CHAPTER 1: MATHEMATICAL BACKGROUND

I- Introduction

The word "physics" originates from Ancient Greek *physikḗ*, meaning "knowledge of nature. Physics is the is the natural science of matter, involving the study of matter, its fundamental constituents, its motion and behaviour through space and time, and the related entities of energy and force. Physics is one of the most fundamental scientific disciplines, its main goal being to understand how the universe behaves.

Physics can be divided into two main classes: classical physics and modern physics:

- Classical physics is generally concerned with matter and energy on the normal scale of observation.
- Modern physics is concerned with the behaviour of matter and energy under extreme conditions or on a very large or very small scale.

In physics, we use two types of quantities: scalar quantities and vector quantities:

- Scalar physical quantities are entirely defined by a number and an appropriate unit; e.g. the mass m of a body, the length l of an object, etc.
- Vector physical quantity is a quantity specified by a number and an appropriate unit plus a direction (geometrically represented by a vector); e.g. velocity $\vec{V}$, weight $\vec{P}$...

Vector is denoted as a symbol with an arrow over the top: $\vec{V}$

II- Reference system

A reference frame is a system of coordinate axes, linked to an observer equipped with a clock. The notion of motion is not absolute but relative to the frame of reference in which it is described.

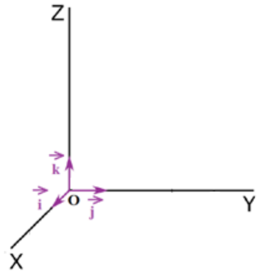
II.1 - Copernicus referential

It is the frame of reference whose origin is at the barycentre of the solar system and whose axes point towards the distant stars known as the "fixed stars".

II.2 - Galilean reference frame

The Galilean reference frames are the reference frames with uniform rectilinear motion relative to Copernicus' reference frame.

In the Galilean reference frame, there are several reference points. One of these is the Cartesian reference frame which consists of an origin O and three orthogonal vectors of unit norm (orthonormal) which are non-collinear ($\vec{i}$, $\vec{j}$, $\vec{k}$). The three vectors determine the three usual directions in space (OX, OY, OZ).



III- Material point

In physics, a point generally refers to an object that has no size or shape and is represented by a single coordinate or location in space. It is used as a basic concept in geometry and can be used to describe the position of an object in space by three coordinates (x, y, z) (in three-dimensional space).

x = **abscissa**, y = **ordinate**, z = **coast**

IV- Physical quantities and units

IV.1 - International System of Units (SI)

Internationally known by the abbreviation **SI** (abbreviated **SI** from its French name *Système International a*nd popularly known as the *metric system)* is the world's most widely used system of measurement. It is established and maintained by the General Conference on Weights and Measures (CGPM).

- MKSA system

The **MKS** system of units is a physical system of measurement that uses the meter, kilogram, and second (MKS) as base units. The MKS system with the ampere as a fourth base unit is sometimes referred to as the **MKSA** system. This system was extended by adding the Kelvin and candela as base units in 1960, thus forming the International System of Units. The mole was added as a seventh base unit in 1971.

Table 1: SI base units.

Quantity	**Name**	**Symbol**
Time	second	**s**
Length	meter	**m**
Mass	kilogram	**kg**
Electric current	ampere	**A**
Temperature	kelvin	**K**
Amount of substance	mole	**mol**
Luminous intensity	candela	**cd**

- CGS system of units

The **centimeter–gram–second system of units** (abbreviated **CGS** or **cgs**) is a variant of the metric system based on the centimeter as the unit of length, the gram as the unit of mass, and the second as the unit of time.

IV.2- Derived units

Derived units are associated with derived quantities; for example, velocity is a quantity that is derived from the base quantities of time and length, and thus the SI derived unit is metre per second (symbol m/s).

Table 2: SI derived units with special names and symbols.

