Chapter IV: Dynamics of a material point

I- Introduction

This chapter is devoted to the fundamental laws of dynamics, the part of ***"classical"*** mechanics. Dynamics studies the motion of bodies in relation to those factors (forces), which cause a particular pattern of motion.

Note that Newton's laws are applied in the reference frame of inertia (Galilean) (already defined in 1st chapter) and if the speed of the bodies is less than the velocity of light.

Reminder:

- ***Copernicus referential:*** Its origin is at the barycentre of the solar system and their axes point towards the distant stars known as the "fixed stars".

- ***Galilean referential:*** is rerefential with uniform rectilinear motion relative to Copernicus referential.

The Galilean referential is also known as the inertial referential.

Any coordinate system that moves at constant velocity relative to an inertial referential can itself be considered as an inertial referential.

II- Newton's laws

In 1665, Newton had the idea that there was a link between Galileo's laws of falling bodies (1610) and Kepler's laws of planetary motion (1618). It took him more than 20 years to make this link by determining a general relationship between cause and motion (2nd Newton's law).

We will first study the laws of motion of a material point, known as Newton's 1st and 2nd laws:

- The first law, the principle of inertia, was published by Galileo in 1632.

- In the second law, better known as the fundamental law of dynamics, Newton introduced the notion of momentum, which combines velocity and mass.

The 3th law is the law of opposition of reciprocal actions between two material points.

Newton's 4th law concerns gravitation. We also give some types of force. We also present the definition of angular momentum, the angular momentum theorem and the conservation of angular momentum**.**

II .1- Newton's first law or law of inertia

In an inertial or Galilean referential, any isolated body (or material point) (not subject to external actions) is either at rest or in uniform rectilinear motion.

Such a body is said to be free or isolated and its motion is said to be free or inertial.

$$\sum \vec{F} = \vec{0} \Rightarrow \begin{cases} \vec{v} = \vec{0} \\ Or \\ \vec{v} = \overrightarrow{cst} \end{cases}$$

For example:

- A body on a table is at rest when no force is exerted on it.

- Given an initial velocity V_0 to a sphere placed on a smooth surface (absence of friction forces), the sphere remains in motion with the same initial velocity along a linear trajectory.

II .2- 2nd law of Newton: conservation of momentum

II .2.1- Mass

When we want to set a body in motion or change its velocity (in modulus or direction), it resists. This ability of bodies to resist changes in their states of rest or motion is called inertia. For example, it is easier to give the same acceleration to an empty lorry than to a loaded one. The physical quantity characterising the inertia of an object is called the inertia mass of an object.

II .2.2- Momentum

The momentum of a particle, denoted by $\vec{P}$ is defined as the product of its mass and its velocity: $\vec{P}$ = m $\vec{V}$

The law of inertia can be stated as follows: " a free material point always moves with a constant momentum: $\vec{P} = \overrightarrow{cst}$

Note:

For a system composed of several material points of mass m_i and velocity $\vec{V_i}$, the total momentum is given by: $\vec{P} = m_1\vec{V_1} + m_2\vec{V_2} + m_3\vec{V_3} + m_4\vec{V_4}$ ...

$$\Rightarrow \vec{P} = \vec{P_1} + \vec{P_2} + \vec{P_3} + \vec{P_4} + \ldots\ldots\ldots\ldots = \Sigma\vec{P_i}$$

II .2.3- Principle of conservation of momentum

Consider two particles of mass m_1 and m_2 and velocity V_1 and V_2 in an isolated system, the total momentum of the system:

$\vec{P} = \vec{P_1} + \vec{P_2}$ ($\vec{P_1} = m\vec{V_1}$, $\vec{P_2} = m\vec{V_2}$)

After the collision of two particles (elastic shock), the total momentum of the system :

$\overrightarrow{P'} = \overrightarrow{P_1'} + \overrightarrow{P_2'}$ ($\overrightarrow{P_1'} = m\overrightarrow{V_1'}$, $\overrightarrow{P_2'} = m\overrightarrow{V_2'}$)

In an isolated system : $\overrightarrow{P'} = \vec{P}$

The total momentum of a system composed of two materials points subject only to their mutual interactions is constant**; this is the principle of conservation of motion.**

Example:

The regress of a firearm: initially, the system (gun + ball) is at rest and the total quantity of movement is zero. After the ball is fired, the gun regresses to counterbalance the amount of movement taken up by the ball.

II .2.4- Newton's second law

Newton's second law is known as the Fundamental Principle of Dynamics (FDP) and is expressed as follows:

"In a Galilean referential, the derivative of the momentum of a material point with respect to time is equal to the sum of the forces applied to this system".

$$\frac{d\vec{P}}{dt} = \sum\vec{F}$$

$$\vec{P} = m\,\vec{V} \Rightarrow \frac{d\vec{P}}{dt} = \frac{d(m\,\vec{V})}{dt} = m\frac{d\,\vec{V}}{dt} = m\,\vec{a}\ (\,m=cst)$$

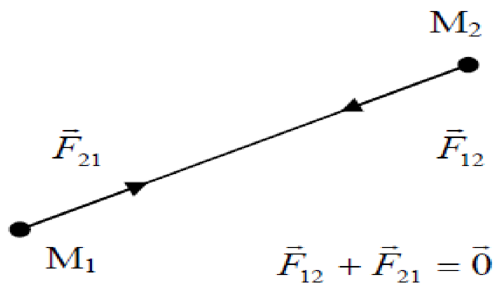
$$\Rightarrow m\,\vec{a} = \sum\vec{F}$$

The fundamental principle of dynamics links the cause of motion (force) to the motion itself.

