Chapter V: Work and energy

I- Definitions

- Let $\vec{G}$ be a vector field: $\vec{G} = G_x\vec{i} + G_y\vec{j} + G_z\vec{k}$
- Let V be a scalar field as a function of x, y and z.
- The total differential of V is defined as: $dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz$
- The nabla operator « $\vec{\nabla}$ » is defined as follows: $\vec{\nabla} = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$
- $\vec{\nabla} = \frac{\partial}{\partial \rho}\overrightarrow{u_\rho} + \frac{1}{\rho}\frac{\partial}{\partial \theta}\overrightarrow{u_\theta} + \frac{\partial}{\partial z}\vec{k}$ (in cylindrical coordinates)

- $\vec{\nabla} = \frac{\partial}{\partial r}\overrightarrow{u_r} + \frac{1}{r}\frac{\partial}{\partial \theta}\overrightarrow{u_\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}\overrightarrow{u_\varphi}$ (in spherical coordinates)

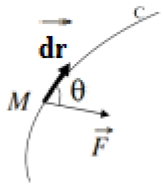
- The following differential operators are defined:

- $\overrightarrow{grad}\,V = \vec{\nabla}V = \frac{\partial V}{\partial x}\vec{i} + \frac{\partial V}{\partial y}\vec{j} + \frac{\partial V}{\partial z}\vec{k}$

- $\mathrm{div}\,\vec{G} = \vec{\nabla}.\vec{G} = \frac{\partial G_x}{\partial x} + \frac{\partial G_y}{\partial y} + \frac{\partial G_z}{\partial z}$ (div = divergence)
- $\overrightarrow{rot}\,\vec{G} = \vec{\nabla}\wedge\vec{G} = \left(\frac{\partial G_z}{\partial y} - \frac{\partial G_y}{\partial z}\right)\vec{i} + \left(\frac{\partial G_x}{\partial z} - \frac{\partial G_z}{\partial x}\right)\vec{j} + \left(\frac{\partial G_y}{\partial x} - \frac{\partial G_x}{\partial y}\right)\vec{k}$
- $\Delta = \vec{\nabla}.\vec{\nabla} = \vec{\nabla}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ (Δ : Laplacien)
- $\Delta V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$
- $\Delta\vec{G} = \left(\frac{\partial^2 G_x}{\partial x^2} + \frac{\partial^2 G_x}{\partial y^2} + \frac{\partial^2 G_x}{\partial z^2}\right)\vec{i} + \left(\frac{\partial^2 G_y}{\partial x^2} + \frac{\partial^2 G_y}{\partial y^2} + \frac{\partial^2 G_y}{\partial z^2}\right)\vec{j} + \left(\frac{\partial^2 G_z}{\partial x^2} + \frac{\partial^2 G_z}{\partial y^2} + \frac{\partial^2 G_z}{\partial z^2}\right)\vec{k}$

II- Work of a force

The elementary work dw of a force $\vec{\mathbf{F}}$ acting on a material point M in an elementary displacement $d\vec{\mathbf{r}}$ along the trajectory (C) is given by :



$$\mathbf{dW = \vec{F}.\,\overrightarrow{dr}}$$

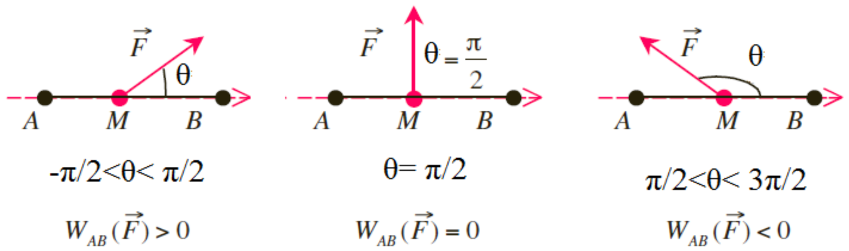
$$\mathbf{dW = F.dr.\cos\theta}$$

The unit of the work is the Joul (J)

If cosθ > 0 (-π/2<θ< π/2): the work is said to be driving (dw > 0) ⇒ The force and displacement are in the same direction.

If cosθ < 0 (π/2<θ< 3π/2) the work is said to be resistive (dw < 0) ⇒ The force is in the opposite direction of movement; it slows the object down.