Name	**Symbol**	**Quantity**	**In SI base units**	**In other SI units**
radian	rad	plane angle	m/m	1
hertz	Hz	frequency	s^{-1}	
newton	N	force, weight	$kg{\cdot}m{\cdot}s^{-2}$	
pascal	Pa	pressure, stress	$kg{\cdot}m^{-1}{\cdot}s^{-2}$	$N/m^2 = J/m^3$
joule	J	energy, work, heat	$kg{\cdot}m^2{\cdot}s^{-2}$	$N{\cdot}m = Pa{\cdot}m^3$
watt	W	power, radiant flux	$kg{\cdot}m^2{\cdot}s^{-3}$	J/s
coulomb	C	electric charge	$s{\cdot}A$	
volt	V	electric potential, voltage, emf	$kg{\cdot}m^2{\cdot}s^{-3}{\cdot}A^{-1}$	W/A = J/C
farad	F	capacitance	$kg^{-1}{\cdot}m^{-2}{\cdot}s^4{\cdot}A^2$	$C/V = C^2/J$
ohm	Ω	resistance, impedance, reactance	$kg{\cdot}m^2{\cdot}s^{-3}{\cdot}A^{-2}$	$V/A = J{\cdot}s/C^2$
siemens	S	electrical conductance	$kg^{-1}{\cdot}m^{-2}{\cdot}s^3{\cdot}A^2$	Ω^{-1}
weber	Wb	magnetic flux	$kg{\cdot}m^2{\cdot}s^{-2}{\cdot}A^{-1}$	$V{\cdot}s$
tesla	T	magnetic flux density	$kg{\cdot}s^{-2}{\cdot}A^{-1}$	Wb/m^2
henry	H	inductance	$kg{\cdot}m^2{\cdot}s^{-2}{\cdot}A^{-2}$	Wb/A
degree Celsius	°C	temperature relative to 273.15 K	K	

IV.3- Dimensional analysis

The **dimensional analysis** is the relationships between different physical quantities by identifying their base quantities (such as length, mass, time, and electric current) and units of measurement (such as meters and grams). It can be used to evaluate scientific formulae.

Quantity	Unit	Dimension symbol
Length	metre (m)	**[L]**
Time	second (s)	**[T]**
Mass	kilogram (kg)	**[M]**
Current	amp (A)	**[I]**
Temperature	Kelvin (K)	**[Θ]**

Any derived quantity G can be expressed as a function of the fundamental quantities M, L, T and I according to the expression:

$$[G] = [M]^a.[L]^b.[T]^c.[I]^d$$

Where *a*, *b*, *c*, *d* are the dimensional exponents.

Example 1

- **Velocity** = length/ time

⇒ [V] = $[L]\ [T]^{-1}$

- **Force** = mass x acceleration = masse x length /time^2

⇒ [F]= $[M]\ [L]\ [T]^{-2}$

- **Energy**= Force*displacement= mass*acceleration*displacement

⇒ [E] = $[M]\ [L]\ [T]^{-2}[L]$= $[M]\ [L]^2\ [T]^{-2}$

Example 2

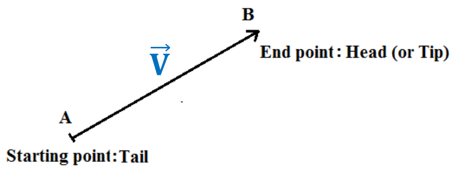
Use dimensional analysis to show that the expressions of kinetic energy $E_k = \frac{1}{2}\ mv^2$ and potential energy $E_p = m.g.h$ are correct and have the same dimension, where v represent velocity, g is gravitational acceleration, h is height and m is mass.

Example 3

The frequency of vibration *f* of a mass *m* at the end of spring that has a stiffness constant *k* is related to *m* and *k* by a relation of the form $f = (constant)m^a k^b$. Use dimensional analysis to find *a* and *b*. it is known that [*f*]=$[T]^{-1}$ and [k] = $[M][T]^{-2}$.