II .3- Newton's third law (principle of reciprocal action)

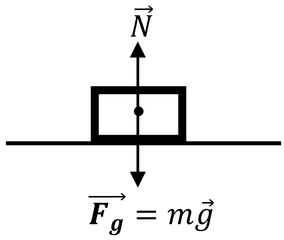
The action of one body on another causes equal in magnitude and opposite in direction reaction of the second one on the first body:

$$\vec{F}_{12} = -\,\vec{F}_{21}\ (F_{12}=F_{21})$$



Example:

A body on a table: $\overrightarrow{\boldsymbol{F_g}} = m\vec{g}$ is the action of the body on the table, $\vec{N}$ is the reaction of the table on the body.



Please note:

For a material system made up of n material points, the internal forces cancel each other out, so we write:

$$\boldsymbol{m}\,\vec{\boldsymbol{a}} = \sum \overrightarrow{\boldsymbol{F}_{\mathbf{ext}}}$$

$\sum \overrightarrow{\boldsymbol{F}_{\mathbf{ext}}}$ the sum of the external forces.

III- Forces

All observed physical phenomena are based on the notion of interaction, during which an exchange of energy, matter or momentum takes place between two distinct physical entities. There are 4 fundamental interactions of different kinds:

- Gravitational interaction.
- The strong nuclear interaction (short range 10^{-15} m) is responsible for the stability of the nucleus.
- The weak nuclear interaction (range 10^{-18} to 10^{-17} m) (radioactivity).
- The electromagnetic interaction acting on any particle with an electric charge.

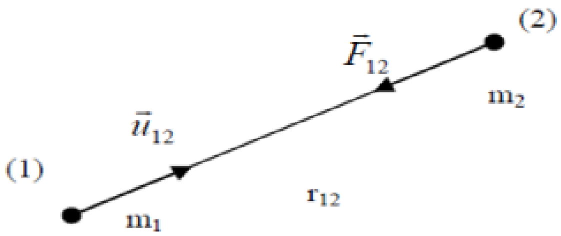
III.1- Gravitational force

Newton's most important contribution to the development of mechanics was the discovery of the law of gravitational interaction (Newton 1867):

Any material point of mass m_1 attracts any material point of mass m_2 with a force directed along the straight line connecting them. This force varies as follows:

$$\overrightarrow{F_{12}} = -G\,\frac{m_1 m_2}{r_{12}^2}\,\overrightarrow{u_{12}}$$

The constant G is called the gravitational constant: $G = 6.672\ 10^{-11}\ N\, m^2 Kg^{-2}$



At the Earth's surface, it is the Earth's gravitational force that is responsible for the fall and attraction of bodies towards the Earth. Consider a solid body of mass m in free fall:

$$\overrightarrow{\boldsymbol{F_g}} = \textbf{-mg}\, \vec{\boldsymbol{u}}$$

$$\vec{\boldsymbol{F}} = \textbf{-G}\, \frac{m_1.m_2}{r^2}\, \vec{\boldsymbol{u}} = \textbf{-G}\, \frac{m_T.m}{(R+Z)^2}\, \vec{\boldsymbol{u}} = \textbf{-G}\, \frac{m_T.m}{R^2}\, \vec{\boldsymbol{u}} \quad (z<<R)$$

$\vec{u}$ moves from the centre of the earth towards its surface.

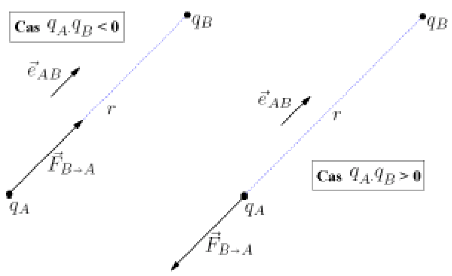
m_T: the mass of the earth, R: the radius of the earth

We find that: g= **G** = 9.81 m/s^2

III.2- Electric force

The electric force reflects the interaction between two point bodies A and B carrying charges q_A and q_B respectively and separated by a distance r. The expression of the force exerted by B on A is:

$$\overrightarrow{\boldsymbol{F_{B/A}}} = \textbf{K}\, \frac{\boldsymbol{q_A.q_B}}{\boldsymbol{r^2}}\, \vec{\boldsymbol{u}}_{\mathbf{AB}}$$



K= 9.10^9 Kg.m^3.A^{-2}.s^{-4} : Coulomb's constant. $\vec{\boldsymbol{u}}_{\mathbf{AB}}$ unit vector along $\overrightarrow{\boldsymbol{AB}}$**.**

If the charges have the same sign, they repel each other. If they have different signs, they attract each other.

