University of M'Sila,
Faculty of Mathematics and Computer Science / Department of Mathematics
Master of **Functional Analysis / S 3**
– ***Espaces fonctionnels de type de Sobolev*** –

<u>Final Exam</u> January 2025

*** We use the notations of the lesson, as $s \in \mathbb{R}$, $1 \leq p, q \leq \infty$, the functions ρ and γ ...**

Exercise 1. (Lesson) **(1)** Give the definitions of :
(a) ρ and γ,
(b) $Q_j f(x)$ and $S_j f(x)$,
(c) the spaces $\dot{B}^u_{\infty,q}(\mathbb{R})$, $B^v_{p,\infty}(\mathbb{R})$, $BMO(\mathbb{R})$.
(2) Write the following two theorems : 1- Nikol'skii's theorem, 2- Paley-Wiener's theorem.
(3) Find the conditions such that $\dot{B}^0_{p,2}(\mathbb{R}) \subset \dot{B}^\alpha_{\infty,q}(\mathbb{R})$.

Exercise 2. Let $p, u \in [1 + \infty]$. Let $\mathcal{A}^u_p$ be the set of all $f \in L^{loc}_p(\mathbb{R})$ such that

$$\|f\|_{\mathcal{A}^u_p} := \sup_{x\in\mathbb{R}} \sup_{r>0} r^{\frac{1}{u}-\frac{1}{p}} \Big(\int_{-r}^{r} |f(x-y)|^p \, \mathrm{d}y \Big)^{1/p} < \infty.$$

(1) Prove that $p > u \Rightarrow \mathcal{A}^u_p = \{0\}$. Define $\mathcal{A}^u_p$.
(2) Prove that $p > q \Rightarrow \mathcal{A}^u_p \hookrightarrow \mathcal{A}^u_q$. Calculate $\|f(\lambda\, \cdot)\|_{\mathcal{A}^u_p}$ for all $\lambda > 0$.
(3) Prove that $\mathcal{A}^\infty_p = L_\infty(\mathbb{R})$. Hint : we recall the Lebesgue's differentiation theorem : if $f \in L^{loc}_1(\mathbb{R})$ then $\lim\limits_{r\to 0} r^{-1} \displaystyle\int_{-r}^{r} |f(x-y) - f(x)| \, \mathrm{d}y = 0$.
(4) Find the conditions for s, r, t, u, p such that $\|f\|_{\dot{B}^s_{r,t}(\mathbb{R})} \leq c\|f\|_{\mathcal{A}^u_p}$.
(5) Let $m \in \mathbb{N}^*$. Using **(4)**, find the conditions for m, s, u, p such that $f^{(m)} \in \mathcal{A}^u_p \Rightarrow f \in \dot{B}^s_{\infty,\infty}(\mathbb{R})$.

Exercise 3. **(1)** Let $u_m(x) := x^m$, $x \in \mathbb{R}$ and $m \in \mathbb{N}^*$. Calculate $\Delta^m_h u_m(x)$.
(2) Let $\psi \in \mathcal{D}(\mathbb{R})$ be a positive function, such that $\operatorname{supp}\psi = [-2, +2]$ and $\psi(x) = x^m$ if $|x| \leq 1$. We put

$$|(\Delta^m_h \psi)(x)| \leq \frac{1}{j^2}, \quad |x| \leq \frac{1}{2}, \quad h = \frac{1}{4m}, \quad \forall j \in \mathbb{N}^*. \qquad \text{(E)}$$

Prove that (E) is impossible.

Scale :
Ex. 1 : 5 pts. = (0.5+1+1) + (1+1) + 0.5
Ex. 2 : 10 pts. = (1+0.5) + (1+1.5) + 2.5 + 1.5 + 2
Ex. 3 : 5 pts. = 2 + 3

Exercice 1. (Lesson) **(1)** Donner les définitions de :

(a) ρ et γ,

(b) $Q_j f(x)$ et $S_j f(x)$,

(c) les espaces $\dot{B}^u_{\infty,q}(\mathbb{R})$, $B^v_{p,\infty}(\mathbb{R})$, $BMO(\mathbb{R})$.

(2) Ecrire les deux théorèmes de : 1- Nikol'skii, 2- Paley-Wiener.

(3) Trouver les conditions tel que $\dot{B}^0_{p,2}(\mathbb{R}) \subset \dot{B}^\alpha_{\infty,q}(\mathbb{R})$.

Exercice 2. Soient $p, u \in [1 + \infty]$. Soit $\mathcal{A}^u_p$ l'ensemble des $f \in L^{loc}_p(\mathbb{R})$ telles que

$$\|f\|_{\mathcal{A}^u_p} := \sup_{x\in\mathbb{R}} \sup_{r>0} r^{\frac{1}{u}-\frac{1}{p}} \Big(\int_{-r}^{r} |f(x-y)|^p \,\mathrm{d}y \Big)^{1/p} < \infty.$$

(1) Démontrer que $p > u \Rightarrow \mathcal{A}^u_p = \{0\}$. Définir $\mathcal{A}^u_p$.

