University of M'Sila,
Faculty of Mathematics and Computer Science / Department of Mathematics
Master of Numerical and Mathematical Analysis / S 1
Distributions Analyse de Fourier 1

Final Exam, January 2025

*We use the notations of the lesson.

Exercise 1. (Lesson) Give the definition of :
– a regular distribution, – the norm of $\mathcal{D}(\mathbb{R})$, – the derivation in $\mathcal{D}'(\mathbb{R})$, – the space $\mathcal{E}'(\mathbb{R})$, – the convolution in $\mathcal{D}'(\mathbb{R})$, – the distribution $\mathrm{vp}\frac{1}{2x-3}$, – the distributions δ_0 and δ_3.

Exercise 2. (Independent questions)
(1) Define the Heaviside function $H(x)$, and calculate $H'(x)$.

(2) Let $f(x) = 1$ if $-1 \leq x \leq 1$ and $f(x) = 0$ if $|x| > 1$. Let $g(x) = 1$ if $x \geq 0$ and $g(x) = 0$ if $x < 0$. Calculate $f * g$.

(3) Calculate $x^m \delta_0''$ and $(x^m \delta_0'') * \delta_0''$ where $m \in \mathbb{N}^*$.

(4) Let $T_j = j\left(\delta_{1/j} - \delta_{-1/j}\right)$, $j \in \mathbb{N}^*$. Calculate $\lim_{j\to\infty} T_j$ in $\mathcal{D}'(\mathbb{R})$.

(5) Let $\varphi \in C^\infty(\mathbb{R})$, $\varphi(x) \geq 0$, increasing $\nearrow$, $\varphi(x) = 0$ if $x \leq -1$, $\varphi(x) = 1$ if $x \geq 1$.

We put $\psi(x) = \begin{cases} \varphi(x-1) & \text{if } x \leq 3 \\ \varphi(5-x) & \text{if } x \geq 3. \end{cases}$

Find the $\operatorname{supp}\psi$. Draw the graphs of φ and of ψ.

Exercise 3. (1) Let $g \in C^\infty(\mathbb{R})$. Calculate $g(x)\delta_{\mathrm{a}}$, where $\mathrm{a} \in \mathbb{R}$.

(2) Let $U = e^x H(x)$ and $V = e^{-3x} H(1-2x)$, where H is the Heaviside function.

Prove that $U, V \in \mathcal{D}'(\mathbb{R})$.
Calculate $U^{(m)}$ in $\mathcal{D}'(\mathbb{R})$, $m \in \mathbb{N}^*$.
Calculate V' and V'' in $\mathcal{D}'(\mathbb{R})$.

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Scale :
Ex. 1 : 4 pts. 0.5 + 0.5 + 0.5 + 0.5 + 0.5 + 0.5 + (0.5+0.5)
Ex. 2 : 10 pts. (0.5+1) + 2 + (1+1) + 2 + (1+0.5+1)
Ex. 3 : 6 pts. 0.5 + ((0.5+0.5) + 2 + (2+0.5))

Exercice 1. (Cours) Donner les définitions de :
– une distribution régulère, – la norme de $\mathcal{D}(\mathbb{R})$, – la dérivation dans $\mathcal{D}'(\mathbb{R})$, – l' espace $\mathcal{E}'(\mathbb{R})$, – la convolution dans $\mathcal{D}'(\mathbb{R})$, – la distribution $\mathrm{vp}\frac{1}{2x-3}$, – les distributions δ_0 et δ_3.

Exercice 2. (Questions indépendantes)
(1) Définir la fonction de Heaviside $H(x)$, et calculer $H'(x)$.

(2) Soit $f(x) = 1$ si $-1 \leq x \leq 1$ et $f(x) = 0$ si $|x| > 1$. Soit $g(x) = 1$ si $x \geq 0$ et $g(x) = 0$ si $x < 0$. Calculer $f * g$.

(3) Calculer $x^m \delta_0''$ et $(x^m \delta_0'') * \delta_0''$ où $m \in \mathbb{N}^*$.

(4) Soit $T_j = j\left(\delta_{1/j} - \delta_{-1/j}\right)$, $j \in \mathbb{N}^*$. Calculer $\lim_{j\to\infty} T_j$ dans $\mathcal{D}'(\mathbb{R})$.

(5) Soit $\varphi \in C^\infty(\mathbb{R})$, $\varphi(x) \geq 0$, croissante $\nearrow$, $\varphi(x) = 0$ si $x \leq -1$, $\varphi(x) = 1$ si $x \geq 1$.