V- Vector

A vector $\vec{\mathbf{V}}$ (or $\overrightarrow{AB}$) is represented by a directed segment (an arrow pointing in the direction of the vector) with a starting point (tail) (A) and an end point (head or tip) (B).



A vector is defined by:

- a length (a magnitude): $\|\vec{V}\|$ ($\|\vec{V}\| = \|\overrightarrow{AB}\|$,

- a support: **the straight line (AB)**,

- a direction (orientation): **from A to B**.

We can define a vector $\vec{V}$ as follows: $\vec{V} = \|\vec{V}\|\,\vec{u}$

$\vec{u}$ is the unit vector with norm 1 ($\|\vec{u}\|$ =1) and the same direction as $\vec{V}$

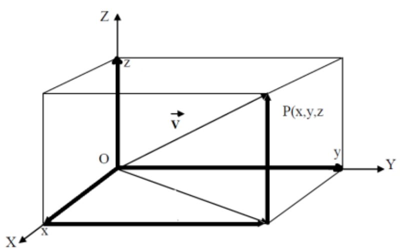
Noted that any vector can be broken down into components x, y and z along the X, Y and Z axes:

$$\vec{V} = \overrightarrow{OP} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k}$$

$$\|\vec{V}\| = V = \left(\sqrt{(x)^2 + (y)^2 + (z)^2}\right)$$

- If the tail is at the Origin O:

$$\vec{V} = \overrightarrow{OP} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k}$$

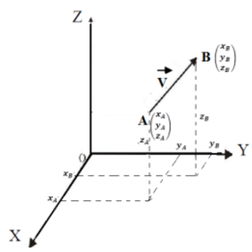


x, y and z are also the coordinates of point P in the direction of OX, OY and OZ, respectively.

- If the tail is not at the Origin:

$$\vec{V} = \overrightarrow{AB} = (x_B - x_A)\vec{i} + (y_B - y_A)\vec{j} + (z_B - z_A)\vec{k}$$

$$(A = \begin{pmatrix} x_A \\ y_A \\ z_A \end{pmatrix},\ B = \begin{pmatrix} x_B \\ y_B \\ z_B \end{pmatrix})$$



Remarks:

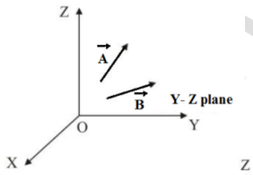
1- The vector $\overrightarrow{BA}$ (-$\vec{V}$) is the opposite vector to the vector $\overrightarrow{AB}$ ($\vec{V}$) (same length, same support, but opposite direction)

2- $\overrightarrow{AA}$ *and* $\overrightarrow{BB}$ is the zero (null) vectors: its magnitude is zero and its direction is arbitrary.

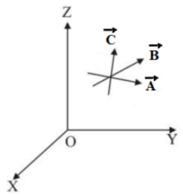
3- The vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are equal or equipollent if they have the same length, the same direction, the same support or parallel supports.

4- Colinear (or linearly dependent) vectors are vectors carried by parallel lines.

5- Coplanar vectors are vectors which lie in the same plane.



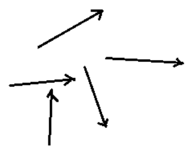
6- Concurrent vectors are the vectors which pass through the common point.



VI- Types of vectors

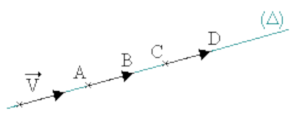
VI.1 - Free vector

It is a vector with a non-specific support. So, its position is not fixed in space.



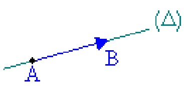
VI.2 - Sliding vector

A vector is called a "sliding vector" if we impose its support (Δ). Example: The vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are representatives of the sliding vector $\vec{V}$.



VI.3 – Localised or bounded vector

A vector is called a "bounded vector" if we fix the point of application A. The position of the vector is completely defined on the support (Δ).



