MOHAMED BOUDIAF UNIVERSITY OF MSILA
**Faculty of Mathematics and Computer Science**
**Computer Science Department**
**University year :**2023/2024

***1.st year bachelor's degree - Semester 2***
***Final Exam in Analysis 2***
***Date : 11/05/2024 Duration : 1 h 30 m***

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**Remarque : Les documents et tout appareil électronique sont strictement interdits.**

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**Exercise 1 : (06 Pt) Consider the following function**

$$f(x) = \frac{\sin x - e^{-x}\ln(x+1)}{1+x}.$$

1. **Write the finite expansion of $f$ in the neighborhood of $0$ at order $3$.**
2. **Deduce the value of the following limit**

$$\lim_{x\to 0}\left(\frac{-e^x \sin x + \ln(x+1)}{x^2(1+x)e^x}\right).$$

**Exercise 2 : (04 Pt)**

1. **Find the constants values $\alpha$ and $\beta$ such that**

$$\frac{x^2}{\sqrt{x^2-1}} = \frac{\alpha}{\sqrt{x^2-1}} + \frac{\sqrt{x^2-1}}{\beta}.$$

2. **Deduce the area of region bounded by the graph of the equation $y = \frac{x^3}{\sqrt{x^2-1}}$ and $x-$axis and the lines $x = \sqrt{2},\ x = 2$.**

**Exercise 3 : (10 Pt)**

1. **Find the primitives of the following function**

$$g(x) = \frac{x^5}{(x-1)(x^2-x+1)}$$

2. **Let the differential equation**

$$y' = \frac{y}{x-1} + \frac{x^5}{(x^2-x+1)} \qquad (1)$$

— **Give the type of the equation** (1).
— **Using the variation of constant method, solve the equation** (1).

3. **Prove that the function $h(x) = x^3 - 1$, is solution of the equation**

$$y' = \frac{y}{x-1} + 2x^2 - x - 1$$

**- Find the solution to this equation that satisfies the initial condition $y(0) = 0$.**

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# SOLUTION OF SECOND EXAM

**Exercise 1 :**

1. **The finite expansion of $f$ in the neighborhood of $0$ at order $3$.**
We have

$$\sin x = x - \frac{x^3}{6} + x^3\epsilon(x), \qquad (0,25)$$

$$e^{-x} = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + x^3\epsilon(x), \qquad (0,25)$$

$$\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} + x^3\epsilon(x), \qquad (0,25)$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^3\epsilon(x). \qquad (0,25) \quad \text{Such that, } \lim_{x\to 0}\epsilon(x) = 0.$$

$$\begin{aligned} e^{-x}\ln(x+1) &= (1 - x + \frac{x^2}{2} - \frac{x^3}{6} + x^3\epsilon(x))(x - \frac{x^2}{2} + \frac{x^3}{3} + x^3\epsilon(x)) \\ &= x - \frac{3}{2}x^2 + \frac{4}{3}x^3 + x^3\epsilon(x) \qquad (0,5) \end{aligned}$$

$$\begin{aligned} \sin x - e^{-x}\ln(x+1) &= (x - \frac{x^3}{6} + x^3\epsilon(x)) - (x - \frac{3}{2}x^2 + \frac{3}{2}x^3 + x^3\epsilon(x)) \\ &= \frac{3}{2}x^2 - \frac{3}{2}x^3 + x^3\epsilon(x). \qquad (0,5) \end{aligned}$$

Then

$$\begin{aligned} f(x) &= (1 - x + x^2 - x^3 + x^3\epsilon(x))(\frac{3}{2}x^2 - \frac{3}{2}x^3 + x^3\epsilon(x)) \\ &= \frac{3}{2}x^2 - 3x^3 + x^3\epsilon(x) \qquad (02) \end{aligned}$$

2. **The value of the following limit**

$$\lim_{x\to 0}(\frac{-e^x\sin x + \ln(x+1)}{x^2(1+x)e^x}) = \frac{0}{0}. \quad \text{I.F}$$

$$\frac{-e^x\sin x + \ln(x+1)}{x^2(1+x)e^x} = \frac{-e^x[\sin x - \ln(x+1)]}{e^x x^2(1+x)} = \frac{-1}{x^2}f(x) = \frac{-3}{2} + 3x + x\epsilon(x) \qquad (01)$$

It results

$$\lim_{x\to 0}(\frac{-e^x\sin x + \ln(x+1)}{x^2(1+x)e^x}) = \lim_{x\to 0}(\frac{-3}{2} + 3x + x\epsilon(x)) = -\frac{3}{2} \qquad (01)$$

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**Exercise 2 :**

1. We have

$$\frac{x^2}{\sqrt{x^2-1}} = \frac{\alpha\beta + x^2 - 1}{\beta\sqrt{x^2-1}} = \frac{\frac{\alpha\beta}{\beta} + \frac{x^2}{\beta} - \frac{1}{\beta}}{\sqrt{x^2-1}}, \qquad (0,5)$$

by identification

$$\begin{cases} \frac{1}{\beta} = 1 \\ \frac{\alpha\beta}{\beta} - \frac{1}{\beta} = 0 \end{cases} \Rightarrow \begin{cases} \beta = 1 & (0,5) \\ \alpha = 1 & (0,5) \end{cases}$$