If θ= π/2⇒ cosθ=0 ⇒ dW=0 $\vec{\mathbf{F}} \perp \overrightarrow{\mathbf{dr}}$ the force perpendicular to the trajectory does not work.



In general, if the material point traverses an arc *AB* on the trajectory, the work along this curve will be the integral of the elementary work:

$$\mathbf{W_{A\to B}}= \int_A^B dW= \int_A^B \vec{\mathbf{F}}.\ \overrightarrow{\mathbf{dr}}= \int_A^B \mathbf{F.\ dr.\ cos}\boldsymbol{\theta}$$

If there are several forces:

$$\mathbf{W_{A\to B}}= = \sum W_i = \int_A^B dW= \int_A^B \sum \vec{F_i}\ .\ \overrightarrow{\mathbf{dr}} \quad \mathbf{(i= 1,\dots\dots\dots n)}$$

Analytical expression of work

a- Using Cartesian coordinates:

$$\vec{\mathrm{F}}= F_x\vec{\mathrm{i}} + F_y\vec{\mathrm{j}} + F_z\vec{\mathrm{k}}$$

$$\overrightarrow{\mathrm{dr}}=dx\vec{\mathrm{i}} + dy\vec{\mathrm{j}} + dz\vec{\mathrm{k}}$$

$$\mathbf{W_{A\to B}}=\int_{x_1}^{x_2} F_x\, dx + \int_{y_1}^{y_2} F_y\, dy + \int_{z_1}^{z_2} F_z\, dz$$

b- Using polar coordinates:

$$\vec{\mathrm{F}}= F_\rho\ \overrightarrow{u_\rho} + F_{\boldsymbol{\theta}}\overrightarrow{u_{\boldsymbol{\theta}}}$$

$$\overrightarrow{\mathrm{dr}}=d\rho\ \overrightarrow{u_\rho} + \rho d\boldsymbol{\theta}\ \overrightarrow{u_{\boldsymbol{\theta}}}$$

$$\mathbf{W_{A\to B}}=\int_{\rho_1}^{\rho_2} F_\rho\, d\rho + \int_{\boldsymbol{\theta}_1}^{\boldsymbol{\theta}_2} F_{\boldsymbol{\theta}}\, \rho d\boldsymbol{\theta}$$

c- Using cylindrical coordinates:

$$\vec{F} = F_\rho\,\overrightarrow{u_\rho} + F_\theta\,\overrightarrow{u_\theta} + F_z\,\vec{k}$$

$$\overrightarrow{dr} = d\rho\,\overrightarrow{u_\rho} + \rho d\boldsymbol{\theta}\,\overrightarrow{u_\theta} + dz\,\vec{k}$$

$$\mathbf{W_{A\to B}} = \int_{\rho_1}^{\rho_2} \boldsymbol{F_\rho\,d\rho} + \int_{\theta_1}^{\theta_2} \boldsymbol{F_\theta\,\rho d\theta} + \int_{z_1}^{z_2} \boldsymbol{F_z\,dz}$$

d- Using intrinsic coordinates:

$$\vec{F} = F_T\,\overrightarrow{u_T} + F_N\,\overrightarrow{u_N}$$

$$\overrightarrow{dr} = dr\,\overrightarrow{u_T}$$

$$\mathbf{W_{A\to B}} = \int_{r_1}^{r_2} \boldsymbol{F_T\,dr}$$

d- Using spherical coordinates:

$$\vec{F} = F_r\,\overrightarrow{u_r} + F_\theta\,\overrightarrow{u_\theta} + \ + F_\varphi\,\overrightarrow{u_\varphi}$$

$$\overrightarrow{dr} = dr\vec{u}_r + r\,d\theta\,\vec{u}_\theta + r\sin\theta\,d\Phi\,\vec{u}_\Phi$$

$$\mathbf{W_{A\to B}} = \int_{r_1}^{r_2} \boldsymbol{F_r\,dr} + \int_{\theta_1}^{\theta_2} \boldsymbol{rF_\theta\,d\theta} + \int_{\varphi_1}^{\varphi_2} r\boldsymbol{F_\varphi}\ \sin\theta\ \boldsymbol{d\varphi}$$

III- Power

Instantaneous power is defined as work per unit of time: $\mathbf{P} = \frac{dw}{dt}$. It is defined by the scalar product of force $\vec{\mathbf{F}}$ and velocity $\vec{\mathbf{V}}$:

$$P = \frac{dw}{dt} = \frac{\vec{F}.\overrightarrow{dr}}{dt} = \vec{F}.\ \vec{V}\ \textbf{(watt)}$$