VII- Vector operations

VII.1- Addition of two vectors

The sum of two vectors is a vector.

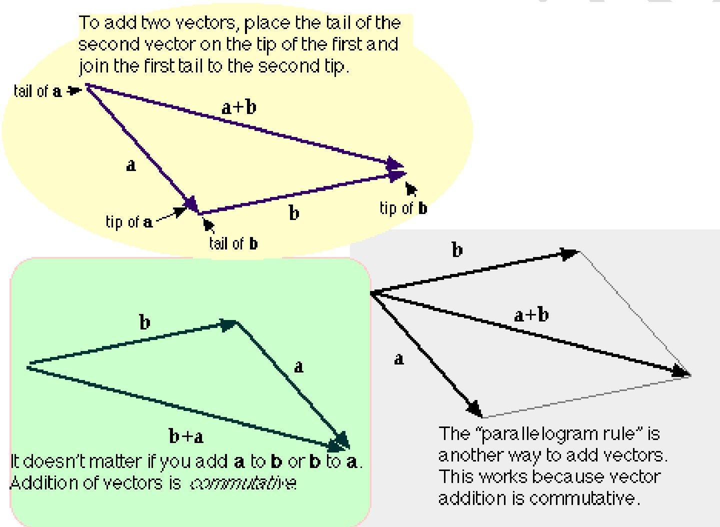
$$\vec{C} = \vec{A} + \vec{B} = (x_A + x_B)\vec{\imath} + (y_A + y_B)\vec{\jmath} + (z_A + z_B)\vec{k}$$

$$\vec{A} = \vec{\imath} + 4\vec{\jmath}$$

$$\vec{B} = 3\vec{\imath} - 2\vec{\jmath}$$

So:

$$\vec{C} = \overrightarrow{4\imath} + 2\vec{\jmath}$$



Commutative law: $\vec{A} + \vec{B} = \vec{B} + \vec{A}$

Associative law: $\vec{A} + (\vec{B} + \vec{C}) = (\vec{A} + \overrightarrow{B)} + \vec{C}$

Distributive law: $\lambda\,(\vec{A} + \vec{B}) = \lambda\,\vec{A} + \lambda\,\vec{B}$

VII.2- Subtraction

To subtract one vector from another, we add the first vector to the opposite of the second vector that needs to be subtracted.

$$\vec{C} = \vec{A} - \vec{B} = \vec{A} + (-\vec{B})$$

$$\vec{C} = \vec{A} - \vec{B} = (x_A - x_B)\vec{\imath} + (y_A - y_B)\vec{\jmath} + (z_A - z_B)\vec{k}$$

Example:

$$\vec{A} = \vec{\imath} + 4\vec{\jmath} \quad , \qquad \vec{B} = 3\vec{\imath} - 2\vec{\jmath}$$

So: $$\vec{C} = \vec{A} - \vec{B} = -2\vec{\imath} + 6\vec{\jmath}$$

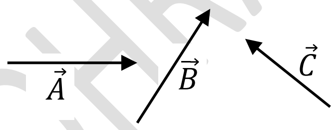
Subtraction is not commutative: $\vec{A} - \vec{B} \neq \vec{B} - \vec{A}$

Distributive law: $\lambda\,(\vec{A} - \vec{B}) = \lambda\,\vec{A} - \lambda\,\vec{B}$

Example

Perform graphically the following vector additions and subtractions, where A, B, and C are the vectors shown in Figure :

(a) $\vec{A} + \vec{B}$, (b) $\vec{A} + \vec{B} + \vec{C}$, (c) $\vec{A} - \vec{B}$, (d) $\vec{A} + \vec{B} - \vec{C}$.



