing the variables $t = x - x_0$.

For example The finite expansion of $f(x) = e^x$ at 1. We make the change variable $t = x - 1$. If x is near to 1 then t is near to 0.

$$e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + ... + \frac{t^n}{n!} + x^n\epsilon(t)$$

$$e^x = e(1 + (x-1) + \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} + ... + \frac{(x-1)^n}{n!} + x^n\epsilon(x-1)), \quad \lim_{x\to 1}\epsilon(x-1) = 0.$$

Exercise:

1. Calculate the finite expansion at 0 of $x \mapsto chx$ by Taylor formula. Find this finite expansions using that $chx = \dfrac{e^x + e^{-x}}{2}$.
2. Calculate the finite expansion at 0 of degree 3 of $\sqrt[3]{1+x}$.
3. Justify the expression of $\frac{1}{1-x}$ using the uniqueness finite expansion of the sum of a geometric sequence.

1.4.3 The finite expansions at Infinity

We said that the function $f : x \mapsto f(x)$ can be represented by a finite expansion at infinity if the function $F : t \mapsto f(\dfrac{1}{x})$ can be represented by finite expansion at zero.

In other words f can be expended at infinity if there exists $a_0, a_1, ..., a_n$ such that

$$f(x) = a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + ... + \frac{a_n}{x^n} + \frac{1}{x^n}\epsilon(\frac{1}{x}),$$

where $\lim\limits_{x\to+\infty}\epsilon(\dfrac{1}{x}) = 0$

Example 1.4.3 *Let the function f defined by: $f : x \mapsto \ln(2 + \dfrac{1}{x})$*

$$f(x) = \ln 2 + \ln(1 + \frac{1}{2x}) = \ln 2 + \frac{1}{2x} - \frac{1}{8x^2} + \frac{1}{24x^3} + ... + \frac{1}{n2^n x^n} + \frac{1}{x^n}\epsilon(\frac{1}{x})$$

This allows us to have a precise idea of the behavior of f in the neighborhood of $+\infty$. When $x \to +\infty$ then $x \to \ln 2$. And the second term is $+\dfrac{1}{2}x$ therefore is positive, this means that the function $f(x)$ tends to $\ln 2$ while remaining above to $\ln 2$.

Remark 1.4.4

1. *The finite expansions at $+\infty$ is also called an asymptotic expansions.*
2. *We said that the function $x \mapsto f(x)$ can be expended at $+\infty$ of degree n is equivalent to $x \mapsto f(\dfrac{1}{x})$ can be expended at 0^+ by finite expansions of degree n.*
3. *We can similarly define what a finite expansions at $-\infty$*

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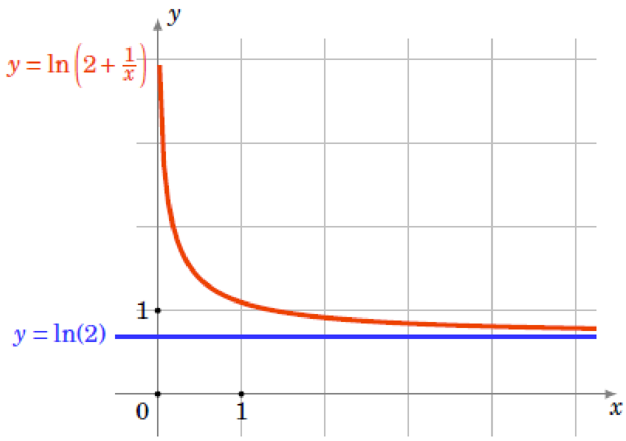

1.5 Applications

1.5.1 Using finite expansions to evaluate limits

The finite expansions provide a good way to understand the behavior of a function near a specified point and so are useful for solving some indeterminate forms. When taking a limit as $x \to 0$, we can often simplify the statement by substituting in finite expansions that we know.

1. The finite expansions is very important for calculating limits with indeterminate forms! It is enough just to note that if

$$f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \ldots \text{ alors } \lim_{x \to x_0} f(x) = a_0.$$

Example:

$$\lim_{x \to 0} \frac{\ln(1+x) - \tan x + \frac{1}{2}\sin^2 x}{3x^2 \sin^2 x}$$

At 0

$$f(x) = \ln(1+x) - \tan x + \frac{1}{2}\sin^2 x = (x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + x^4\epsilon(x)) - (x - \frac{x^3}{3} + x^4\epsilon(x))$$

$$+\frac{1}{2}(x - \frac{x^3}{6} + x^4\epsilon(x))^2$$

and

$$g(x) = 3x^2 \sin^2 x = 3x^2(x + x\epsilon(x))^2 = 3x^4 + x^4\epsilon(x)$$

Also

$$\frac{f(x)}{g(x)} = \frac{-\frac{5}{12}x^4 + x^4\epsilon(x)}{3x^4 + x^4\epsilon(x)}$$

Then

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = -\frac{5}{36}$$

Note: by calculating the finite expansions at a lower order (2 for example), we would not have been able to conclude, because we would have obtained $\frac{f(x)}{g(x)} = \frac{x^2\epsilon(x)}{x^2\epsilon(x)}$, which

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does not remove the indeterminacy. Generally, we calculate the finite expansion at the lowest possible order, and if this is not enough, we increase the order (and therefore the precision of the approximation).

2. Suppose a function $f : I \to \mathbb{R}$ admitting a finite expansion at x_0 : then

$$f(x) = a_0 + a_1(x - x_0) + a_k(x - x_0)^k + (x - x_0)^k \epsilon(x)$$

where k is the smallest integer greater than 2 such that the coefficient a_k either non-zero. then the tangent equation of the curve of f at x_0 is $y = a_0 + a_1(x - x_0)$ and the relative position of the curve compared to (or, in relation to) the tangent for x near to x_0 is given by the sign $f(x) - y$ i.e. the sign of $a_k(x - x_0)^k$.

Example 1.5.1 *Let f the function defined by:*

$$f(x) = x^4 - 2x^3 + 1.$$

- *Finding the tangent at $\frac{1}{2}$ of the curve of f et specify the relative position of the curve in relation to the tangent.*
  We have $f"(\frac{1}{2}) = -3 \neq 0$ et $k = 2$
  We deduce the finite expansion at $\frac{1}{2}$ by Taylor formula

$$f(x) = f(\frac{1}{2}) + f'(\frac{1}{2})(x - \frac{1}{2}) + f"(\frac{1}{2})(x - \frac{1}{2})^2 + (x - \frac{1}{2})^2 \epsilon(x)$$

  So the tangent at $\frac{1}{2}$ is $y = \frac{13}{16} - (x - \frac{1}{2})$ and the curve of f is bellow to the tangent because $f(x) - y = (-\frac{3}{2} + \epsilon(x))(x - \frac{1}{2})^2$ is negative around to $x = \frac{1}{2}$
- *Let's find the inflection points.*
  The inflection points are to be found among the solutions of $f"(x) = 0$. therefore $x = 0$ and $x = 1$.
  ** The finite expansions at 0 : is $f(x) = 1 - 2x^3 + x^4$ (it's just a matter of writing the monomials in increasing degrees!). The tangent equation at x-axis point 0 is $y = 1$ (the horizontal tangent). because $-3x^2$ change the sign at 0 then 0 is inflexion point of f.*
  ** The finite expansion at $x = 1$: is $f(x) = -2(x - 1) + 2(x - 1)^3 + (x - 1)^4$. The tangent equation at x-axis point 1 is $y = -2(x - 1)$. because $2(x - 1)^3$ change the sign at 1, 1 also is an inflexion point of f.*

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