Chapter II: Kinematics of the material point

I-Introduction

Kinematics is a subfield of physics, developed in classical mechanics. **Kinematics** is the study of the motion of objects without reference to the forces that caused the motion.

II- characteristics of the motion

In kinematics, the two fundamental concepts are space and time, because the motion takes place in space as a function of time. Mathematically solving kinematics problems in physics will involve understanding, calculating, and measuring several physical quantities:

- Position vector ($\overrightarrow{OM}$): determines the object's physical location in space relative to an origin in a defined coordinate system.

- Velocity vector ($\vec{V}$): which determines the variation in magnitude and position of the position vector.

- Acceleration vector ($\vec{a}$): which determines the variation in magnitude and position of the velocity vector.

II.1- Position vector

The position of an object $\overrightarrow{OM}$ is given by its displacement relative to O. It changes with time (Fig.1).

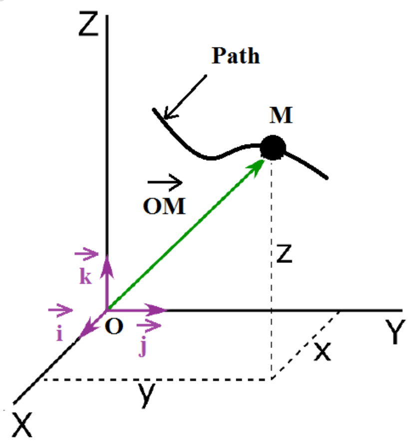


Fig.1

$\vec{\imath}, \vec{\jmath}$ *and* $\vec{k}$: unit vectors.

x, y and z: point coordinates.

II.1.1- Path of motion

The path followed by the object is the set of successive positions or line along which point P moves in space.

Parametric equation of the path: After removing time, we get the relations between the x, y, z coordinates.

II.1.2- Displacement

The displacement is a vector quantity. It's the distance in a given direction. So, it's a vector from the starting point to the end point:

$$\Delta \overrightarrow{\boldsymbol{OM}} = \overrightarrow{OM_2} - \overrightarrow{OM_1} = \overrightarrow{\boldsymbol{M_1M_2}}$$

II.2- Velocity vector

Velocity vector is a vector quantity that characterizes the rate of change in the position of a body in space ([V] =m/s). The direction of velocity is the same as the direction of motion.

II.2.1- Average velocity vector

The average velocity vector $\vec{V}$ between M_1 and M_2 (or between two times t_1 and t_2) is defined as the ratio of the displacement $\Delta\overrightarrow{OM} = \overrightarrow{OM_2} - \overrightarrow{OM_1}$ to the time interval $\Delta t = t_2 - t_1$ (Fig. 2).

That is:

$$\vec{\boldsymbol{V}}_{\boldsymbol{avg}} = \frac{\overrightarrow{OM_2} - \overrightarrow{OM_1}}{t_2 - t_1} = \frac{\overrightarrow{\mathrm{M_1M_2}}}{t_2 - t_1} = \frac{\boldsymbol{\Delta\overrightarrow{OM}}}{\boldsymbol{\Delta t}}$$

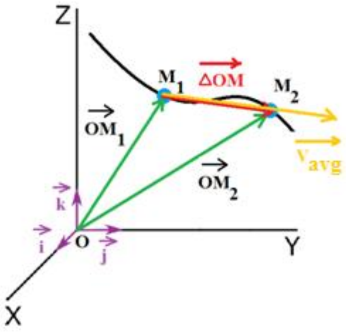


Fig.2

II.2.2- Instantaneous velocity vector

The instantaneous velocity $\vec{V}$ is defined as the limiting value of the ratio $\frac{\Delta\overrightarrow{OM}}{\Delta t}$ as Δt approaches zero.

Mathematically, $\vec{V}$ can be expressed as:

$$\vec{\boldsymbol{V}} = \lim_{\Delta t \to 0} \frac{\Delta\overrightarrow{OM}}{\Delta t} = \frac{\mathbf{d}\overrightarrow{\boldsymbol{OM}}}{\boldsymbol{dt}}$$

II.3- Acceleration vector

The acceleration vector is vector quantity that characterizes the variation of the velocity vector with respect to time.

II.3.1- Average acceleration vector

The average acceleration $\vec{a}_{avg}$ is defined as the ratio of the change in velocity $\Delta\vec{V} = \vec{V_2} - \vec{V_1}$ to the time interval $\Delta t = t_2 - t_1$. That is:

$$\vec{\boldsymbol{a}}_{\boldsymbol{avg}} = \frac{\vec{V_2}-\vec{V_1}}{t_2-t_1} = \frac{\boldsymbol{\Delta\vec{V}}}{\boldsymbol{\Delta t}}$$

II.3.2- Instantaneous acceleration vector

Instantaneous acceleration is defined as the limiting value of the ratio $\frac{\Delta\vec{V}}{\Delta t}$ when Δt approaches zero. It is defined as follows:

$$\vec{\boldsymbol{a}} = \lim_{\Delta t\to 0}\frac{\Delta\vec{V}}{\Delta t} = \frac{\mathbf{d}\vec{\boldsymbol{V}}}{\boldsymbol{dt}} = \frac{\boldsymbol{d^2\overrightarrow{OM}}}{\boldsymbol{dt^2}}$$

Note

1- Depending on the shape of the path, the motion is classified:

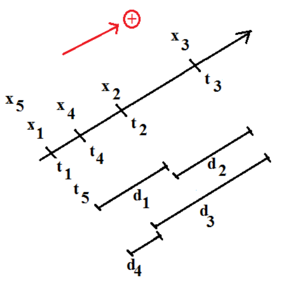
- Linear when the path is straight line.
- Curvilinear when the path isn't straight line.

2- If the motion is unidirectional (one direction), for example in the direction of the axis (OX), the velocity can be expressed as follow:

$$V_{avg} = \frac{x_2-x_1}{t_2-t_1} = \frac{\Delta x}{\Delta t}$$

Note that the velocity is a vector quantity.

Example:



$$\boldsymbol{V_{t_1,t_3}} = \frac{\boldsymbol{x_3 - x_1}}{\boldsymbol{t_3 - t_1}} > \mathbf{0}$$

$$\boldsymbol{V_{t_3,t_4}} = \frac{\boldsymbol{x_4 - x_3}}{\boldsymbol{t_4 - t_3}} < 0$$

$$\boldsymbol{V_{t_1,t_5}} = \frac{\boldsymbol{x_5-x_1}}{\boldsymbol{t_5-t_1}} = \mathbf{0} \quad (x_5 = x_1)$$

Or, the average speed $\boldsymbol{S_{avg}}$ is: $\boldsymbol{S_{avg}} = \frac{\boldsymbol{traveled\ distance}}{\Delta t} \Rightarrow \boldsymbol{S_{avg}(t_1, t_5)} = \frac{\boldsymbol{d_1+d_2+d_3+d_4}}{\boldsymbol{t_5-t_1}}$

So, the speed is the distance travelled per unit time (the speed is positive scalar quantity).

III- Motions in various coordinate systems and bases

Before studying the motion of a system, it is necessary to indicate the coordinate system and bases in which the motion (the position vector, velocity vector and acceleration vector) will be describe.