VII.3- Scalar product (or the Dot Product)

The scalar product of two vectors $\vec{A}$ *and* $\vec{B}$ (denoted $\vec{A}.\vec{B}$($\vec{A}$ *scalar* $\vec{B}$)) is defined as the product of the magnitude of $\vec{A}$ *and* $\vec{B}$ by the cosine of the angle between the two vectors :

$$\vec{A}.\vec{B} = A.\,B.\,\cos\theta$$

Remarks

1- The product $\vec{A}.\vec{B}$ is a scalar and not a vector.

2- The scalar product satisfies the following laws:

2.a- $\vec{A}.\vec{B} = \vec{B}.\vec{A}$ (the scalar product is commutative).

2.b- $\vec{A}.(\vec{B} + \vec{C}) = \vec{A}.\vec{B} + \vec{A}.\vec{C}$ (the scalar product is distributive).

2.c- $\lambda\,(\vec{A}.\vec{B}) = (\lambda\,\vec{A})\,.\,\vec{B} = \vec{A}\,.\,(\lambda\,\vec{B})$, λ is a scalar value.

3- If $\vec{A}.\vec{B}=0$ and if $\vec{A}$ *and* $\vec{B}$ are zero vectors, then are **perpendicular**.

4- The orthogonal unit vectors $(\vec{\imath}, \vec{\jmath}, \vec{k})$ which form the Cartesian basis satisfy :

$\vec{\imath}.\vec{\imath} = \vec{\jmath}.\vec{\jmath} = \vec{k}.\vec{k} = 1$ and $\vec{\imath}.\vec{\jmath} = \vec{\jmath}.\vec{k} = \vec{k}.\vec{\imath} = 0$

5- The analytical expression for the scalar product is:

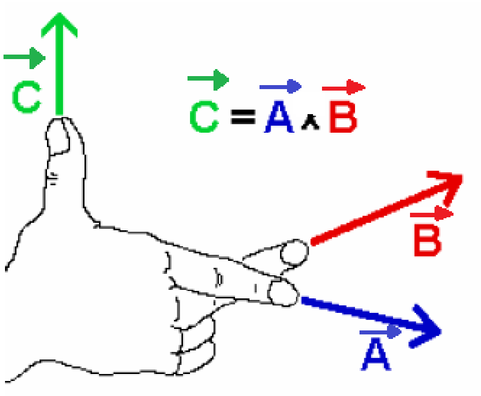
$$\vec{A}.\vec{B} = (x_A.\vec{i} + y_A.\vec{j} + z_A.\vec{k}).(x_B.\vec{i} + y_B.\vec{j} + z_B.\vec{k}) = x_A.x_B + y_A.y_B + z_A.z_B$$

VII.4- Vector product (or the Cross Product)

The vector product of two vectors $\vec{A}$ *et* $\vec{B}$ (denoted $\vec{A} \wedge \vec{B}$($\vec{A}$ *vectorial* $\vec{B}$)) is a vector . It is defined as the product of the moduli of $\vec{A}$ *and* $\vec{B}$ by the sine of the angle between the two vectors:

$$\vec{A} \wedge \vec{B} = (A.\ B.\ \sin\theta).\ \vec{u} = \vec{C}$$

where $\vec{u}$ is a unit vector indicating the direction of $\vec{A} \wedge \vec{B}$ which is perpendicular to the plane formed by $\vec{A}$ *and* $\vec{B}$. Noted that $\vec{A}$, $\vec{B}$ *and* $\vec{u}$ (*or* $\vec{C}$) form a direct trihedron (i.e., a corkscrew that which turns from $\vec{A}$ to $\vec{B}$ advances in the direction of $\vec{u}$ (or $\vec{C}$) or we can use the Right Hand Rule.



Index finger in the direction of the first vector.

Middle finger in the direction of the second vector

Thumb points in the direction of the cross product.

Note that:

1- If $\vec{A}$ *and* $\vec{B}$ are parallel, $\vec{A} \wedge \vec{B} = \vec{0}$.