On pose $\psi(x) = \begin{cases} \varphi(x-1) & \text{si} \quad x \leq 3 \\ \varphi(5-x) & \text{si} \quad x \geq 3. \end{cases}$

Trouver le $\operatorname{supp}\psi$. Tracer les graphes de φ et de ψ.

Exercice 3. **(1)** Soit $g \in C^\infty(\mathbb{R})$. Calculer $g(x)\delta_{\mathrm{a}}$, où $\mathrm{a} \in \mathbb{R}$.

(2) Soient $U = e^x H(x)$ et $V = e^{-3x} H(1-2x)$, où H est la fonction de Heaviside.

Démontrer que $U, V \in \mathcal{D}'(\mathbb{R})$.
Calculer $U^{(m)}$ dans $\mathcal{D}'(\mathbb{R})$, $m \in \mathbb{N}^*$.
Calculer V' and V'' dans $\mathcal{D}'(\mathbb{R})$.

CORRECTION

4 pts. **Exercise 1** : See the lesson.

10 pts. **Exercise 2** : (1) • $\langle H, \varphi\rangle = \int_0^\infty \varphi(x)\,dx, \ \forall \varphi \in \mathcal{D}(\mathbb{R})$.

1.5 • $H' = \delta_{x=0}$.

2 (2) $f*g(x) = \int_{-\infty}^{\infty} f(x-y)\,g(y)\,dy = \int_0^\infty f(x-y)\,dy$

$= \int_{-\infty}^{x} f(z)\,dz =: A$

$x<-1 \Rightarrow A = 0$

$-1 \le x \le 1 \Rightarrow A = \left(\int_{-\infty}^{-1} 0 + \int_{-1}^{x} 1\right) dz = 0 + x+1 = x+1$

$x>1 \Rightarrow A = \left(\int_{-\infty}^{-1} 0 + \int_{-1}^{1} 1 + \int_1^x 0\right) dz = 0+2+0 = 2$

$$\Rightarrow f*g(x) = \begin{cases} 0, & x<-1 \\ x+1, & -1\le x \le 1 \\ 2, & x>1. \end{cases}$$

2 (3) • $\langle x^m \delta_0'', \varphi\rangle = \langle \delta_0'', x^m\varphi\rangle = (-1)^2\langle \delta_0, (x^m\varphi)''\rangle = (x^m\varphi(x))''_{|x=0}$

$m=1 \Rightarrow (x\varphi(x))''_{|x=0} = (2\varphi'(x) + x\varphi''(x))_{|x=0} = 2\varphi'(0)$

$m\geq 2 \Rightarrow (x^m\varphi(x))''_{|x=0} = \left(m(m-1)x^{m-2}\varphi(x) + 2mx^{m-1}\varphi'(x) + x^m\varphi''(x)\right)_{|x=0}$

$$= \begin{cases} 0, & m\geq 3 \\ 2\varphi(0), & m=2 \end{cases}$$

$$\langle x^m\delta_0'', \varphi\rangle = \begin{cases} 2\varphi'(0), & m=1 \\ 2\varphi(0), & m=2 \\ 0, & m\geq 3 \end{cases} \Rightarrow x^m\delta_0'' = \begin{cases} -2\delta_0', & m=1 \\ 2\delta_0, & m=2 \\ 0, & m\geq 3. \end{cases}$$

• $(x^m\delta_0'') * \delta_0'' = (x^m\delta_0'' * \delta_0)''$

$$= (x^m\delta_0'')'' * \delta_0 = (x^m\delta_0'')'' = \begin{cases} -2\delta_0''', & m=1 \\ 2\delta_0'', & m=2 \\ 0, & m\geq 3. \end{cases}$$

2 (4) $\langle T_j, \varphi\rangle = j\left(\varphi(\tfrac{1}{j}) - \varphi(-\tfrac{1}{j})\right), \ \forall \varphi \in \mathcal{D}(\mathbb{R})$

$\varphi(1/j) = \varphi(0) + \frac{1}{j}\varphi'(0) + \frac{1}{2j^2}\varphi''(\theta/j), \quad 0<\theta<1,$

$\varphi(-1/j) = \varphi(0) - \frac{1}{j}\varphi'(0) + \frac{1}{2j^2}\varphi''(-\theta/j)$

2|3

$$\Rightarrow \langle T_j | \varphi \rangle = 2\varphi'(0) + \frac{1}{2j}\left(\varphi''(\theta/j) - \varphi''(-\theta/j)\right)$$

