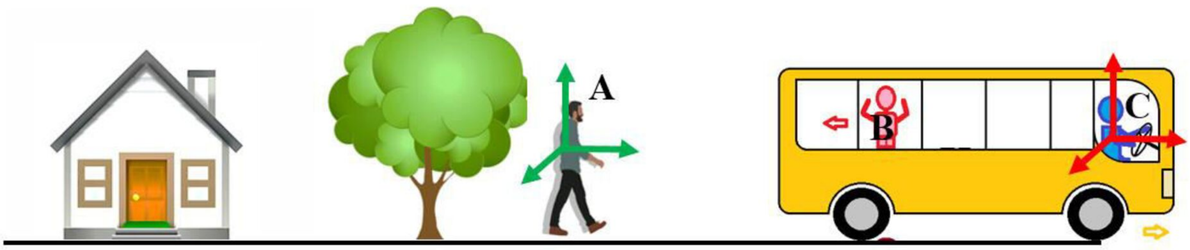
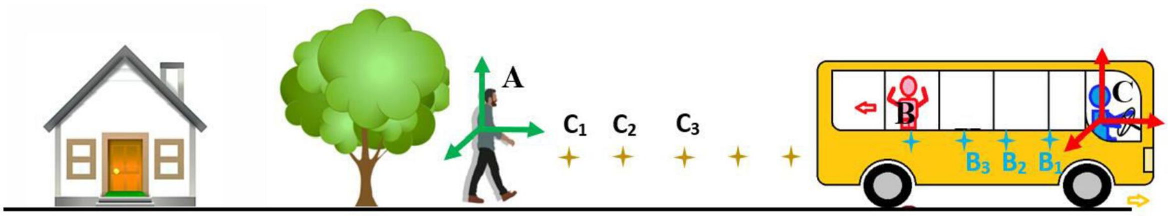
3) Relative motion

In this section, we will focus on studying the motion of a material particle relative to two reference frames, one of which is stationary (R'), and the other is moving (R).

As an example of relative motion, a person (A) sits under a tree to bid farewell to his friend (B), who is moving inside a bus that passes in front of him. In this case, the observer (A) sees that the movement of his friend (B) is composed of two movements: his accompanying movement and the one resulting from the movement of the bus. And its own movement inside the bus.



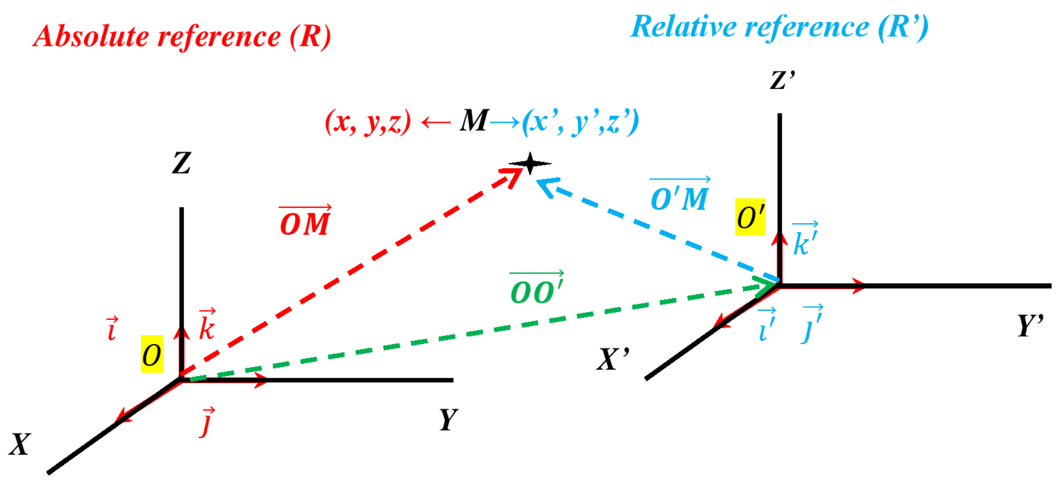
*As indicated in the figure below, the motion of person B (the passenger on the bus) with respect to person A (the observer) is a composite) superposition**)** motion. Therefore, their movement is influenced by the **positions**, **velocities**, **accelerations**, and the **path** of the moving reference, which is the bus. From the diagram, it is evident that the bus occupies different positions (C_1, C_2, C_3, etc.) concerning the stationary reference (observer). Simultaneously, person B moves to different positions (B_1, B_2, B_3,…, etc.) concerning the moving reference (observer C inside the bus).*