2- The orthogonal unit vectors ($\vec{\imath}, \vec{\jmath}, \vec{k}$) form a direct trihedron, they satisfy:

$\vec{\imath} \wedge \vec{\imath} = \vec{\jmath} \wedge \vec{\jmath} = \vec{k} \wedge \vec{k} = \vec{0}$

$\vec{\imath} \wedge \vec{\jmath} = \vec{k}$, $\vec{\jmath} \wedge \vec{k} = \vec{\imath}$, $\vec{k} \wedge \vec{\imath} = \vec{\jmath}$ (circular permutation)

3- The analytical expression of the vector product:

$$\vec{A} \wedge \vec{B} = (x_A.\vec{\imath} + y_A.\vec{\jmath} + z_A.\vec{k}) \wedge (x_B.\vec{\imath} + y_B.\vec{\jmath} + z_B.\vec{k})$$

$$\vec{A} \wedge \vec{B} = (\ y_A.z_B - z_A.y_B)\vec{\imath} + (\ z_A.x_B - x_A.z_B)\vec{\jmath} + (\ x_A.y_B - y_A.x_B)\vec{k}$$

Or, by using the determinant:

$$\vec{A}\wedge\vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_A & y_A & z_A \\ x_B & y_B & z_B \end{vmatrix} = \begin{vmatrix} y_A & z_A \\ y_B & z_B \end{vmatrix}\vec{i} - \begin{vmatrix} x_A & z_A \\ x_B & z_B \end{vmatrix}\vec{j} + \begin{vmatrix} x_A & y_A \\ x_B & y_B \end{vmatrix}\vec{k}$$

$$\vec{A}\wedge\vec{B} = (\, y_A.z_B - z_A.y_B)\vec{i} + (\, z_A.x_B - x_A.z_B)\vec{j} + (\, x_A.y_B - y_A.x_B)\vec{k}$$

4-The vector product is not commutative: $\vec{A}\wedge\vec{B} = -\vec{B}\wedge\vec{A}$.

5-The vector product is distributive: $\vec{A}\wedge(\overrightarrow{B+C}) = \vec{A}\wedge\vec{B} + \vec{A}\wedge\vec{C}$

6- The double vector product : $\vec{A}\wedge(\overrightarrow{\vec{B}\wedge\vec{C}}) = \vec{B}.(\vec{A}\ .\overrightarrow{C})$- $\vec{C}.(\vec{A}\ .\vec{B})$

7- $|\vec{A}\wedge\vec{B}|$ is the area of a parallelogram with sides $\vec{A}$ *and* $\vec{B}$.

VII.5- Mixed product

A mixed product of three vectors $(\vec{A},\vec{B}\ and\ \vec{C})$is defined by:

$$\vec{A}.(\vec{B}\wedge\vec{C}) = \begin{vmatrix} x_A & y_A & z_A \\ x_B & y_B & z_B \\ x_C & y_C & z_C \end{vmatrix}$$

This mixed product represents the volume of a parallelepiped with sides $\vec{A},\vec{B}\ and\ \vec{C}$.

We also have $\vec{A}.(\vec{B}\wedge\vec{C}) = \vec{B}.(\overrightarrow{\vec{C}\wedge A}) = \vec{C}.(\vec{A}\ \wedge\vec{B})$ (circular permutation).

VII.6- Derivatives of vectors

Let be $\vec{A}(t)$ the vector function in terms of time (t): $\vec{A}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$,

The first derivative of with respect to t is defined by:

$$\frac{d\vec{A}(t)}{dt} = \frac{dx(t)}{dt}\vec{i} + \frac{dy(t)}{dt}\vec{j} + \frac{dz(t)}{dt}\vec{k}$$

The second derivative is :

$$\frac{d^2\vec{A}(t)}{dt^2} = \frac{d}{dt}\left(\frac{d\vec{A}(t)}{dt}\right) = \frac{d^2x(t)}{dt^2}\vec{i} + \frac{d^2y(t)}{dt^2}\vec{j} + \frac{d^2z(t)}{dt^2}\vec{k}$$