IV- Kinetic energy

The elementary work of the force $\vec{\mathbf{F}}$ acting on a material point can be written as:

$$dw = \vec{F}\,.\,\overrightarrow{dr} = m\,\vec{a}.\,\overrightarrow{dr} = m\,\frac{d\vec{V}}{dt}\,.\,\vec{V}\,dt = m\,\vec{V}.\,d\vec{V} = d\left(\frac{mv^2}{2}\right) = d(E_k)$$

$$(\vec{F} = m\,\vec{a}\ ,\quad \vec{v} = \frac{d\vec{r}}{dt}\ )$$

$\mathbf{E_k} = \frac{mv^2}{2}$ is the kinetic energy of the material point.

$P = mV \Rightarrow E_k = \frac{P^2}{2m}$ (**P : momentum (motion quantity)**)

Kinetic energy theorem

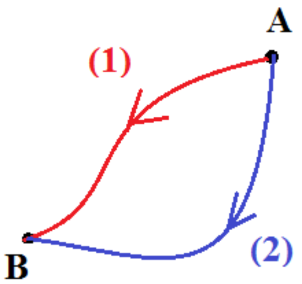
The total work of the forces exerted on a material point between two instants t_1 and t_2 is equal to the variation in the kinetic energy of the point between these two instants:

$$W_{A\rightarrow B} = \int_A^B dW = = \int_A^B d(E_k) = E_k(B) - E_k(A) = \frac{mv_B^2}{2} - \frac{mv_A^2}{2} = \Delta E_k$$

V- Conservative forces

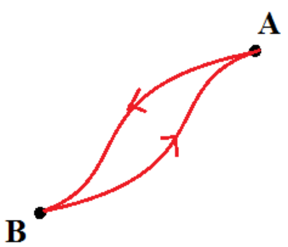
A force is said to be conservative if its work does not depend on the path followed:

$$W_{A \overset{1}{\rightarrow} B} = W_{A \overset{2}{\rightarrow} B}$$



In other words, the total work on a closed path is zero:

$$W_{A\rightarrow A} = \oint_A^A dW = \oint_A^A \vec{F} \cdot \overrightarrow{dr} = 0 \Rightarrow E_k(A, t_1) = E_k(A, t_2) = \ldots\ldots\ldots\ldots\ldots\ldots E_k(A, t_n)$$



VI- Potential energy

A conservative force is a force derived from a potential:

$$\vec{F} = -\overrightarrow{\mathbf{grad}}\, E_p$$

$$\overrightarrow{\text{grad}}\, E_p = \vec{\nabla} E_p = \frac{\partial E_p}{\partial x}\vec{i} + \frac{\partial E_p}{\partial y}\vec{j} + \frac{\partial E_p}{\partial z}\vec{k}$$

E_p is the potential or potential energy. Potential energy is defined within one additive constant. In general, a reference 'origin' position is defined for which E_p=0, and the variation in potential energy is measured, not its absolute value.

Note 1

If the force $\vec{F}$ is a conservative force : $\overrightarrow{rot}\ \vec{F} = \vec{0}$

$$\overrightarrow{rot}\ \vec{F} = \vec{\nabla} \wedge \vec{F} = \vec{\nabla} \wedge (-\overrightarrow{grad}\ E_p) = -(\vec{\nabla} \wedge \vec{\nabla})\ E_p = \vec{0}$$

Note 2

We have: $dW = \vec{F} . \overrightarrow{dr}$

For a conservative force: $\vec{F} = -\overrightarrow{grad}\ E_p \Rightarrow dW = -\overrightarrow{grad}\ E_p\ .\ \overrightarrow{dr}$

$$\Rightarrow dW = -\left(\frac{\partial E_p}{\partial x}\vec{i} + \frac{\partial E_p}{\partial y}\vec{j} + \frac{\partial E_p}{\partial z}\vec{k}\right).\left(dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}\right)$$

$$\Rightarrow dW = -\left(\frac{\partial E_p}{\partial x}dx + \frac{\partial E_p}{\partial y}dy + \frac{\partial E_p}{\partial z}dz\right)$$