3-1) The absolute reference (stationary) and the relative reference (moving)

- *The stationary reference frame is called the **absolute** frame (R(O, x, y, z)), its origin **O**, and its unit vectors $\vec{i}$, $\vec{j}$, and $\vec{k}$ remain constant relative to the origin O over time.*
- *The coordinates of the particle **M** with respect to the absolute reference are denoted as x, y, z*

$$\left.\frac{d\,\vec{\imath}}{dt}\right|_R = \vec{0},\ \left.\frac{d\,\vec{\jmath}}{dt}\right|_R = \vec{0},\left.\frac{d\,\vec{k}}{dt}\right|_R = \vec{0}$$



- *The relative reference frame, is a moving reference with respect to the absolute frame, its origin O', and its unit vectors $\vec{\imath'}$, $\vec{\jmath'}$, and $\vec{k'}$ remain constant relative to the origin O' over time, and are not constant relative to the absolute frame.*

$$\left.\frac{d\,\vec{\imath'}}{dt}\right|_{R'} = \vec{0},\ \left.\frac{d\,\vec{\jmath'}}{dt}\right|_{R'} = \vec{0},\left.\frac{d\,\vec{k'}}{dt}\right|_{R'} = \vec{0}$$

$$\left.\frac{d\,\vec{\imath'}}{dt}\right|_{R} \neq \vec{0},\ \left.\frac{d\,\vec{\jmath'}}{dt}\right|_{R} \neq \vec{0},\left.\frac{d\,\vec{k'}}{dt}\right|_{R} \neq \vec{0}$$

- *The coordinates of the particle **M** with respect to the relative reference R' are denoted as x', y', z'.*

✓ *The motion of M concerning R is labelled as absolute motion.*

✓ *The motion of M concerning R' is identified as relative motion.*

✓ *The motion of R' in relation to R is termed entrained motion.*

3-2) Analysing Particle Motion within an Absolute Frame of Reference R(O, x, y, z)

The expression provides the position vector, velocity, and acceleration for the moving particle M in the fixed or absolute frame (R) as follows:

3-2-1) Position Vector

Position vector of the particle M with respect to the absolute (fixed) frame (R) is expressed as:

$$\overrightarrow{OM} = \overrightarrow{r_a} = x\,\vec{\imath} + y\vec{\jmath} + z\,\vec{k}$$

3-2-2) Velocity vector (Absolute Velocity $\overrightarrow{V_a}(t)$)

Velocity vector of the particle M with respect to the absolute frame (R) is called absolute velocity $\overrightarrow{V_a}(t)$ and is expressed as:

$$\overrightarrow{V_a}(t) = \left.\frac{d(\overrightarrow{r_a})}{dt}\right|_R = \left.\frac{d(\overrightarrow{OM})}{dt}\right|_R = \frac{d(x\,\vec{\imath} + y\vec{\jmath} + z\,\vec{k})}{dt} = \frac{dx}{dt}\vec{\imath} + \frac{dy}{dt}\vec{\jmath} + \frac{dz}{dt}\vec{k}$$

Where

$$\left.\frac{d\,\vec{\imath}}{dt}\right|_R = \vec{0},\ \left.\frac{d\,\vec{\jmath}}{dt}\right|_R = \vec{0},\ \left.\frac{d\,\vec{k}}{dt}\right|_R = \vec{0}$$

$$\overrightarrow{V_a}(t) = \frac{dx}{dt}\vec{\imath} + \frac{dy}{dt}\vec{\jmath} + \frac{dz}{dt}\vec{k} = \dot{x}\,\vec{\imath} + \dot{y}\,\vec{\jmath} + \dot{z}\,\vec{k}$$

$$\overrightarrow{V_a}(t) = V_{a,x}\vec{\imath} + V_{a,y}\vec{\jmath} + V_{a,z}\vec{k}\ \text{ where } \begin{cases} V_{a,x} = \dfrac{dx}{dt} \\ V_{a,y} = \dfrac{dy}{dt} \\ V_{a,z} = \dfrac{dz}{dt} \end{cases}$$