(2) Démontrer que $p > q \Rightarrow \mathcal{A}^u_p \hookrightarrow \mathcal{A}^u_q$. Calculer $\|f(\lambda\,\cdot)\|_{\mathcal{A}^u_p}$ pour tout $\lambda > 0$.

(3) Démontrer que $\mathcal{A}^\infty_p = L_\infty(\mathbb{R})$. Indication : on rappelle le théorème de différentiation de Lebesgue :

si $f \in L^{loc}_1(\mathbb{R})$ alors $\displaystyle\lim_{r\to 0} r^{-1} \int_{-r}^{r} |f(x-y) - f(x)| \,\mathrm{d}y = 0.$

(4) Trouver les conditions sur s, r, t, u, p tel que $\|f\|_{\dot{B}^s_{r,t}(\mathbb{R})} \le c\|f\|_{\mathcal{A}^u_p}$.

(5) Soit $m \in \mathbb{N}^*$. En utilisant **(4)**, Trouver les conditions sur m, s, u, p tel que

$f^{(m)} \in \mathcal{A}^u_p \Rightarrow f \in \dot{B}^s_{\infty,\infty}(\mathbb{R})$.

Exercice 3. **(1)** Soit $u_m(x) := x^m$, $x \in \mathbb{R}$ et $m \in \mathbb{N}^*$. Calculer $\Delta^m_h u_m(x)$.

(2) Soit $\psi \in \mathcal{D}(\mathbb{R})$ une fonction positive, telle que $\operatorname{supp}\psi = [-2, +2]$ et $\psi(x) = x^m$ si $|x| \le 1$. On pose

$$|(\Delta^m_h \psi)(x)| \le \frac{1}{j^2}, \quad |x| \le \frac{1}{2}, \quad h = \frac{1}{4m}, \quad \forall j \in \mathbb{N}^*. \qquad \text{(E)}$$

Démontrer que (E) est impossible.

CORRECTION

5 pts. **Exercise 1** : See the lesson. (5)

10 pts. **Exercise 2** : **(1)** let $f \in A_p^u$, i.e. $\|f\|_{A_p^u} < +\infty$. (0,5)

$$\Rightarrow \left(\int_{-r}^{r} |f(x-y)|^p \, dy\right)^{1/p} \le \|f\|_{A_p^u} \, r^{1/p - 1/u}$$

$\frac{1}{p} - \frac{1}{u} < 0$ $\quad \lim_{r \to +\infty} r^{\frac{1}{p} - \frac{1}{u}} = 0 \Rightarrow \int_{-\infty}^{\infty} |f(x-y)|^p \, dy = 0$ (0,5)

1 i.e. $f \equiv 0$ a.e.

0,5 • $A_p^u = \{ f \in L_p^{loc} : \text{st. } \|f\|_{A_p^u} < +\infty \}$ and $1 \le p \le u \le \infty$. (0,5)

1 **(2)** • $p > q$ let $f \in A_p^u$. By Hölder ineq. $V = \frac{p}{q} > 1$

$$\int_{-r}^{r} |f(x-y)|^q \, dy \le \left(\int_{-r}^{r} |f(x-y)|^p \, dy\right)^{q/p} (2r)^{1 - q/p}$$ (0,5)

$$r^{\frac{1}{u} - \frac{1}{q}} \left(\int_{-r}^{r} |f(x-y)|^q \, dy\right)^{\frac{1}{q}} \le 2^{\frac{1}{q} - \frac{1}{p}} \, r^{\frac{1}{u} - \frac{1}{p}} \left(\int_{-r}^{r} |f(x-y)|^p \, dy\right)^{\frac{1}{p}}$$

taking "$\sup_{r>0} \sup_{x \in \mathbb{R}}$" : $\|f\|_{A_q^u} \le 2^{\frac{1}{q} - \frac{1}{p}} \|f\|_{A_p^u}$. (0,5)

1,5 • $$r^{\frac{1}{u} - \frac{1}{p}} \left(\int_{-r}^{r} |f(\lambda(x-y))|^p \, dy\right)^{\frac{1}{p}} = (\lambda r)^{\frac{1}{u} - \frac{1}{p}} \left(\int_{-\lambda r}^{\lambda r} |f(\lambda x - z)|^p \, dz\right)^{1/p} \lambda^{-\frac{1}{u}}$$

we put $\lambda r = r'$ and $\lambda x = x'$ (0,5)

taking $\sup_{r' > 0} \sup_{x' \in \mathbb{R}}$ we have

$$r^{\frac{1}{u} - \frac{1}{p}} \left(\int_{-r}^{r} |f(\lambda(x-y))|^p \, dy\right)^{\frac{1}{p}} \le \lambda^{-\frac{1}{u}} \|f\|_{A_p^u}$$ (1)

$$\Rightarrow \|f(\lambda \cdot)\|_{A_p^u} \le \lambda^{-\frac{1}{u}} \|f\|_{A_p^u}, \quad \forall \lambda > 0$$

Remark: We can prove $\|f(\lambda \cdot)\|_{A_p^u} = \lambda^{-1/u} \|f\|_{A_p^u}$ with the same method.