III.1- Cartesian coordinate system

The Cartesian coordinates system is orthonormal and it consists of three axes (OX,OY,OZ). The directions of (OX,OY,OZ) are determined by three unit vectors $(\vec{i}, \vec{j}, \vec{k})$ which are fixed in the observation frame of reference (neither the norm, nor the support, nor the direction of these vectors change with time).

III.1.1- Motion in one dimension: rectilinear motion

A rectilinear (linear) motion is one-dimensional motion along a straight line. It can be described mathematically using only one spatial dimension.
To locate an object in one-dimensional space, we find its position with respect to the origin of an axis, such as the x-axis shown in figure 3.

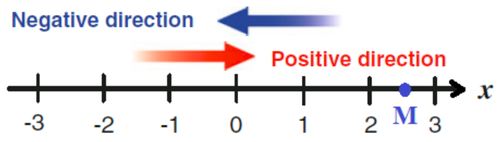


Fig. 3

The position vector is defined by the origin point O and the component x:

$$\overrightarrow{OM} = x\,\vec{i}$$

$$\overrightarrow{OM}\begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix}$$

$$\|\overrightarrow{OM}\| = OM = |x|$$

- The displacement of M: $\Delta\overrightarrow{OM} = \overrightarrow{OM_2} - \overrightarrow{OM_1} = \Delta x\,\vec{i} = (x_2 - x_1)\,\vec{i}$.

- The elementary displacement of M: $d\overrightarrow{OM} = dx\,\vec{i}$

- The velocity vector: $\vec{V} = \frac{d\overrightarrow{OM}}{dt} = \frac{dx}{dt}\vec{i} = \dot{x}\,\vec{i}$

$$\|\vec{V}\| = V = |\dot{x}|$$

- The acceleration vector: $\vec{a} = \frac{d^2\overrightarrow{OM}}{dt^2} = \frac{d^2x}{dt^2}\vec{i} = \ddot{x}\,\vec{i}$

$$\|\vec{a}\| = a = |\ddot{x}|$$

III.1.1.1- Uniform rectilinear motion

It is characterized by constant velocity (zero acceleration): $\boldsymbol{a = 0 \;\; and \;\; V = V_0 = C^{te}}$

$$V = V_0 = \frac{dx}{dt} \Rightarrow dx = V_0 dt \Rightarrow \int_{x_0}^{x} dx = \int_0^t V_0 dt \Rightarrow x - x_0 = V_0 t$$

$$\Rightarrow \boldsymbol{x = x_0 + V_0 t}$$

If $\boldsymbol{V_0 = 0}$, the object is stationary.

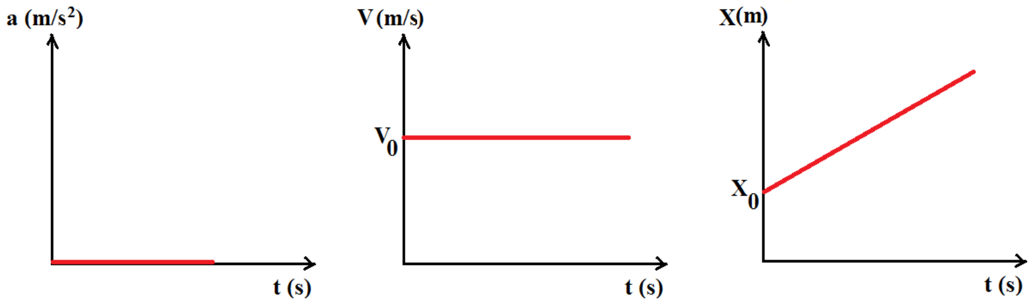


Fig.4

III.1.1.2- Non-uniform linear motion

It can be uniformly accelerated (or retarded) rectilinear motion. It is characterised by variable velocity (non-zero acceleration) (Fig.5).

$$\boldsymbol{a = a_0 = \frac{dV}{dt} \Rightarrow dV = a_0 dt \Rightarrow \int_{V_0}^{V} dV = \int_0^t a_0 dt \Rightarrow V - V_0 = a_0 t}$$

$$\Rightarrow \boldsymbol{V = V_0 + a_0 t}$$

$$\boldsymbol{V = \frac{dx}{dt} \Rightarrow dx = V dt \Rightarrow \int_{x_0}^{x} dx = \int_0^t (V_0 + a_0 t) dt \Rightarrow x - x_0 = V_0 t + \frac{1}{2} a_0 t^2}$$

$$\boldsymbol{x = \frac{1}{2} a_0 t^2 + V_0 t + x_0} \quad \textbf{(equation of a parabola)}$$

- If acceleration and velocity are in the same direction ($\vec{V}.\vec{a} > 0$)the movement is accelerated.
- If acceleration and speed are in opposite directions ($\vec{V}.\vec{a} < 0$), the movement is retarded.

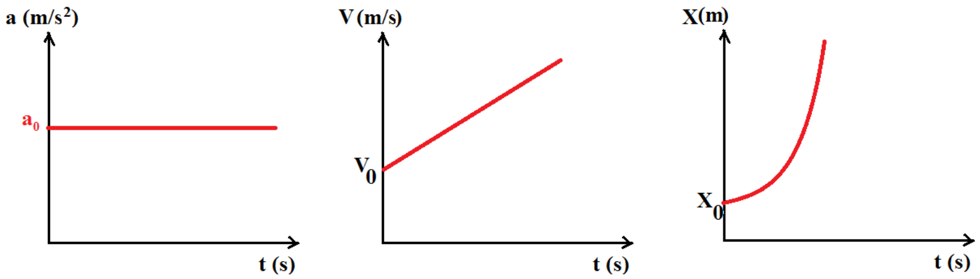


Fig.5

- Free Fall

Due to gravity, it is well known that all dropped objects near the Earth's surface will accelerate downward with a nearly constant acceleration when the effect of air resistance is very small and can be neglected. We use the term "free fall" for this motion and the same will be applied to objects that are either thrown up or down.

The acceleration due to gravity (g) is very close to 9.8 m/s^2 near the Earth's surface. Noted that:

- The motion is along the vertical *y*-axis.
- The free-fall acceleration is negative if the motion and the *y*-axis is upward: $a = -g$.
- The free-fall acceleration is positive if the motion and the *y*-axis is downward: $a = +g$.

Exercise

A ball is dropped from a tall building, as shown in figure 6. Find the following for the ball's motion:

1- its acceleration, 2- the distance it falls in 2 s, 3- its velocity after falling 15 m,

4- the time it takes to fall 25 m, 5- the time it takes to reach a velocity of 29.4 m/s.

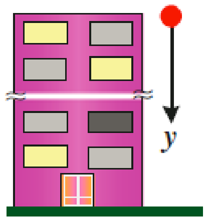


Fig.6

Solution

1- Finding its acceleration.

Since (+ y) is downward, then the ball's acceleration is positive (downward): $a = g = 9.8$ m/s^2

2- Finding the distance it falls in 2 s.