We have $\frac{1}{2j}\left|\varphi''\left(\frac{\theta}{j}\right) - \varphi''\left(-\frac{\theta}{j}\right)\right| \leq \frac{1}{j}\|\varphi''\|_\infty \xrightarrow[j\to+\infty]{} 0$

then $\langle T_j, \varphi \rangle \longrightarrow 2\varphi'(0) = \langle -2\delta_0', \varphi \rangle$

$$\lim_{j\to\infty} T_j = -2\delta_0' \text{ in } \mathcal{D}'(\mathbb{R}).$$

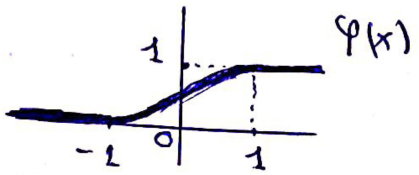
2,5 **(5)** $\varphi(x-1) = 0$: $x-1 \leq -1$: $x \leq 0$

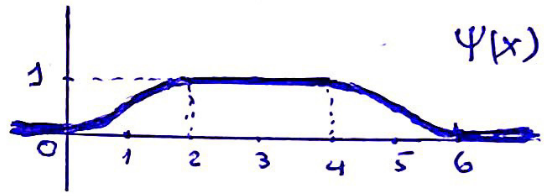
$\varphi(x-1) = 1$: $x-1 \geq 1$: $x \geq 2$ and $x \leq 3$

$\varphi(5-x) = 0$: $5-x \leq -1$: $x \geq 6$

$\varphi(5-x) = 1$: $5-x \geq 1$: $x \leq 4$ and $x \geq 3$

$\Rightarrow \operatorname{supp} \psi \subset [0,6]$ and $\psi(x) = 1$ if $2 \leq x \leq 4$.





6 pts. Exercise 3 : **(1)** $\langle g\delta_a | \varphi \rangle = \langle \delta_a | g\varphi \rangle$ since $g \in C^\infty$

0.5 then $\langle g\delta_a | \varphi \rangle = g(a)\varphi(a) = \langle g(a)\delta_a | \varphi \rangle$

$$g(x)\delta_a = g(a)\delta_a$$

1 **(2)** • $U, V \in \mathcal{D}'(\mathbb{R})$ easy (linear and since e^x, e^{-3x} are in $L^1_{Loc}(\mathbb{R}^+)$ and $L^1_{Loc}(]-\infty, \frac{1}{2}])$.

2 • $H' = \delta_{x=0}$ (see Ex.2)

$$U' = e^x H(x) + e^x \delta_0 = e^x H(x) + e^0 \delta_0$$

$$U' = e^x H(x) + \delta_0 = U + \delta_0$$

$$U'' = U' + \delta' = U + \delta_0 + \delta_0'$$

$$\cdots \quad U^{(m)} = U + \delta_0 + \delta_0' + \cdots + \delta_0^{(m-1)}.$$

2 • $V' = -3e^{-3x} H(1-2x) + e^{-3x}\left(H(1-2x)\right)'$

$\left(H(1-2x)\right)' = ?$ we have $H(1-2x) = \begin{cases} 1, & x \leq 1/2 \\ 0, & x > 1/2 \end{cases}$

$\Rightarrow \langle H(1-2x) \mid \varphi \rangle = \int_{-\infty}^{1/2} \varphi(x)\,dx$

$\langle (H(1-2x))' \mid \varphi \rangle = -\langle H(1-2x) \mid \varphi' \rangle$

$= -\int_{-\infty}^{1/2} \varphi'(x)\,dx = -\varphi(1/2) = \langle -\delta_{1/2} \mid \varphi \rangle$

$\Rightarrow (H(1-2x))' = -\delta_{1/2}$, in $\mathcal{D}'(\mathbb{R})$

$V' = -3e^{-3x} H(1-2x) + e^{-3x}\delta_{1/2}$,

$= -3e^{-3x} H(1-2x) - e^{-3/2}\delta_{1/2}$ (see (1))

$$\underline{V' = -3V - e^{-3/2}\delta_{1/2}.}$$

0.5 • $V'' = -3V' - e^{-3/2}\delta'_{1/2}$

$= -3\left(-3V - e^{-3/2}\delta_{1/2}\right) - e^{-3/2}\delta'_{1/2}$

$$\underline{= 9V + 3e^{-3/2}\delta_{1/2} - e^{-3/2}\delta'_{1/2}.}$$