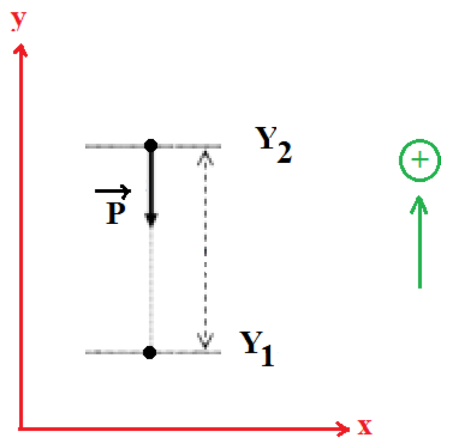
$$\Rightarrow \mathbf{dW = -\,dE_p \Rightarrow W_{1\to 2} = -\,\Delta E_p = E_{p1} - E_{p2}}$$

VII- Examples of conservative forces and potential energies

1- Potential energy of a body in a uniform gravity field

$$\vec{F} = \vec{P} = -\,mg\,\vec{j}$$

$$\overrightarrow{dr} = dy\,\vec{j}$$



$$\mathbf{W_{y1\to y2} = -\int_{y_1}^{y_2} mg\,dy = -\,mg\,(y_2 - y_1)}$$

If $y_1 = y_2 \Rightarrow W_{y1\to y1} = -\,mg\,(y_1 - y_1) = 0 \Rightarrow$ $\vec{P}$ is a conservative force. Therefore:

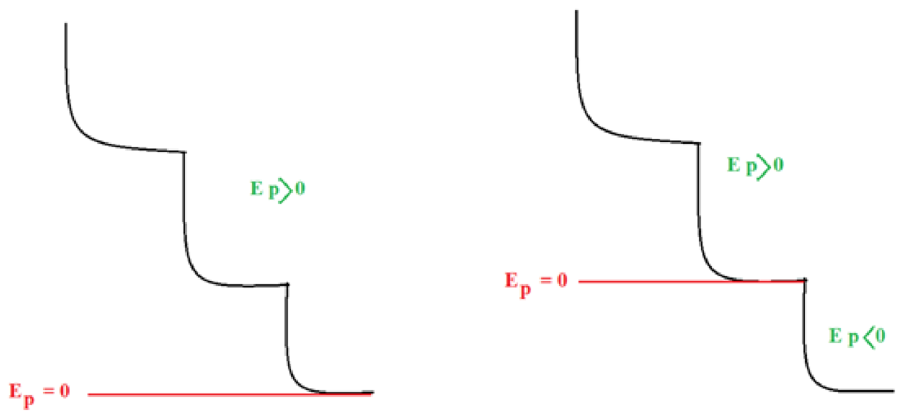
$$\vec{P} = -\,\overrightarrow{\textbf{grad}}\,E_p$$

$$-mg\,\vec{j} = -\left(\frac{\partial E_p}{\partial x}\vec{i} + \frac{\partial E_p}{\partial y}\vec{j} + \frac{\partial E_p}{\partial z}\vec{k}\right) \Rightarrow \quad mg = \frac{\partial E_p}{\partial y} \;\Rightarrow dE_p = mg\,dy \Rightarrow \mathbf{E_p = \;mg\,y + C^{ste}}$$

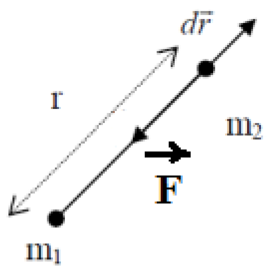
To determine the constant, we choose a Reference position for which Ep is zero. Therefore:

$$\mathbf{E_p = \;mg\,y}$$

y: the vertical position *y* (or the height) of the particle relative to the reference position *y* = 0



2- Potential energy from the gravitational attraction of two material points



$$\vec{F} = -G\,\frac{m_1 m_2}{r^2}\,\overrightarrow{U_r} \quad , \qquad \overrightarrow{dr} = dr\,\overrightarrow{U_r}$$

$$dW = -\,dE_p$$

$$-\,dE_p = dW = \vec{F}\,.\,\overrightarrow{dr}$$

$$E_p = -\int \vec{F}\,.\,\overrightarrow{dr}$$

$$E_p = \int G\,\frac{m_1 m_2}{r^2}\,dr$$

$$\boldsymbol{E_p = -G\,\frac{m_1 m_2}{r} + C^{ste}}$$

Or

$$\vec{F} = -grad\, E_p$$

$$-G\,\frac{m_1 m_2}{r^2}\,\vec{u_r} = -\left(\frac{\partial E_p}{\partial r}\vec{u}_r + \frac{1}{\mathrm{r}}\frac{\partial E_p}{\partial \theta}\vec{u}_\theta + \frac{1}{\mathrm{r}\sin\theta}\,\frac{\partial E_p}{\partial \varphi}\,\vec{u}_\varphi\right)$$