So if (λ) is a scalar function and if $\vec{A}$ and $\vec{B}$ are vectors, then:

$$\frac{d}{dt}(\lambda\vec{A}) = \frac{d\lambda}{dt}\vec{A} + \lambda\frac{d\vec{A}}{dt}$$

$$\frac{d}{dt}(\vec{A}.\vec{B}) = \frac{d\vec{A}}{dt}\vec{B} + \vec{A}\frac{d\vec{B}}{dt}$$

$$\frac{d}{dt}(\vec{A} \wedge \vec{B}) = \frac{d\vec{A}}{dt} \wedge \vec{B} + \vec{A} \wedge \frac{d\vec{B}}{dt}$$

VII.7- Vector integrals

Let the vector $\vec{A}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$ which is a vector function of t. We define an integral of $\vec{A}(t)$ by: $\int\vec{A}(t)dt = \vec{i}\int x(t)dt + \vec{j}\int y(t)dt + \vec{k}\int z(t)dt$

VIII- Error in a function of several variables

The error in a function of several variables is approximately the sum of the absolute values of all the partial derivatives, each multiplied by the corresponding errors. Let f(x,y,z) function of three variables (x, y and z) and Δx, Δy, Δz the accuracy (the measurement errors) of the devices that measured x, y and z. We can estimate the absolute error $\Delta f(x,y,z)$ by moving from the differential $df(x,y,z)$ to the error $\Delta f(x,y,z)$:

$$df(x,y,z) = \frac{\partial f(x,y,z)}{\partial x}\,dx + \frac{\partial f(x,y,z)}{\partial y}\,dy + \frac{\partial f(x,y,z)}{\partial z}\,dz$$

$\frac{\partial f(x,y,z)}{\partial x}$ is the partial derivative of $f(x,y,z)$ with respect to x.

So, the absolute error $\Delta f(x,y,z)$ is:

$$\Delta f(x,y,z) = \left|\frac{\partial f(x,y,z)}{\partial x}\right|\Delta x + \left|\frac{\partial f(x,y,z)}{\partial y}\right|\Delta y + \left|\frac{\partial f(x,y,z)}{\partial z}\right|\Delta z$$

The relative error is the quotient: $\frac{\Delta f(x,y,z)}{f(x,y,z)}$

This number is expressed in%.

Example

1) $f(x, y) = xy$

$$df(x,y) = \frac{\partial f(x,y)}{\partial x}\, dx + \frac{\partial f(x,y)}{\partial y}\, dy$$

$$df(x,y) = y\, dx + x\, dy$$

$$\Delta f(x,y) = |y|\, \Delta x + |x|\, \Delta y$$

2) A runner travels the distance x in certain time t:

x= 50.0 ± 0.1 m

t = 6.00 ± 0.01 s.

So, the velocity of runner is: $V = \frac{x}{t} = \frac{50}{6} = 8.3333\ m/s$

So V is function of x and t.

$$dV(x,t) = \frac{\partial V(x,t)}{\partial x}\, dx + \frac{\partial V(x,t)}{\partial t}\, dt$$

$$dV(x,t) = \frac{1}{t}\, dx - \frac{x}{t^2}\, dt$$

$$\Delta \boldsymbol{V}(\boldsymbol{x},\boldsymbol{t}) = \left|\frac{\boldsymbol{1}}{\boldsymbol{t}}\right| \Delta \boldsymbol{x} + \left|-\frac{\boldsymbol{x}}{\boldsymbol{t}^2}\right| \Delta \boldsymbol{t}$$

$$\Delta V(x,t) = \frac{1}{6}\, 0.1 + \frac{50}{6^2}\, 0.01 = 0.03\, \frac{m}{s}$$

So, we write: **V= (8.33+0.03) m/s**

The relative error is: $\frac{\Delta V}{V} = \frac{0.03}{8.33} = \mathbf{0.0036}$

The relative error is therefore **0.36 %.**