We have $a = g\ ,\ V_0 = 0\ ,\ y_0 = 0 \Rightarrow \quad V = gt \quad , \quad y = \frac{1}{2}gt^2$

$$y\,(2) = \frac{1}{2}9,8 \times 2^2 \Rightarrow y\,(2) = 19,6\ m$$

3- Finding its velocity after falling 15 m.

$$V = gt \Rightarrow t = \frac{V}{g} \Rightarrow y = \frac{1}{2}gt^2 = \frac{1}{2}g\left(\frac{V}{g}\right)^2 \Rightarrow y = \frac{1}{2}\frac{V^2}{g} \Rightarrow V^2 = 2gy \Rightarrow V = \sqrt{2gy}$$

$$y = 15\ m \Rightarrow V = \sqrt{2 \times 9,8 \times 15} \Rightarrow V = 17,2\ m/s$$

4- Finding the time it takes to fall 25 m.

$$y = \frac{1}{2}gt^2 \Rightarrow t = \sqrt{\frac{2y}{g}}$$

$$y = 25\ \mathrm{m} \Rightarrow t = \sqrt{\frac{2\times 23}{9.8}} \Rightarrow t = 2,3\ s$$

5- Finding the time it takes to reach a velocity of 29.4 m/s.

$$V = gt \Rightarrow t = \frac{V}{g} = \frac{29.4}{9,8} \Rightarrow t = 3\ s$$

III.1.2- Motion in two dimensions

The motion of a particle is described in a plane. Each point M is marked by its coordinates (x, y) in the base $(\vec{\imath}, \vec{\jmath})$ (Fig.7).

The position vector is defined by the origin point O and the components (x,y):

$$\overrightarrow{OM} = x\vec{\imath} + y\vec{\jmath}$$

$$\overrightarrow{OM}\begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

$$\|\overrightarrow{OM}\| = OM = \sqrt{x^2 + y^2}$$

$$\overrightarrow{OM} = \|\overrightarrow{OM}\|\vec{u}$$

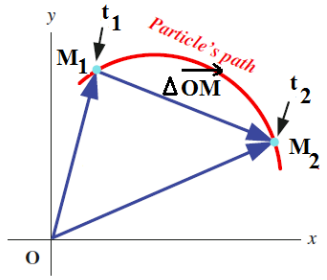


Fig.7

$\vec{u}$ is the unit vector: $\vec{u} = \dfrac{\overrightarrow{OM}}{\|\overrightarrow{OM}\|}$

- The displacement of M: $\Delta\overrightarrow{OM} = \overrightarrow{OM_2} - \overrightarrow{OM_1} = \Delta x\,\vec{\imath} + \Delta y\,\vec{\jmath}$
- The elementary displacement of M: $d\overrightarrow{OM} = dx\,\vec{\imath} + dy\,\vec{\jmath}$
- The elementary surface: **ds = dxdy**.
- The velocity vector: $\vec{V} = \dfrac{d\overrightarrow{OM}}{dt} = \dfrac{dx}{dt}\vec{\imath} + \dfrac{dy}{dt}\vec{\jmath} = \dot{x}\vec{\imath} + \dot{y}\vec{\jmath}$

$$\|\vec{V}\| = V = \sqrt{\dot{x}^2 + \dot{y}^2}$$

- The acceleration vector: $\vec{a} = \dfrac{d^2\overrightarrow{OM}}{dt^2} = \dfrac{d^2x}{dt^2}\vec{\imath} + \dfrac{d^2y}{dt^2}\vec{\jmath} = \ddot{x}\vec{\imath} + \ddot{y}\vec{\jmath}$

$$\|\vec{a}\| = a = \sqrt{\ddot{x}^2 + \ddot{y}^2}$$

- Projectile motion

Any object that is thrown into the air is called a *projectile*.

Let us assume that at $t = 0$ the projectile leaves the origin (i.e. $x_0 = y_0 = 0$) with initial velocity $\vec{V_0}$ that makes an angle θ_0 with the positive x direction as in Fig.8.

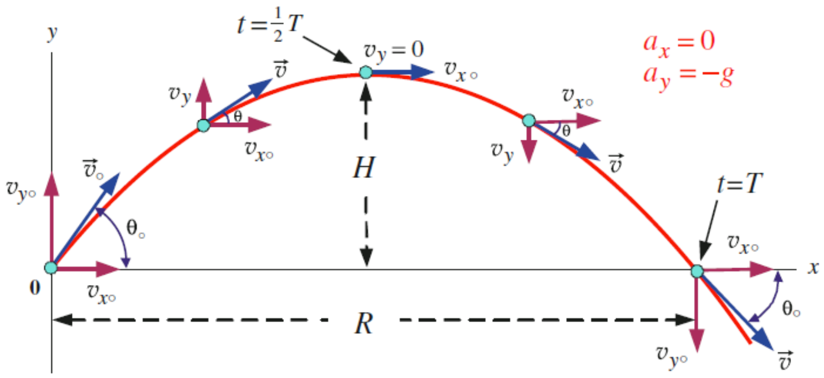


Fig.8

$$a_x = 0 \text{ and } a_y = -g$$

$$\vec{V_0} = V_{x_0}\vec{i} + V_{y_0}\vec{j} \quad , V_{x_0} = V_0 \cos\theta_0 \;,\; V_{y_0} = V_0 \sin\theta_0$$

We decompose the horizontal motion and vertical motion as described below:

$$a_x = 0 \Rightarrow V_x = V_{x_0} = V_0 \cos\theta_0 \Rightarrow x = V_{x_0}\, t \Rightarrow x = (V_0 \cos\theta_0)\, t$$

$$a_y = -g \Rightarrow V_y = V_{y_0} - g\,t \Rightarrow V_y = V_0 \sin\theta_0 - g\,t \Rightarrow y = (V_0 \sin\theta_0)\, t - \frac{1}{2} g\, t^2$$

- Calculus of the horizontal range R (the distance traveled by the projectile when it returns to $y = 0$) after time $t = T$

Set $x = R$ at time $t = T$ *and* $y = 0$

$$R = (V_0 \cos\theta_0)\, T$$

$$y = (V_0 \sin\theta_0)\, T - \frac{1}{2} g\, T^2 = 0$$

$$\Rightarrow T = \frac{2\, V_0 \sin\theta_0}{g}$$

$$R = (V_0 \cos\theta_0)\, T \Rightarrow R = (V_0 \cos\theta_0) \frac{2\, V_0 \sin\theta_0}{g} \quad , 2\sin\theta_0 \cos\theta_0 = \sin 2\,\theta_0$$

$$R = \frac{V_0^2 \sin 2\theta_0}{g}$$

- Calculus of maximum height H:

  we set $V_y = 0$

$$V_y = V_0 \sin\theta_0 - g\,t = 0 \Rightarrow t = \frac{V_0 \sin\theta_0}{g}$$

$$\Rightarrow H = (V_0 \sin\theta_0) \frac{V_0 \sin\theta_0}{g} - \frac{1}{2} g \left(\frac{V_0 \sin\theta_0}{g}\right)^2$$

$$H = \frac{V_0^2 \sin^2\theta_0}{2g}$$

- Equation of the Trajectory:

$$x = (V_0 \cos\theta_0)\, t \Rightarrow t = \frac{x}{V_0 \cos\theta_0}$$

$$y = (V_0 \sin\theta_0)\left(\frac{x}{V_0 \cos\theta_0}\right) - \frac{1}{2} g \left(\frac{x}{V_0 \cos\theta_0}\right)^2 \quad \Rightarrow y = -\left(\frac{g}{2V_0^2 \cos^2\theta_0}\right)x^2 + (\mathrm{tg}\,\theta_0)\, x$$

This can be written in the form $y = ax^2 + bx$, which is the equation of a *parabola* that passes through the origin.