3-2-3) Acceleration vector (Absolute Acceleration $\overrightarrow{a_a}(t)$)

Acceleration vector of the particle M with respect to the absolute frame (R) is called absolute acceleration $\overrightarrow{a_a}(t)$ and is defined as:

$$\overrightarrow{a_a}(t) = \left.\frac{d(\overrightarrow{V_a})}{dt}\right|_R = \left.\frac{d^2(\overrightarrow{OM})}{d^2t}\right|_R = \frac{d(\dot{x}\,\vec{\imath} + \dot{y}\,\vec{\jmath} + \dot{z}\,\vec{k})}{dt} = \frac{d(V_{a,x})}{dt}\vec{\imath} + \frac{d(V_{a,y})}{dt}\vec{\jmath} + \frac{d(V_{a,z})}{dt}\vec{k}$$

$$\overrightarrow{a_a}(t) = \ddot{x}\,\vec{\imath} + \ddot{y}\,\vec{\jmath} + \ddot{z}\,\vec{k}$$

$$\overrightarrow{a_a}(t) == a_{a,x}\vec{\imath} + a_{a,y}\vec{\jmath} + a_{a,z}\vec{k}\ \text{ where } \begin{cases} a_{a,x} = \dfrac{d(V_{a,x})}{dt} \\ a_{a,y} = \dfrac{d(V_{a,y})}{dt} \\ a_{a,z} = \dfrac{d(V_{a,z})}{dt} \end{cases}$$

3-3) Analysing Particle Motion within an Relative Frame of Reference R'(O', x', y', z')

The expression of the position vector, velocity, and acceleration for the moving particle M in the relative (moving) frame (R') are given as follows:

3-3-1) Position Vector

Position vector of the particle M with respect to the relative (moving) frame (R') is given as:

$$\overrightarrow{O'M} = \vec{r_r} = x'\vec{\imath'} + y'\vec{\jmath'} + z'\vec{k'}$$

3-3-2) Velocity vector (Relative Velocity $\vec{V_r}(t)$)

Velocity vector of the particle M with respect to the absolute frame (R') is called absolute velocity $\vec{V_r}(t)$ *and is expressed as:*

$$\vec{V_r}(t) = \frac{d(\vec{r_r})}{dt}\bigg|_{R'} = \frac{d(\overrightarrow{O'M})}{dt}\bigg|_{R'} = \frac{d(x'\vec{\imath'} + y'\vec{\jmath'} + z'\vec{k'})}{dt} = \frac{dx'}{dt}\vec{\imath'} + \frac{dy'}{dt}\vec{\jmath'} + \frac{dz'}{dt}\vec{k'}$$

Where

$$\frac{d\vec{\imath'}}{dt}\bigg|_{R'} = \vec{0},\ \frac{d\vec{\jmath'}}{dt}\bigg|_{R'} = \vec{0},\ \frac{d\vec{k'}}{dt}\bigg|_{R'} = \vec{0}$$

$$\vec{V_r}(t) = \frac{dx'}{dt}\vec{\imath'} + \frac{dy'}{dt}\vec{\jmath'} + \frac{dz'}{dt}\vec{k'} = \dot{x}'\vec{\imath'} + \dot{y}'\vec{\jmath'} + \dot{z}'\vec{k'}$$

$$\vec{V_r}(t) == V_{r,x'}\vec{\imath'} + V_{r,y'}\vec{\jmath'} + V_{r,z'}\vec{k'} \text{ where } \begin{cases} V_{r,x'} = \frac{dx'}{dt} \\ V_{r,y'} = \frac{dy'}{dt} \\ V_{r,z'} = \frac{dz'}{dt} \end{cases}$$

3-3-3) Acceleration vector (Relative Acceleration $\vec{a_r}(t)$)