$$G\,\frac{m_1 m_2}{r^2}\,\vec{u_r} = \frac{\partial E_p}{\partial r}\vec{u}_r$$

$$G\,\frac{m_1 m_2}{r^2} = \frac{\mathrm{d}E_p}{dr}$$

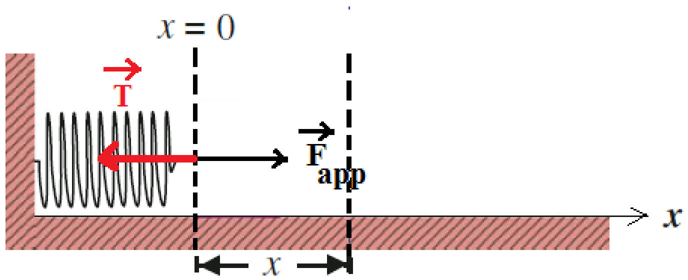
$$\mathrm{d}E_p = G\,\frac{m_1 m_2}{r^2}\,\mathrm{dr}$$

$$E_p = -G\,\frac{m_1 m_2}{r} + C^{ste}$$

Generally, we take $r = \infty$ as the reference position and $C^{ste}=0$; then :

$$E_p = -G\,\frac{m_1 m_2}{r}$$

3- Elastic Potential Energy



$$\vec{T} = -k.x\,\vec{\imath}$$

$$\vec{T} = -grad\, E_p$$

$$-k.x\,\vec{\imath} = -\left(\frac{\partial E_p}{\partial x}\vec{\imath} + \frac{\partial E_p}{\partial y}\vec{\jmath} + \frac{\partial E_p}{\partial z}\vec{k}\right)$$

$$k.x = \frac{\mathrm{d}E_p}{dx}$$

$$\mathrm{d}E_p = k.x\,dx$$

$$E_p = \frac{1}{2}k.x^2 + C^{ste}$$

VIII- Mechanical energy

The mechanical energy of a system is given by the sum of kinetic energy and potential energy.

$$E_M = E_K + E_p$$

We have seen that for conservative forces:

$$W_{1\to 2} = \Delta E_k = E_{k2} - E_{k1}$$

$$\mathbf{W_{1\to2} = -\Delta Ep = E_{p1} - E_{p2}}$$

$$\mathbf{\Rightarrow E_{k2} - E_{k1} = E_{p1} - E_{p2}}$$

$$\mathbf{\Rightarrow E_{k2} + E_{p2} = E_{k1} + E_{p1}}$$

$$\Rightarrow \mathbf{E_{M2} = E_{M1}}$$

$$\mathbf{\Rightarrow \Delta E_M = 0}$$

This relationship indicates that the mechanical energy of a system subject to conservative forces remains constant - the "law of conservation of mechanical energy".

IX - Non-conservative forces

In the general case, the forces acting on a system can be divided into conservative forces (which derive from a potential) and forces $\vec{F}$ which do not derive from a potential (frictional forces, for example) and can be written as :

$$\vec{F}_{tot} = \vec{F} + \vec{F}_f$$

$$\mathbf{W_{1\to2} = \int_1^2 (\vec{F} + \vec{F}_f) \cdot \overrightarrow{dr}}$$

$$\mathbf{W_{1\to2} = W_{\vec{F}} + W_{\vec{F}_f}}$$

$\vec{\mathbf{F}}$ is a conservative force : $\mathbf{W}_{\vec{F}} = -\Delta E_p = E_{p1} - E_{p2}$

$$W_{1\to2} = \Delta E_c \Rightarrow E_{c2} - E_{c1} = E_{p1} - E_{p2} + W_{\vec{F}_f}$$

$$\Rightarrow (E_{c2} + E_{p2}) - (E_{c1} + E_{p1}) = W_{\vec{F}_f}$$

$$\Rightarrow E_{M2} - E_{M1} = W_{\vec{F}_f}$$

$$\Rightarrow \mathbf{\Delta E_M = W_{\vec{F}_f}}$$