III.1.3- Motion in three dimensions

Each point M is marked by its coordinates (x, y, z) in the base $(\vec{\imath}, \vec{\jmath}, \vec{k})$ (Fig.9).

The position vector is defined by the origin point O and the components (x,y,z):

$$\overrightarrow{OM} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$$

$$\overrightarrow{OM}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\|\overrightarrow{OM}\| = OM = \sqrt{x^2 + y^2 + z^2}$$

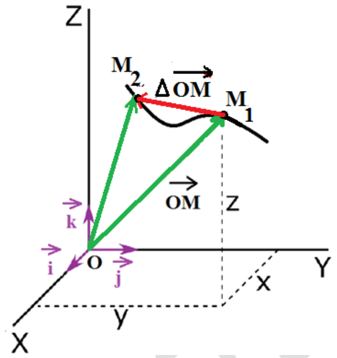


Fig.9

$$\overrightarrow{OM} = \|\overrightarrow{OM}\|\vec{u}$$

$\vec{u}$ is the unit vector: $\vec{u} = \frac{\overrightarrow{OM}}{\|\overrightarrow{OM}\|}$

- The displacement of M: $\Delta\overrightarrow{OM} = \overrightarrow{OM_2} - \overrightarrow{OM_1} = \Delta x\,\vec{\imath} + \Delta y\,\vec{\jmath} + \Delta z\,\vec{k}$. So:
- The elementary displacement of M: $d\overrightarrow{OM} = dx\,\vec{\imath} + dy\,\vec{\jmath} + dz\,\vec{k}$
- The elementary surface: ds = dxdy (or: ds = dydz, or: ds = dzdx).
- The elementary volume dv = dxdydz.
- The velocity vector: $\vec{V} = \frac{d\overrightarrow{OM}}{dt} = \frac{dx}{dt}\vec{\imath} + \frac{dy}{dt}\vec{\jmath} + \frac{dz}{dt}\vec{k} = \dot{x}\vec{\imath} + \dot{y}\vec{\jmath} + \dot{z}\vec{k}$

$$\|\vec{V}\| = V = \left(\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\right)$$

- The acceleration vector: $\vec{a} = \frac{d^2\overrightarrow{OM}}{dt^2} = \frac{d^2x}{dt^2}\vec{\imath} + \frac{d^2y}{dt^2}\vec{\jmath} + \frac{d^2z}{dt^2}\vec{k} = \ddot{x}\vec{\imath} + \ddot{y}\vec{\jmath} + \ddot{z}\vec{k}$

$$\|\vec{a}\| = a = \left(\sqrt{\ddot{x}^2 + \ddot{y}^2 + \ddot{z}^2}\right)$$

III.2- Polar coordinates system

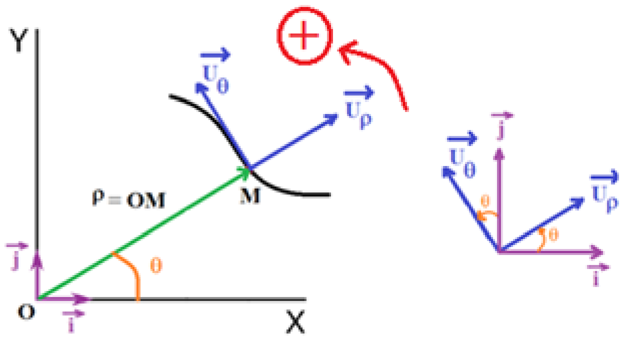
It is orthonormal. It consists of two unit vectors $(\vec{u}_\rho, \vec{u}_\theta)$ which move with time.

Each point M is identified by its coordinates (ρ, θ) in the base $(\vec{u}_\rho, \vec{u}_\theta)$ (Fig. 10).

θ is the angle between $\overrightarrow{OM}$ and $\vec{\imath}$.

$\vec{u}_\theta$ is perpendicular to $\vec{u}_\rho$.

Fig. 10



Each point M is identified by its coordinates (ρ, θ) in the base $(\vec{u}_\rho, \vec{u}_\theta)$.

The position vector is defined by:

$$\overrightarrow{OM} = \rho\vec{u}_\rho$$

$$\overrightarrow{OM}\begin{pmatrix}\rho \\ \theta\end{pmatrix}$$

$$\|\overrightarrow{OM}\| = OM = \rho$$

In the Cartesian reference frame: $\overrightarrow{OM} = x\vec{\imath} + y\vec{\jmath}$

So we have:

$$\begin{cases} x = \rho cos\theta \\ y = \rho sin\theta \end{cases}$$

$$tg\theta = \frac{y}{x} \quad , \qquad \rho = \sqrt{x^2 + y^2}$$

$$\vec{u}_\rho = cos\theta\vec{\imath} + sin\theta\vec{\jmath}$$

$$\vec{u}_\theta = -sin\theta\vec{\imath} + cos\theta\vec{\jmath}$$

Noted that: $\vec{u}_\theta = \frac{d\vec{u}_\rho}{d\theta}$

- The elementary displacement of M: $d\vec{M} = d\rho\vec{u}_\rho + \rho d\theta\vec{u}_\theta$

- The elementary surface: $ds = \rho\, d\rho d\theta$

The velocity vector: $\vec{V} = \frac{d\overrightarrow{OM}}{dt} = \frac{d(\rho\vec{u}_\rho)}{dt} = \frac{d\rho}{dt}\vec{u}_\rho + \rho\frac{d\theta}{dt}\vec{u}_\theta \Rightarrow \vec{V} = \dot{\rho}\vec{u}_\rho + \rho\dot{\theta}\vec{u}_\theta$

$$\|\vec{V}\| = V = \sqrt{\dot{\rho}^2 + (\rho\dot{\theta})^2}$$

The acceleration vector: $\vec{a} = \frac{d\vec{V}}{dt} = \frac{d(\dot{\rho}\vec{u}_\rho + \rho\dot{\theta}\vec{u}_\theta)}{dt}$

$$\vec{a} = \frac{d\dot{\rho}}{dt}\vec{u}_\rho + \dot{\rho}\frac{d\vec{u}_\rho}{dt} + (\frac{d\rho}{dt})\dot{\theta}\vec{u}_\theta + \rho\frac{d\dot{\theta}}{dt}\vec{u}_\theta + \rho\dot{\theta}\frac{d\vec{u}_\theta}{dt}$$

$$\vec{a} = \ddot{\rho}\vec{u}_\rho + \dot{\rho}\,\dot{\theta}\vec{u}_\theta + \dot{\rho}\dot{\theta}\vec{u}_\theta + \rho\ddot{\theta}\vec{u}_\theta - \rho\dot{\theta}^{2}\vec{u}_\rho$$

$$\vec{a} = (\ddot{\rho} - \rho\dot{\theta}^2)\vec{u}_\rho + (2\dot{\rho}\,\dot{\theta} + \rho\ddot{\theta})\vec{u}_\theta$$

$$\|\vec{a}\| = a = \left(\sqrt{(\ddot{\rho} - \rho\dot{\theta}^2)^2 + (2\dot{\rho}\,\dot{\theta} + \rho\ddot{\theta})^2}\right)$$

III.3- Cylindrical coordinates system

It is orthonormal. It consists of three unit vectors $(\vec{u}_\rho, \vec{u}_\theta, \vec{k})$ which the two vectors $(\vec{u}_\rho, \vec{u}_\theta)$ varies with time while $\vec{k}$ is invariable (Fig.11).