The relative Acceleration vector $\vec{a_r}(t)$ *of the particle M with respect to the relative frame (R') is written as:*

$$\vec{a_r}(t) = \frac{d(\vec{V_r})}{dt}\bigg|_{R'} = \frac{d^2(\overrightarrow{O'M})}{d^2t}\bigg|_{R'} = \frac{d(\dot{x}'\vec{\imath}' + \dot{y}'\vec{\jmath'} + \dot{z}'\vec{k'})}{dt}$$

$$= \frac{d(V_{r,x'})}{dt}\vec{\imath'} + \frac{d(V_{r,y'})}{dt}\vec{\jmath'} + \frac{d(V_{r,z'})}{dt}\vec{k'}$$

$$\vec{a_r}(t) = \ddot{x}'\,\vec{\imath}' + \ddot{y}'\,\vec{\jmath}' + \ddot{z}'\,\vec{k}'$$

$$\vec{a_a}(t) = a_{r,x'}\vec{\imath}' + a_{r,y'}\vec{\jmath}' + a_{r,z'}\vec{k}' \text{ where } \begin{cases} a_{r,x'} = \dfrac{d(V_{r,x'})}{dt} \\ a_{r,y'} = \dfrac{d(V_{r,y'})}{dt} \\ a_{r,z'} = \dfrac{d(V_{r,z'})}{dt} \end{cases}$$

3-4) Relation between relative and absolute motions

Based on the principles of composition, it is possible to derive the mathematical formula that establishes the connections between the components of the position vector (velocity and acceleration) for the moving particle M in the absolute frame (R) and the components of the position vector (velocity and acceleration) for the moving particle M in a relative frame (R').

3-4-1) Relation between position Vectors ($\overrightarrow{OM}$ and $\overrightarrow{O'M}$)

The position vector ($\overrightarrow{OM}$) for the particle M in the absolute frame (R) is equal to the sum of the position vector ($\overrightarrow{O'M}$) for particle M in the relative frame (R) and the origin position vector ($\overrightarrow{OO'}$) of the relative frame with respect to the absolute frame. The relation between both vectors can be written according to the following expression:

$$\overrightarrow{OM} = \overrightarrow{OO'} + \overrightarrow{O'M}$$

$$\overrightarrow{OM} = x\,\vec{\imath} + y\vec{\jmath} + z\,\vec{k} = \overrightarrow{OO'} + x'\,\vec{\imath}' + y'\,\vec{\jmath}' + z'\,\vec{k}'$$

$$\overrightarrow{OM} = x\,\vec{\imath} + y\vec{\jmath} + z\,\vec{k} = x_{O'}\vec{\imath} + y_{O'}\vec{\jmath} + z_{O'}\vec{k} + x'\,\vec{\imath}' + y'\,\vec{\jmath}' + z'\,\vec{k}'$$

Where ($x_{O'}, y_{O'}, z_{O'}$) are the coordinates of the origin O' with respect to the absolute reference frame (R).

In the case of irregular motion of the moving frame (R'), the unit vectors are unequal ($\vec{\imath} \neq \vec{\imath}', \vec{\jmath} \neq \vec{\jmath}'$, and $\vec{k} \neq \vec{k}'$)

3-4-2) Relation between velocity Vectors ($\vec{V_a}$ and $\vec{V_r}$)

When we derive the position vector for the moving particle M with respect to the absolute reference(R), we obtain the expression for the absolute velocity vector as follows:

$$\vec{V_a}(t) = \frac{d(\overrightarrow{OM})}{dt}\bigg|_R = \frac{d(\overrightarrow{OO'} + \overrightarrow{O'M})}{dt}\bigg|_R = \frac{d(\overrightarrow{OO'})}{dt}\bigg|_R + \frac{d(\overrightarrow{O'M})}{dt}\bigg|_R$$

$$\vec{V_a}(t) = \frac{d(\overrightarrow{OO'})}{dt}\bigg|_R + \frac{d(x'\vec{\imath'} + y'\vec{\jmath'} + z'\vec{k'})}{dt}\bigg|_R$$