1 (3) • $A_p^{\infty} \supset L_{\infty}$:

$$r^{\frac{1}{\infty}-\frac{1}{p}}\left(\int_{-r}^{r}|f(x-y)|^p\,dy\right)^{\frac{1}{p}} \leq \|f\|_{\infty}\, r^{-\frac{1}{p}}\left(\int_{-r}^{r}dx\right)^{\frac{1}{p}} = 2^{\frac{1}{p}}\|f\|_{\infty}. \quad (1)$$

with sup : $\|f\|_{A_p^{\infty}} \leq 2^{1/p}\|f\|_{\infty}$.

1,5 • $A_p^{\infty} \subset L_{\infty}$:

$$|f(x)| = \frac{1}{2r}\int_{-r}^{r}|f(x)|\,dy \leq \frac{1}{2r}\int_{-r}^{r}|f(x)-f(x-y)|\,dy + \frac{1}{2r}\int_{-r}^{r}|f(x-y)|\,dy$$

$$\leq \frac{1}{2r}\int_{-r}^{r}|f(x)-f(x-y)|\,dy + \underbrace{2^{-1/p}\cdot r^{-1/p}\left(\int_{-r}^{r}|f(x-y)|^p\,dy\right)^{1/p}}_{\text{Hölder inequality}} \quad (0,5)$$

with $\sup\limits_{r>0}\ \sup\limits_{x\in\mathbb{R}}$

$$|f(x)| \leq \underbrace{\frac{1}{2r}\int_{-r}^{r}|f(x)-f(x-y)|\,dy}_{\longrightarrow 0 \text{ when } r\to 0 \text{ (Lebesgue diff thm.)}} + 2^{-1/p}\|f\|_{A_p^{\infty}} \quad (1)$$

$\Rightarrow$ $|f(x)| \leq 2^{-1/p}\|f\|_{A_p^{\infty}}$, $\forall\, x\in\mathbb{R}$ a.e.

1,5 (4) We change f by $f(\lambda\cdot)$ $(\lambda>0)$ then

$$\lambda^{s-1/r}\|f\|_{\dot{B}^{s}_{r,t}} \leq c\,\lambda^{-1/u}\|f\|_{A_p^{u}}, \quad \forall\, \lambda>0 \quad (1)$$

then $s-\frac{1}{r} = -\frac{1}{u}$ i.e. $s = \frac{1}{r}-\frac{1}{u}$. (0,5)

2 (5) $\|f\|_{\dot{B}^{s}_{\infty,\infty}} = \|f^{(m)}\|_{\dot{B}^{s-m}_{\infty,\infty}}$ (see the lesson) (1,5)

$\leq c\,\|f^{(m)}\|_{A_p^{u}}$ with $s-m = \frac{1}{\infty}-\frac{1}{u}$ (0,5)

then $s = m-\frac{1}{u}$.

5 pts. **Exercise 3 :**

2 (1) We know (see the lesson)

$$\Delta_h^m(u) = 0, \quad \forall u \text{ poly of } d^\circ u \leq m-1.$$ (0,5)

$$\Delta_h(x^m) = (x+h)^m - x^m = C_m^1 h x^{m-1} + g(x), \quad g \text{ poly } d^\circ = m-2$$

$$\Delta_h^m(x^m) = \Delta^{m-1}(\Delta_h x^m) = \Delta_h^{m-1}(C_m^1 h x^{m-1}) + \underbrace{\Delta_h^{m-1} g(x)}_{=0}$$ (0,5)

$$= C_m^1 h \cdot \Delta_h^{m-1} x^{m-1}$$

We put $v_m := \Delta_h^m x^m$ then $v_m = m h v_{m-1}$; $m \geq 2$

$v_1 = h$, $v_2 = 2 h v_1 = 2h^2$, $v_3 = 3 h v_2 = 3! h^3$, ... (1)

... $v_m = m! h^m$.

i.e $\Delta_h^m x^m = m! h^m$.

We can also calculate $\Delta_h^1 x$, $\Delta_h^2 x^2$, $\Delta_h^3 x^3$, ... etc

3 (2) $\Delta_h^m \varphi(x) = \sum_{k=0}^{m} (-1)^{m-k} C_m^k \varphi(x+kh)$

$$\left.\begin{array}{l} -\frac{1}{2} \leq x \leq \frac{1}{2} \\ h = \frac{1}{4m},\ 0 \leq k \leq m \end{array}\right\} \Rightarrow -1 < -\tfrac{1}{2} \leq x + kh < \tfrac{1}{2} + m \cdot \tfrac{1}{4m} < 1$$ (1)

$$\Rightarrow \varphi(x+kh) = (x+kh)^m$$

then $\Delta_h^m \varphi(x) = \Delta_h^m u_m(x) = m! h^m = m! \left(\frac{1}{4m}\right)^m$ (1)

i.e.

$|x| \leq \frac{1}{2}$, $h = \frac{1}{4m}$ $\quad |\Delta_h^m \varphi(x)| = m! (4m)^{-m} > 0.$

in (1) We take $j \to +\infty$ then $|\Delta_h^m \varphi(x)| = 0$ (1)

a contradiction.