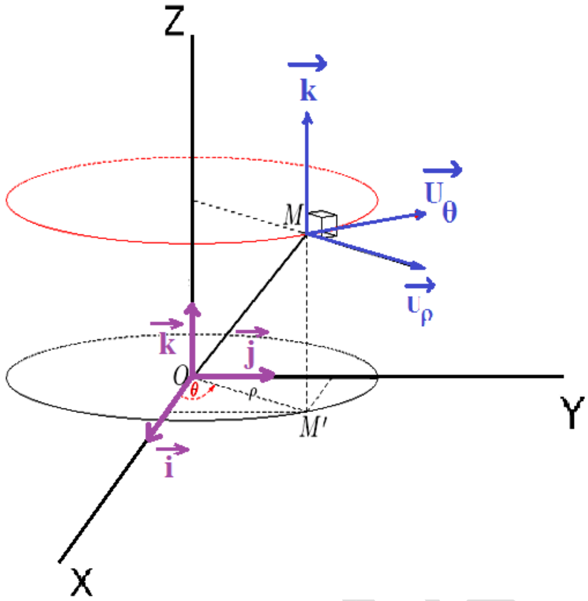


Fig.11

Each point M is identified by its coordinates (ρ, θ, z) in the base $(\vec{u}_\rho, \vec{u}_\theta, \vec{k})$.

θ is the angle between $\overrightarrow{OM'}$ and $\vec{\imath}$, M' is the projection of M in the plane (xoy).

The position vector is defined from the origin point O and the components (ρ, θ, z):

$$\overrightarrow{OM} = \overrightarrow{OM}' + \overrightarrow{M'M} \Rightarrow \overrightarrow{OM} = \rho\vec{u}_\rho + z\vec{k}$$

$$\overrightarrow{OM}\begin{pmatrix}\rho\\ \theta\\ z\end{pmatrix}$$

$$\|\overrightarrow{OM}\| = OM = \sqrt{\rho^2 + z^2}$$

In the Cartesian reference frame: $\overrightarrow{OM} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$

So we have :

$$\begin{cases} x = \rho cos\theta \\ y = \rho sin\theta \\ z = z \end{cases}$$

$$OM = \sqrt{\rho^2 + z^2} \quad , \quad \rho = \sqrt{x^2 + y^2} \quad , \quad tg\theta = \frac{y}{x}$$

- The elementary displacement of M: $d\overrightarrow{OM} = d\rho\, \vec{u}_\rho + \rho d\theta\, \vec{u}_\theta + dz\, \vec{k}$
- The elementary surface: $ds = \rho\, d\theta dz$ (the side area)

- The elementary volume: $dv = \rho\, d\rho d\theta dz$

The velocity vector : $\vec{V}=\frac{d\overrightarrow{OM}}{dt}=\frac{d(\rho\vec{u}_\rho+z\vec{k})}{dt}=\frac{d\rho}{dt}\vec{u}_\rho+\rho\frac{d\theta}{dt}\vec{u}_\theta+\dot{z}\vec{k}\Rightarrow\vec{V}=\dot{\rho}\,\vec{u}_\rho+\rho\dot{\theta}\,\vec{u}_\theta+\dot{z}\,\vec{k}$

$$\|\vec{V}\| = V = \left(\sqrt{\dot{\rho}^2+(\rho\dot{\theta})^2+\dot{z}^2}\right)$$

- The acceleration vector: $\vec{a}=\frac{d\vec{V}}{dt}=\frac{d(\dot{\rho}\vec{u}_\rho+\rho\dot{\theta}\vec{u}_\theta++\dot{z}\vec{k})}{dt}$

$$\vec{a}=\frac{d\dot{\rho}}{dt}\vec{u}_\rho+\dot{\rho}\frac{d\vec{u}_\rho}{dt}+(\frac{d\rho}{dt})\dot{\theta}\vec{u}_\theta+\rho\frac{d\dot{\theta}}{dt}\vec{u}_\theta+\rho\dot{\theta}\frac{d\vec{u}_\theta}{dt}+\frac{d\dot{z}}{dt}\vec{k}$$

$$\vec{a}=\ddot{\rho}\vec{u}_\rho+\dot{\rho}\,\dot{\theta}\vec{u}_\theta+\dot{\rho}\dot{\theta}\vec{u}_\theta+\rho\ddot{\theta}\vec{u}_\theta-\rho\dot{\theta}^2\vec{u}_\rho+\ddot{z}\,\vec{k}$$

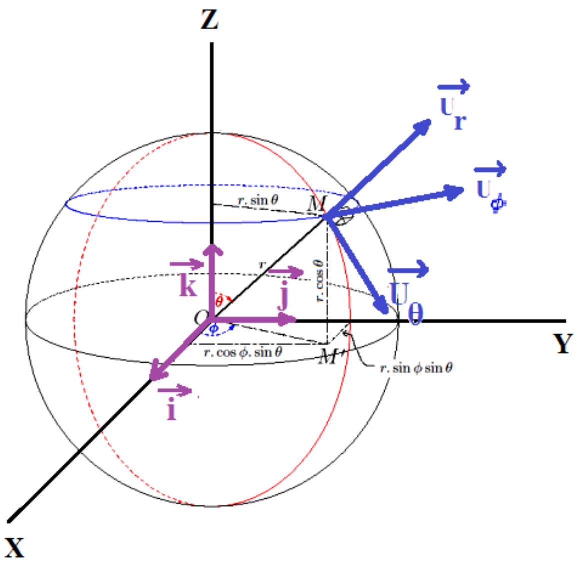
$$\vec{a}=(\ddot{\rho}-\rho\dot{\theta}^2)\vec{u}_\rho+(2\dot{\rho}\,\dot{\theta}+\rho\ddot{\theta})\vec{u}_\theta+\ddot{z}\,\vec{k}$$

$$\|\vec{a}\| = a = \sqrt{\left(\ddot{\rho}-\rho\dot{\theta}^2\right)^2+(2\dot{\rho}\,\dot{\theta}+\rho\ddot{\theta})^2+\ddot{z}^2}$$

III.4- Spherical coordinates system

The spherical reference frame is orthonormal. It consists of three unit vectors $(\vec{u}_r,\vec{u}_\theta,\vec{u}_\Phi)$ which vary with time.