The unit vectors ($\vec{\imath'}, \vec{\jmath'}$, and $\vec{k'}$) of the relative frame are constant with respect to the origin O' of the relative frame (R')($\frac{d\,\vec{\imath'}}{dt}\Big|_{R'} = \vec{0}$, $\frac{d\,\vec{\jmath'}}{dt}\Big|_{R'} = \vec{0}$, $\frac{d\,\vec{k'}}{dt}\Big|_{R'} = \vec{0}$), however, they are not constant with respect to the absolute frame (R). Therefore, the derivative of these unit vectors ($\vec{\imath'}, \vec{\jmath'}$, and $\vec{k'}$) with respect to the absolute frame (R) does not equal the zero vector ($\frac{d\,\vec{\imath'}}{dt}\Big|_R \neq \vec{0}$, $\frac{d\,\vec{\jmath'}}{dt}\Big|_R \neq \vec{0}$, $\frac{d\,\vec{k'}}{dt}\Big|_R \neq \vec{0}$).

$$\vec{V_a}(t) = \frac{d(\overrightarrow{OO'})}{dt}\bigg|_R + \frac{d(x')}{dt}\vec{\imath'} + x'\frac{d(\vec{\imath'})}{dt} + \frac{d(y')}{dt}\vec{\jmath'} + y'\frac{d(\vec{\jmath'})}{dt} + \frac{d(z')}{dt}\vec{k'} + z'\frac{d(\vec{k'})}{dt}$$

We rearrange the equation, and we get the following:

$$\vec{V_a}(t) = \underbrace{\frac{d(\overrightarrow{OO'})}{dt}\bigg|_R + x'\frac{d\vec{\imath'}}{dt} + y'\frac{d\vec{\jmath'}}{dt} + z'\frac{d\vec{k'}}{dt}}_{\vec{V_e}} + \underbrace{\frac{dx'}{dt}\vec{\imath'} + \frac{dy'}{dt}\vec{\jmath'} + \frac{dz'}{dt}\vec{k'}}_{\vec{V_r}}$$

$$\vec{V_a}(t) = \underbrace{\frac{d(\overrightarrow{OO'})}{dt}\bigg|_R + x'\frac{d\vec{\imath'}}{dt} + y'\frac{d\vec{\jmath'}}{dt} + z'\frac{d\vec{k'}}{dt}}_{\vec{V_e}} + \underbrace{\dot{x}'\vec{\imath'} + \dot{y}'\vec{\jmath'} + \dot{z}'\vec{k'}}_{\vec{V_r}}$$

Hence, the absolute velocity vector ($\vec{V_a}(t)$) of the moving particle M is the sum of the relative ($\vec{V_r}(t)$) velocity vector and the entrained velocity vector $\vec{V_e}(t)$.

$$\vec{V_a}(t) = \vec{V_r}(t) + \vec{V_e}(t) \quad where \begin{cases} \vec{V_e}(t) = \frac{d(\overrightarrow{OO'})}{dt}\bigg|_R + x'\frac{d\vec{\imath'}}{dt} + y'\frac{d\vec{\jmath'}}{dt} + z'\frac{d\vec{k'}}{dt} \\ \vec{V_r}(t) = \frac{dx'}{dt}\vec{\imath'} + \frac{dy'}{dt}\vec{\jmath'} + \frac{dz'}{dt}\vec{k'} \end{cases}$$

We can further develop the expression for the entrained velocity

$$\frac{d(\vec{\imath'})}{dt}\bigg|_R = \frac{d\theta}{dt}\vec{\jmath'} \text{ where } \frac{d\theta}{dt} = \omega \text{ (angular velocity) and } \vec{\jmath'} = \vec{k'} \wedge \vec{\imath'}$$

$$\left.\frac{d(\vec{\imath'})}{dt}\right|_R = \frac{d\theta}{dt}\vec{\jmath'} = \omega\,\vec{k'} \wedge \vec{\imath'} = \vec{\omega} \wedge \vec{\imath'} \quad where = \omega\,\vec{k'} = \vec{\omega}$$

Using the same method we obtain

$$\left.\frac{d(\vec{\jmath'})}{dt}\right|_R = \vec{\omega} \wedge \vec{\jmath'} \quad and \quad \left.\frac{d(\vec{k'})}{dt}\right|_R = \vec{\omega} \wedge \vec{k'}$$