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2. **We denote by $D$ the area of region bounded by the graph of the equation $y = \frac{x^3}{\sqrt{x^2-1}}$ and $x-$axis and the lines $x=\sqrt{2}$, $x=2$.** Then

$$D = \int_{\sqrt{2}}^{2} \frac{x^3}{\sqrt{x^2-1}}dx \qquad (0,5)$$

$$D = \int_{\sqrt{2}}^{2} x(\frac{x^2}{\sqrt{x^2-1}})dx = \int_{\sqrt{2}}^{2} \frac{x}{\sqrt{x^2-1}}dx + \int_{\sqrt{2}}^{2} x\sqrt{x^2-1}dx \qquad (0,5)$$

$$D = [\sqrt{x^2-1}]_{\sqrt{2}}^{2} + \frac{1}{3}[(x^2-1)^{\frac{3}{2}}]_{\sqrt{2}}^{2} = 2\sqrt{3} - \frac{4}{3}. \qquad (01,5)$$

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**Exercise 3 :**

1. **The primitives of the following function**

$$g(x) = \frac{x^5}{(x-1)(x^2-x+1)}$$

The euclidean division of $g(x)$

$$g(x) = x^2 + 2x + 2 + \frac{x^2-2x+2}{(x-1)(x^2-x+1)} \qquad (01)$$

Using the decomposition of simple elements of 1-st and 2-nd degree

$$\frac{x^2-2x+2}{(x-1)(x^2-x+1)} = \frac{a}{x-1} + \frac{bx+c}{x^2-x+1} \qquad (0,5)$$

after calculation and by identification, we obtain

$$\begin{cases} a+b=1 \\ -a-b+c=-2 \\ a-c=2 \end{cases} \Leftrightarrow \begin{cases} a=1 \\ b=0 \\ c=-1 \end{cases} \qquad (0,75)$$

Then

$$\frac{x^2-2x+2}{(x-1)(x^2-x+1)} = \frac{1}{x-1} - \frac{1}{x^2-x+1}$$

Integrating

$$\int \frac{x^2-2x+2}{(x-1)(x^2-x+1)}dx = \int \frac{1}{x-1}dx - \int \frac{1}{x^2-x+1}dx \qquad (0,25)$$

Calculation of

$$\int \frac{1}{x^2-x+1}dx = \int \frac{dx}{(x-1)^2+\frac{3}{4}} = \frac{4}{3}\int \frac{dx}{1+(\frac{2x-1}{\sqrt{3}})^2}$$

$$= \frac{2}{\sqrt{3}}\int \frac{dt}{1+t^2}, \quad \text{with } t = \frac{2x-1}{\sqrt{3}}$$

$$= \frac{2}{\sqrt{3}}\arctan t + C = \frac{2}{\sqrt{3}}\arctan\frac{2x-1}{\sqrt{3}} + C.$$

Then

$$\int g(x)dx = \frac{x^3}{3} + x^2 + 2x + \ln|x-1| - \frac{2}{\sqrt{3}}\arctan\frac{2x-1}{\sqrt{3}} + K. \qquad (02$$

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2. **Let the differential equation**

$$y' = \frac{y}{x-1} + \frac{x^5}{(x^2-x+1)} \qquad (1)$$

— The type of the equation (1) is : 1-st order linear differential equation. (0,5)
— **Using the variation of constant method, for solve the equation** (1).
a) general solution without second member (homogeneous solution) :

$$y' = \frac{y}{x-1} \Leftrightarrow \frac{dy}{y} = \frac{dx}{x-1}$$

By transition of the integral

$$\int \frac{dy}{y} = \int \frac{dx}{x-1} \Rightarrow y_H = C(x-1), \quad C \in \mathbb{R}. \qquad (01)$$

b) Solution with second member : using the variation of constant method.
We make

$$y = C(x)(x-1) \Leftrightarrow y' = C'(x)(x-1) + C(x)$$

We replace in the equation (1)

$$C'(x)(x-1) + C(x) - C(x) = \frac{x^5}{(x^2-x+1)} \Leftrightarrow C'(x) = \frac{x^5}{(x-1)(x^2-x+1)}$$

From the question 1 we can deduce

$$C(x) = \frac{x^3}{3} + x^2 + 2x + \ln|x-1| - \frac{2}{\sqrt{3}} \arctan \frac{2x-1}{\sqrt{3}} + K, \quad K \in \mathbb{R}$$

Then

$$y = K(x-1) + (x-1)(\frac{x^3}{3} + x^2 + 2x + \ln|x-1| - \frac{2}{\sqrt{3}} \arctan \frac{2x-1}{\sqrt{3}}). \qquad (01,5)$$

3. **Prove that the function** $h(x) = x^3 - 1$, **is solution of the equation**

$$y' = \frac{y}{x-1} + x^2 - x + 1$$

We have $h'(x) = 3x^2$ and $h(x) = (x-1)(x^2+x+1)$ then we replace in the equation

$$3x^2 = \frac{(x-1)(x^2+x+1)}{x-1} + 2x^2 - x - 1. \quad \text{true.} \qquad (01)$$

So, $h$ is a particular solution of the equation.
- **The solution to this equation that satisfies the initial condition** $y(0) = 0$.
We have the general solution is

$$y = C(x-1) + x^3 - 1.$$

We have $y(0) = 0 \Leftrightarrow -C - 1 = 0 \Leftrightarrow C = -1 \qquad (0,5)$

$$\text{Then} \quad y = x^3 - x. \qquad (01)$$

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