Each point M is identified by its coordinates (r, θ, Φ) in the base $(\vec{u}_r,\vec{u}_\theta,\vec{u}_\Phi)$ (Fig. 12).



$\theta = (\widehat{oz,oM}) = [0,\pi]$, $\Phi = (\widehat{ox,oM'}) = [0,2\pi]$

Fig.12

The position vector is defined from the origin point O and the components (r, θ, Φ):

$$\overrightarrow{\mathbf{OM}} = \mathbf{r}\vec{\mathbf{u}}_{\mathbf{r}}$$

$$\overrightarrow{\mathbf{OM}}\begin{pmatrix}\mathbf{r}\\ \boldsymbol{\theta}\\ \boldsymbol{\Phi}\end{pmatrix}, \quad (\mathbf{0} \le \boldsymbol{\Phi} \le \mathbf{2\pi}) \ \text{ and } (\mathbf{0} \le \boldsymbol{\theta} \le \boldsymbol{\pi})$$

$$\left\|\overrightarrow{\mathbf{OM}}\right\| = \mathbf{OM} = \mathbf{r}$$

$$\overrightarrow{\mathbf{OM}} = \overrightarrow{\mathbf{OM'}} + \overrightarrow{\mathbf{M'M}}$$

$$\overrightarrow{OM'} = r'(\cos\Phi\vec{\imath} + \sin\Phi\vec{\jmath})$$

$$r' = r\sin\theta$$

$$\overrightarrow{OM'} = r\sin\theta(\cos\Phi\vec{\imath} + \sin\Phi\vec{\jmath}) = r\sin\theta\cos\Phi\vec{\imath} + r\sin\theta\sin\Phi\vec{\jmath}$$

$$\overrightarrow{M'M} = r\cos\theta\vec{k}$$

So :

$$\overrightarrow{OM} = r\sin\theta\cos\Phi\vec{\imath} + r\sin\theta\sin\Phi\vec{\jmath} + r\cos\theta\vec{k}$$

In the Cartesian reference frame: $\overrightarrow{OM} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$

We deduce that:

$$\begin{cases} \mathbf{x} = \mathbf{r}\sin\boldsymbol{\theta}\cos\boldsymbol{\Phi} \\ \mathbf{y} = \mathbf{r}\sin\boldsymbol{\theta}\sin\boldsymbol{\Phi} \\ \mathbf{z} = \mathbf{r}\cos\boldsymbol{\theta} \end{cases}$$

$$\overrightarrow{\mathbf{OM}} = \mathbf{r}\sin\boldsymbol{\theta}\cos\boldsymbol{\Phi}\vec{\mathbf{i}} + \mathbf{r}\sin\boldsymbol{\theta}\sin\boldsymbol{\Phi}\vec{\mathbf{j}} + \mathbf{r}\cos\boldsymbol{\theta}\vec{\mathbf{k}} = \mathbf{r}(\sin\boldsymbol{\theta}\cos\boldsymbol{\Phi}\vec{\mathbf{i}} + \sin\boldsymbol{\theta}\sin\boldsymbol{\Phi}\vec{\mathbf{j}} + \cos\boldsymbol{\theta}\vec{\mathbf{k}}) = \mathbf{r}\vec{\mathbf{u}}_{\mathbf{r}}$$

$$\Rightarrow \overrightarrow{\mathbf{OM}} = \mathbf{r}\,\vec{\mathbf{u}}_{\mathbf{r}}$$

$\mathbf{r} = \sqrt{\mathbf{x^2 + y^2 + z^2}}$, $\boldsymbol{\rho} = \sqrt{\mathbf{x^2 + y^2}}$, $\mathbf{tg}\boldsymbol{\Phi} = \frac{\mathbf{y}}{\mathbf{x}}$, $\mathbf{tg}\boldsymbol{\theta} = \frac{\boldsymbol{\rho}}{\mathbf{z}} = \frac{\sqrt{\mathbf{x^2+y^2}}}{\mathbf{z}}$

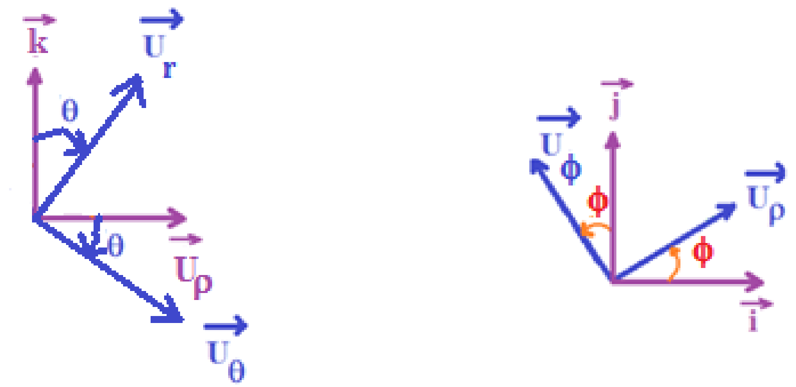


Fig.13

$$\vec{u}_\rho = \cos\Phi\vec{i} + \sin\Phi\vec{j}$$

$$\vec{u}_\varphi = -\sin\Phi\vec{i} + \cos\Phi\vec{j}$$

$$\vec{u}_\theta = \cos\theta\,\vec{u}_\rho - \sin\theta\,\vec{k} \Rightarrow \vec{u}_\theta = \cos\theta\cos\Phi\,\vec{i} + \cos\sin\Phi\,\vec{j} - \sin\theta\vec{k}$$

$$\vec{u}_r = \sin\theta\,\vec{u}_\rho + \cos\theta\,\vec{k} \Rightarrow \vec{u}_r = \sin\theta\cos\Phi\,\vec{i} + \sin\theta\sin\Phi\,\vec{j} + \cos\theta\,\vec{k}$$

$\vec{u}_r$ is the radial unit vector.

$$\frac{\partial\vec{u}_r}{\partial\theta} = \vec{u}_\theta = \cos\theta\cos\Phi\,\vec{i} + \cos\sin\Phi\,\vec{j} - \sin\theta\vec{k}$$

$\vec{u}_\theta$ *is the ortho* $-$ *radial vector.*

$$\vec{u}_\Phi = \frac{1}{\cos\theta}\frac{\partial\vec{u}_\theta}{\partial\Phi} = \frac{1}{\sin\theta}\frac{\partial\vec{u}_r}{\partial\Phi} = -\sin\Phi\vec{i} + \cos\Phi\vec{j}$$

- The elementary displacement of M: $d\overrightarrow{OM} = dr\vec{u}_r + rd\vec{u}_r = dr\vec{u}_r + r(\frac{\partial\vec{u}_r}{\partial\theta}d\theta + \frac{\partial\vec{u}_r}{\partial\Phi}d\Phi)$

$$d\overrightarrow{OM} = dr\vec{u}_r + r(\,d\theta\,\vec{u}_\theta + \sin\theta\,d\Phi\,\vec{u}_\Phi)$$

- The elementary surface: $dS = r^2\sin\theta\,d\theta\,d\Phi$
- The elementary volume $dV = r^2 dr\sin\theta\,d\theta\,d\Phi$