And therefore the entrained velocity can be written:

$$\vec{V_e}(t) = \left.\frac{d(\overrightarrow{OO'})}{dt}\right|_R + x'\vec{\omega} \wedge \vec{\imath'} + y'\,\vec{\omega} \wedge \vec{\jmath'} + z'\,\vec{\omega} \wedge \vec{k'}$$

$$\vec{V_e}(t) = \left.\frac{d(\overrightarrow{OO'})}{dt}\right|_R + \vec{\omega} \wedge (x'\vec{\imath'}) + \vec{\omega} \wedge (y'\,\vec{\jmath'}) + \vec{\omega} \wedge (z'\,\vec{k'})$$

$$\vec{V_e}(t) = \left.\frac{d(\overrightarrow{OO'})}{dt}\right|_R + \vec{\omega} \wedge \left(x'\vec{\imath'} + y'\,\vec{\jmath'} + z'\,\vec{k'}\right)$$

$$\vec{V_e}(t) = \left.\frac{d(\overrightarrow{OO'})}{dt}\right|_R + \vec{\omega} \wedge \overrightarrow{O'M}$$

The entrained velocity vector ($\vec{V_e}(t)$) of the relative reference frame is related, on one hand, to the velocity vector of the origin O' with respect to the absolute reference frame, and on the other hand, to the rotation of the unit vectors of the relative reference frame with respect to the absolute reference frame. Therefore, we must take into account both the translational and rotational motions of the relative reference frame.

3-4-3) Relation between acceleration Vectors ($\vec{a_a}$ and $\vec{a_r}$)

We derive the expression for absolute velocity with respect to time then, we simplify the obtained expressions as follows:

$$\vec{a_a}(t) = \frac{d}{dt}\left\{\left.\frac{d(\overrightarrow{OO'})}{dt}\right|_R + x'\frac{d\vec{\imath'}}{dt} + y'\frac{d\vec{\jmath'}}{dt} + z'\frac{d\vec{k'}}{dt} + \dot{x}'\,\vec{\imath'} + \dot{y}'\,\vec{\jmath'} + \dot{z}'\,\vec{k'}\right\}$$

$$\vec{a_a}(t) = \left.\frac{d^2(\overrightarrow{OO'})}{d^2t}\right|_R + \frac{dx'}{dt}\frac{d\vec{\imath'}}{dt} + x'\frac{d^2\vec{\imath'}}{d^2t} + \frac{dy'}{dt}\frac{d\vec{\jmath'}}{dt} + y'\frac{d^2\vec{\jmath'}}{d^2t} + \frac{dz'}{dt}\frac{d\vec{k'}}{dt} + z'\frac{d^2\vec{k'}}{d^2t}$$

$$+d\frac{\dot{x}'}{dt}\vec{\imath'}+\dot{x}'\frac{d\vec{\imath'}}{dt}+d\frac{\dot{y}'}{dt}\vec{\jmath'}+\dot{y}'\frac{d\vec{\jmath'}}{dt}+d\frac{\dot{z}'}{dt}\vec{k'}+\dot{z}'\frac{d\vec{k'}}{dt}$$

We further simplify this equation and rearrange its terms, we obtain the following expression:

$$\vec{a_a}(t)=\left.\frac{d^2(\overrightarrow{OO'})}{d^2t}\right|_R+\dot{x}'\frac{d\vec{\imath'}}{dt}+\dot{y}'\frac{d\vec{\jmath'}}{dt}+\dot{z}'\frac{d\vec{k'}}{dt}+x'\frac{d^2\vec{\imath'}}{d^2t}+y'\frac{d^2\vec{\jmath'}}{d^2t}+z'\frac{d^2\vec{k'}}{d^2t}$$

$$+\ddot{x}'\vec{\imath'}+\ddot{y}'\vec{\jmath'}+\ddot{z}'\vec{k'}+\dot{x}'\frac{d\vec{\imath'}}{dt}+\dot{y}'\frac{d\vec{\jmath'}}{dt}+\dot{z}'\frac{d\vec{k'}}{dt}$$

Finally, we obtain the following formula:

$$\vec{a_a}(t)=\underbrace{\left.\frac{d^2(\overrightarrow{OO'})}{d^2t}\right|_R+x'\frac{d^2\vec{\imath'}}{d^2t}+y'\frac{d^2\vec{\jmath'}}{d^2t}+z'\frac{d^2\vec{k'}}{d^2t}}_{\vec{a_e}}+\underbrace{2\left(\dot{x}'\frac{d\vec{\imath'}}{dt}+\dot{y}'\frac{d\vec{\jmath'}}{dt}+\dot{z}'\frac{d\vec{k'}}{dt}\right)}_{\vec{a_c}}+\underbrace{\ddot{x}'\vec{\imath'}+\ddot{y}'\vec{\jmath'}+\ddot{z}'\vec{k'}}_{\vec{a_r}}$$

Consequently, the absolute acceleration vector ($\vec{a_a}$) for the moving particle M is the summation of the relative acceleration vector ($\vec{a_r}$), the entrained acceleration vector ($\vec{a_e}$), and the Coriolis acceleration vector ($\vec{a_c}$).

$$\vec{a_a}(t)=\vec{a_r}(t)+\vec{a_e}(t)+\vec{a_c}(t)\quad where\begin{cases}\vec{a_r}(t)=d\frac{\dot{x}'}{dt}\vec{\imath'}+d\frac{\dot{y}'}{dt}\vec{\jmath'}+d\frac{\dot{z}'}{dt}\vec{k'}\\ \vec{a_e}(t)=\left.\frac{d^2(\overrightarrow{OO'})}{d^2t}\right|_R+x'\frac{d^2\vec{\imath'}}{d^2t}+y'\frac{d^2\vec{\jmath'}}{d^2t}+z'\frac{d^2\vec{k'}}{d^2t}\\ \vec{a_c}(t)=2\left(\dot{x}'\frac{d\vec{\imath'}}{dt}+\dot{y}'\frac{d\vec{\jmath'}}{dt}+\dot{z}'\frac{d\vec{k'}}{dt}\right)\end{cases}$$

We can further simplify the Coriolis acceleration vector $\vec{a_c}(t)$ and the entrained acceleration vector $\vec{a_e}(t)$ as follows:

- *The Coriolis acceleration vector $\vec{a_c}(t)$*

  Using these relations

$$\left.\frac{d(\vec{\imath'})}{dt}\right|_R=\vec{\omega}\wedge\vec{\imath'}\ ,\quad \left.\frac{d(\vec{\jmath'})}{dt}\right|_R=\vec{\omega}\wedge\vec{\jmath'}\quad and\quad \left.\frac{d(\vec{k'})}{dt}\right|_R=\vec{\omega}\wedge\vec{k'}$$

$$\vec{a_c}(t)=2\left(\dot{x}'\frac{d\vec{\imath'}}{dt}+\dot{y}'\frac{d\vec{\jmath'}}{dt}+\dot{z}'\frac{d\vec{k'}}{dt}\right)=2\left(\dot{x}'\vec{\omega}\wedge\vec{\imath'}+\dot{y}'\vec{\omega}\wedge\vec{\jmath'}+\dot{z}'\vec{\omega}\wedge\vec{k'}\right)$$

$$\vec{a_c}(t) = 2\left(\vec{\omega} \wedge (\dot{x}'\vec{i'} + \dot{y}'\vec{j'} + \dot{z}'\vec{k'})\right) = 2\vec{\omega} \wedge \vec{V_r}$$

- *The entrained acceleration vector* $\vec{a_e}(t)$

$$\vec{a_e}(t) = \left.\frac{d^2(\overrightarrow{OO'})}{d^2t}\right|_R + x'\frac{d^2\vec{i'}}{d^2t} + y'\frac{d^2\vec{j'}}{d^2t} + z'\frac{d^2\vec{k'}}{d^2t}$$

Using these relations

$$\begin{cases} \dfrac{d^2\vec{i'}}{d^2t} = \dfrac{d}{dt}\left(\vec{\omega} \wedge \vec{i'}\right) = \dfrac{d\vec{\omega}}{dt} \wedge \vec{i'} + \vec{\omega} \wedge \dfrac{d\vec{i'}}{dt} = \dfrac{d\vec{\omega}}{dt} \wedge \vec{i'} + \vec{\omega} \wedge \vec{\omega} \wedge \vec{i'} \\ \dfrac{d^2\vec{j'}}{d^2t} = \dfrac{d}{dt}\left(\vec{\omega} \wedge \vec{j'}\right) = \dfrac{d\vec{\omega}}{dt} \wedge \vec{j'} + \vec{\omega} \wedge \dfrac{d\vec{j'}}{dt} = \dfrac{d\vec{\omega}}{dt} \wedge \vec{j'} + \vec{\omega} \wedge \vec{\omega} \wedge \vec{j'} \\ \dfrac{d^2\vec{k'}}{d^2t} = \dfrac{d}{dt}\left(\vec{\omega} \wedge \vec{k'}\right) = \dfrac{d\vec{\omega}}{dt} \wedge \vec{k'} + \vec{\omega} \wedge \dfrac{d\vec{k'}}{dt} = \dfrac{d\vec{\omega}}{dt} \wedge \vec{k'} + \vec{\omega} \wedge \vec{\omega} \wedge \vec{k'} \end{cases}$$