- The velocity vector : $\vec{V} = \frac{d\overrightarrow{OM}}{dt} = \frac{d(r\vec{u}_r)}{dt} = \frac{dr}{dt}\vec{u}_r + r\frac{d\vec{u}_r}{dt} = \frac{dr}{dt}\vec{u}_r + r(\frac{\partial\vec{u}_r}{\partial\theta}\frac{d\theta}{dt} + \frac{\partial\vec{u}_r}{\partial\Phi}\frac{d\Phi}{dt})$

$$\vec{V} = \dot{r}\vec{u}_r + r(\dot{\theta}\vec{u}_\theta + \dot{\Phi}\sin\theta\,\vec{u}_\Phi)$$

$$\|\vec{V}\| = V = \sqrt{\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\Phi}^2(\sin\theta)^2}$$

- The acceleration vector: $\vec{a} = \frac{d\vec{V}}{dt} = \frac{d}{dt}\left(\dot{r}\vec{u}_r + r\left(\dot{\theta}\vec{u}_\theta + \dot{\Phi}\sin\theta\,\vec{u}_\Phi\right)\right)$

$$\vec{a} = \begin{cases} a_r = \ddot{r} - r\dot{\Phi}^2(\sin\theta)^2 - r\dot{\theta}^2 \\ a_\theta = 2\dot{r}\dot{\theta} + r\ddot{\theta} - r\dot{\Phi}^2\sin\theta\cos\theta \\ a_\Phi = 2\dot{r}\dot{\Phi}\sin\theta + r\ddot{\Phi}\sin\theta + 2r\dot{\theta}\dot{\Phi}\cos\theta \end{cases}$$

$$\|\vec{a}\| = a = \left(\sqrt{{a_r}^2 + {a_\theta}^2 + {a_\Phi}^2}\right)$$

III.5- Intrinsic coordinate system (Frenet system)

The intrinsic coordinate system for each point of the trajectory is defined as a system of reference formed by ***two axes*** $(\overrightarrow{u_T}, \overrightarrow{u_N})$ (Fig.14):

- ***Tangent axis***: its direction is *tangent* to the trajectory and is positive in the same direction than the velocity at that point. It is defined by the unit vector $\overrightarrow{u_T}$
- ***Normal axis***: it is *perpendicular* to the trajectory and is positive toward the center of curvature of the trajectory. It is defined by the unit vector $\overrightarrow{u_N}$

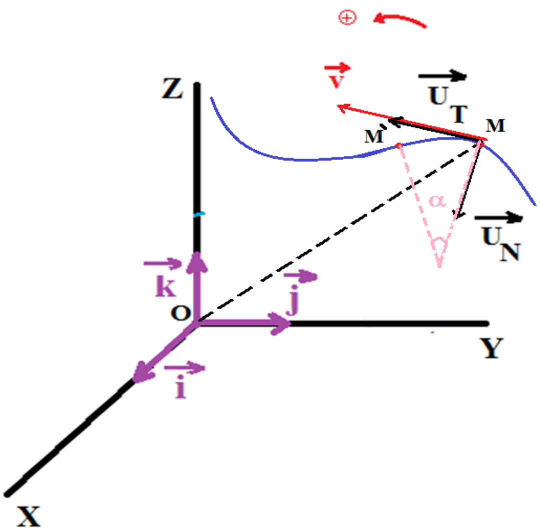


Fig.14

- Curvilinear abscissa

In this frame of reference, we define the curvilinear abscissa ***S*** of the point M along the trajectory as being equal to the length of the arc $\widehat{MM'}$. Noting that:

$\overrightarrow{u_N} = \frac{d\overrightarrow{u_T}}{ds}$, $\Re$ is the radius of curvature.

The velocity vector: $\vec{V} = V\,\overrightarrow{u_T},\ V = \frac{dS}{dt}$

The acceleration vector:

$$\vec{a} = \frac{d\vec{V}}{dt} = \frac{d(V\,\overrightarrow{u_T})}{dt} = \frac{dV}{dt}\vec{u}_T + V\frac{d\vec{u}_T}{dt}$$

$$\frac{d\vec{u}_T}{dt} = \frac{d\vec{u}_T}{d\alpha}\frac{d\alpha}{dt} = \overrightarrow{u_N}\frac{d\alpha}{dt}$$

$$d\vec{u}_T = \overrightarrow{u_N}\,d\alpha$$

$$dS = \Re d\alpha$$

$$d\vec{u}_T = \vec{u_N}\,\frac{dS}{\Re}$$

$$\vec{u_N} = \Re\,\frac{d\vec{u}_T}{dS}$$

$$\frac{d\vec{u}_T}{dt} = \frac{d\vec{u}_T}{dS}\,\frac{dS}{dt}$$

$$\frac{d\vec{u}_T}{dt} = \frac{d\vec{u}_T}{dS}\,V$$

$$\frac{d\vec{u}_T}{dt} = \frac{V}{\Re}\,\vec{u_N}$$

$$\vec{a} = \frac{dV}{dt}\vec{u}_T + V\frac{d\vec{u}_T}{dt}$$

$$\vec{a} = \frac{dV}{dt}\vec{u}_T + V\frac{V}{\Re}\vec{u}_N$$

$$\vec{a} = \frac{dV}{dt}\vec{u}_T + \frac{V^2}{\Re}\vec{u}_N$$

$$\vec{a} = a_T\vec{u}_T + a_N\vec{u}_N$$

$$\begin{cases} a_T = \dfrac{dV}{dt} \text{ ; tangential acceleration} \\ a_N = \dfrac{V^2}{\Re} \text{ ; normal acceleration} \end{cases}$$

$$\|\vec{a}\| = a = \sqrt{{a_T}^2 + {a_N}^2}$$

IV- Curvilinear movement

IV.1- Circular movement

In this case, the trajectory is not a straight line, but a circle of radius R (R is constant) (Fig.15).

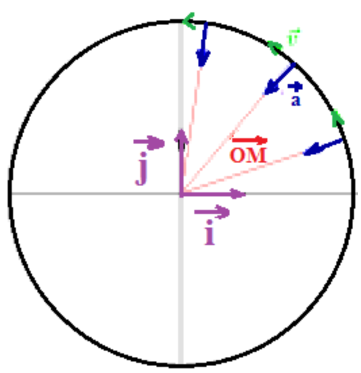


Fig. 15

Using polar coordinates: $\overrightarrow{OM} = \rho\vec{u}_\rho = R\,\vec{u}_\rho$

The velocity vector $\vec{V} = \frac{d\overrightarrow{OM}}{dt} = \frac{d(R\vec{u}_\rho)}{dt} = \frac{dR}{dt}\vec{u}_\rho + R\frac{d\theta}{dt}\vec{u}_\theta \quad (R = constant \Rightarrow \frac{dR}{dt} = 0)$

$$\vec{V} = R\dot{\theta}\vec{u}_\theta$$

$$\|\vec{V}\| = V = R\dot{\theta} = Rw \text{ } (\dot{\theta} = w \text{ the angular velocity})$$