$$\vec{a_e}(t) = \left.\frac{d^2(\overrightarrow{OO'})}{d^2t}\right|_R + x'\left(\frac{d\vec{\omega}}{dt} \wedge \vec{i'} + \vec{\omega} \wedge \left(\vec{\omega} \wedge \vec{i'}\right)\right) + y'\left(\frac{d\vec{\omega}}{dt} \wedge \vec{j'} + \vec{\omega} \wedge (\vec{\omega} \wedge \vec{j'})\right) + z'\left(\frac{d\vec{\omega}}{dt} \wedge \vec{k'} + \vec{\omega} \wedge (\vec{\omega} \wedge \vec{k'})\right)$$

$$\vec{a_e}(t) = \left.\frac{d^2(\overrightarrow{OO'})}{d^2t}\right|_R + \left(\frac{d\vec{\omega}}{dt} \wedge x'\vec{i'} + \vec{\omega} \wedge \left(\vec{\omega} \wedge x'\vec{i'}\right)\right) + \left(\frac{d\vec{\omega}}{dt} \wedge y'\vec{j'} + \vec{\omega} \wedge (\vec{\omega} \wedge y'\vec{j'})\right) + \left(\frac{d\vec{\omega}}{dt} \wedge z'\vec{k'} + \vec{\omega} \wedge (\vec{\omega} \wedge z'\vec{k'})\right)$$

$$\vec{a_e}(t) = \left.\frac{d^2(\overrightarrow{OO'})}{d^2t}\right|_R + \left(\frac{d\vec{\omega}}{dt} \wedge x'\vec{i'}\right) + \left(\frac{d\vec{\omega}}{dt} \wedge y'\vec{j'}\right) + \left(\frac{d\vec{\omega}}{dt} \wedge z'\vec{k'}\right) + \vec{\omega} \wedge \left(\vec{\omega} \wedge x'\vec{i'}\right) + \vec{\omega} \wedge \left(\vec{\omega} \wedge y'\vec{j'}\right)$$
$$+ \left(\vec{\omega} \wedge (\vec{\omega} \wedge z'\vec{k'})\right)$$

$$\vec{a_e}(t) = \left.\frac{d^2(\overrightarrow{OO'})}{d^2t}\right|_R + \left(\frac{d\vec{\omega}}{dt} \wedge (x'\vec{i'} + y'\vec{j'} + z'\vec{k'}\right) + \vec{\omega} \wedge \left(\vec{\omega} \wedge (x'\vec{i'} + y'\vec{j'} + z'\vec{k'}\right)$$

$$\vec{a_e}(t) = \left.\frac{d^2(\overrightarrow{OO'})}{d^2t}\right|_R + \left(\frac{d\vec{\omega}}{dt} \wedge (\overrightarrow{O'M}\right) + \vec{\omega} \wedge \left(\vec{\omega} \wedge (\overrightarrow{O'M}\right)$$

Note: If the relative reference moves in a translation motion with respect to the absolute reference ($\vec{\omega} = \vec{0}$), then:

- $\vec{V_e}(t) = \left.\frac{d(\overrightarrow{OO'})}{dt}\right|_R$
- $\vec{a_c}(t) = 2\left(\vec{\omega} \wedge (\dot{x}'\vec{i'} + \dot{y}'\vec{j'} + \dot{z}'\vec{k'})\right) = \vec{0}$
- $\vec{a_e}(t) = \left.\frac{d^2(\overrightarrow{OO'})}{d^2t}\right|_R$