The acceleration vector: $\vec{a} = \frac{d\vec{V}}{dt} = \frac{d(R\dot{\theta}\,\vec{u}_\theta)}{dt}$

$$= R\frac{d\dot{\theta}}{dt}\vec{u}_\theta + R\dot{\theta}\frac{d\vec{u}_\theta}{dt}$$

$$\vec{a} = R\ddot{\theta}\vec{u}_\theta - R\dot{\theta}^2\vec{u}_\rho$$

$$\vec{a} = R(-\dot{\theta}^2\vec{u}_\rho + \ddot{\theta}\vec{u}_\theta)$$

$$\vec{a} = R(-\dot{\theta}^2\vec{u}_\rho + \alpha\vec{u}_\theta) \text{ } (\ddot{\theta} = \alpha \text{ angular acceleration})$$

$$\|\vec{a}\| = a = R\sqrt{(\dot{\theta}^2)^2 + (\alpha)^2}$$

- Using intrinsic coordinates

$$\vec{u}_\rho = -\vec{u}_N, \quad \vec{u}_\theta = \vec{u}_T$$

$$\vec{V} = R\,\dot{\theta}\,\vec{u}_T$$

The acceleration vector: $\vec{a} = R(\dot{\theta}^2\vec{u}_N + \ddot{\theta}\vec{u}_T)$

$$\begin{cases} a_T = \frac{dV}{dt} = R\ddot{\theta} \\ a_N = \frac{V^2}{\Re} = R\dot{\theta}^2 = \sqrt{a^2 - a_T^2} \end{cases}$$

$\frac{V^2}{\Re} = R\dot{\theta}^2 \Rightarrow \frac{(R\dot{\theta})^2}{\Re} = R\dot{\theta}^2 \Rightarrow \Re = R$ (radius of curvature = radius of the circle)

The curvilinear abscissa:

$V = \frac{dS}{dt} = R\frac{d\theta}{dt} \Rightarrow ds = R\,d\theta \Rightarrow \int_{s_0}^{s} dS = \int_{\theta_0}^{\theta} Rd\theta \Rightarrow S - S_0 = R(\theta - \theta_0)$

$$\Rightarrow S = R(\theta - \theta_0) + S_0$$

IV.2- Uniform circular motion

$\dot{\theta} = w_0 = C^{te}$, $\ddot{\theta} = 0$

$$\dot{\theta} = \frac{d\theta}{dt} = w_0 \Rightarrow \int_{\theta_0}^{\theta} d\theta = \int_0^t w_0 dt \Rightarrow \theta - \theta_0 = w_0 t$$

$$\Rightarrow \theta = w_0 t + \theta_0$$

In polar coordinates : $\vec{V} = R w_0 \vec{u}_\theta$

$$\vec{a} = R(-\dot{\theta}^2 \vec{u}_\rho + \ddot{\theta}\vec{u}_\theta) = -R{w_0}^2 \vec{u}_\rho$$

In intrinsic coordinates: $\vec{V} = R w_0 \vec{u}_T$

$$\vec{a} = R{w_0}^2 \vec{u}_N$$

The curvilinear abscissa: $s = R(\theta - \theta_0) + s_0 = R(w_0 t) + s_0$

IV.2.1- Accelerated uniform circular motion

$\ddot{\theta} = \alpha_0 = \dot{w} = C^{te} \Rightarrow \frac{dw}{dt} = \alpha_0 \Rightarrow \int_{w_0}^{w} dw = \int_0^t \alpha_0 dt \Rightarrow w - w_0 = \alpha_0 t$

$$\Rightarrow w = \alpha_0 t + w_0$$

$$W = \frac{d\theta}{dt} \Rightarrow \int_{\theta_0}^{\theta} d\theta = \int_0^t w dt \Rightarrow \int_{\theta_0}^{\theta} d\theta = \int_0^t (\alpha_0 t + w_0) dt$$

$$\Rightarrow \theta - \theta_0 = \frac{1}{2}\alpha_0 t^2 + w_0 t \Rightarrow \theta = \frac{1}{2}\alpha_0 t^2 + w_0 t + \theta_0$$

$$\vec{V} = R\dot{\theta}\vec{u}_T = Rw\vec{u}_T \Rightarrow \vec{V} = R(\alpha_0 t + w_0)\,\vec{u}_T$$

$$\begin{cases} a_T = \dfrac{dV}{dt} = R\alpha_0 \\ a_N = \dfrac{V^2}{\Re} = \dfrac{V^2}{R} = R(\alpha_0 t + w_0)^2 \end{cases}$$

$$S = R(\theta - \theta_0) + S_0 \Rightarrow S = R(\frac{1}{2}\alpha_0 t^2 + w_0 t) + S_0$$

IV.2.2- Angular rotation velocity vector

Let the plane of motion be the (xoy) plane and (oz) the axis of rotation. The angular velocity vector of rotation, or simply vector rotation is $\vec{w}$:

$$\vec{w} = w\vec{k}$$

$$\vec{V} = Rw\vec{u}_\theta = Rw\,(\vec{k} \wedge \vec{u}_\rho) = w\vec{k} \wedge R\vec{u}_\rho$$

$$\overrightarrow{OM} = R\vec{u}_\rho \quad \Rightarrow \quad \vec{V} = \vec{w} \wedge \overrightarrow{OM} \quad \Rightarrow \quad \frac{d\overrightarrow{OM}}{dt} = \vec{w} \wedge \overrightarrow{OM}$$

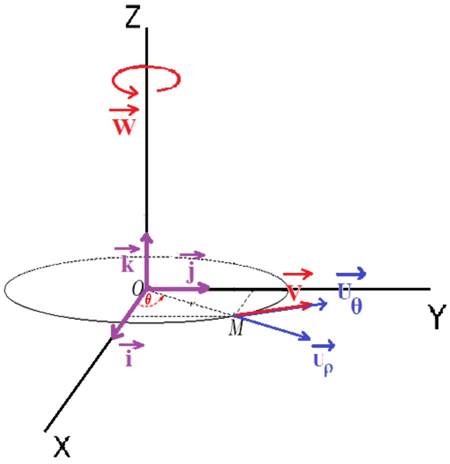


Fig.16

IV.3- Harmonic motion (rectilinear sinusoidal)

The motion of a solid is said to be rectilinear and sinusoidal if its time law is written in the form:

$$x(t) = x_m sin(\omega t + \varphi_0)$$

x: is also called the elongation of the solid at time t (m).

x_m : is the amplitude of the movement (m).

$\varphi = (\omega t + \varphi_0)$ is the phase at time t (rad).

φ_0 : initial phase, at t = 0 (rad).

ω: is the pulsation of the movement (rad.s^{-1}).

Rectilinear motion is periodic and sinusoidal with period $T = \frac{w}{2\pi}$ **(s)** and a frequency $f = \frac{1}{T} = \frac{2\pi}{w}$ **(Hz).**

$$V(t) = \frac{dx}{dt} = \omega x_m cos(\omega t + \varphi_0)$$

$$a(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2} = -\omega^2 x_m sin(\omega t + \varphi_0) \Rightarrow a = -\omega^2 x$$