III.3- Electromagnetic force

The force experienced by an electric charge placed in fields $\vec{E}$ (electric) and $\vec{B}$ (magnetic) is called the electromagnetic or Lorentz force:

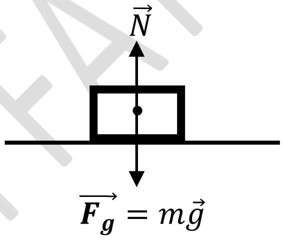
$$\vec{F} = q\left(\vec{E} + \vec{v} \wedge \vec{B}\right)$$

III.4- Contact forces

When a body is in contact with a solid or a fluid, there are contact actions. These are generally in the form of fairly simple phenomenological expressions, drawn from experience and valid only in a certain experimental context.

III.4.1- Normal Force ($\vec{N}$)

If an object placed on a table, the table exerts an upward action force $\vec{N}$ (normal force) on the object. The reaction of the table ($\vec{N}$) on the object m is distributed over the entire table-object contact surface and represents the resultant of all the actions exerted on the contact surface. The normal force $\vec{N}$ is the force that prevents the object from falling through the table, and can have any value up to the point of breaking the table.



The object is in equilibrium: $\vec{F_g} + \vec{N} = \vec{0} \Rightarrow \vec{F_g} = -\vec{N}$

III.4.2- Forces of friction

The forces of friction are forces that occur when an object moves. Friction opposes the movement of moving objects. There are two types of friction:

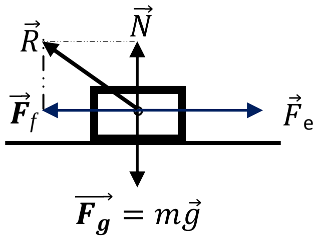
- solid friction (solid-solid contact),
- viscous friction (solid-fluid contact).

III.4.2.1- Solid friction (solid-solid contact)

The solid object, placed on a solid support, is in motion under the action of the driving force $\vec{F}_e$. A force opposite to the driving force $\vec{F}_e$ will appear and will slow down the motion; this force called the force of friction (frictional force) or simply friction: $\vec{\mathbf{f}}_f$. It is always in the opposite direction to the motion (velocity). When the friction is ignored, the surface is said to be *frictionless*.

So we can decompose the reaction force of the support $\vec{\boldsymbol{R}}$ into the force normal to the support $\vec{\boldsymbol{N}}$ and the tangent force $\vec{\mathbf{f}}_f$ called the friction force:

$$\vec{\mathbf{R}} = \vec{\mathbf{N}} + \vec{\mathbf{f}}_f$$



Experimentally, the force of friction F_f can be considered proportional to the modulus of the normal reaction force N: $F_f = \mu N$

μ : coefficient of friction: a constant that depends on the nature of the contact surface.

There are two types of coefficient of friction:

- μ_s is the static coefficient of friction which, when multiplied by N, gives the minimum force necessary to set in motion two bodies initially in contact and immobile relative to each other.

- μ_c is the kinetic coefficient of friction which, when multiplied by N, gives the force required to maintain two bodies in uniform relative motion.

When we apply a force $\vec{\mathbf{F}}_e$ on the object to move it to the right, the object will remain immobile if $\vec{\mathbf{F}}_e$ is not large enough. The frictional force $\vec{\mathbf{f}}_f$ acts to the left and **keeps the object immobile**. We call this frictional force the **force of static friction**. If we increase $\vec{\mathbf{F}}_e$, the static frictional force increases $\vec{\mathbf{f}}_s$, while the object remains at rest. When the applied force $\vec{\mathbf{F}}_e$ **reaches a certain value**, the object will be on the verge of slipping and the frictional force will be maximum, $\mathbf{f}_{f,\,max}$. When $\mathbf{F}_e$ **exceeds *fs*,max, the object moves to the right**. When the object is in motion, the frictional force becomes less than $f_{s,max}$ and is called the **force of kinetic friction** f_c.

Note that: $\mu_s > \mu_c$.

The values of the dimensionless coefficients μ_s and μ_c depend on the nature of the surfaces, not on their areas.

III.4.2.2- Viscous friction (in liquids)

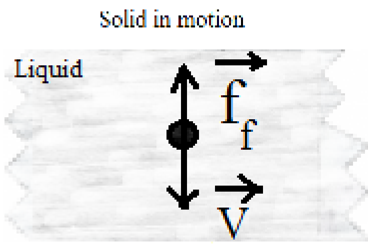
When an object moves through a fluid (liquid or gas) at a relatively low velocity, the force of friction is proportional to the velocity, and in the opposite direction:

$$\vec{f}_f = -k\,\eta\,\vec{V} \qquad \text{(Formula valid only for low velocities)}$$

k: positive coefficient linked to the shape of the object.

η is the coefficient of viscosity ($Kg.m^{-1}.s^{-1}$) of the fluid.

$\vec{V}$: object velocity vector .

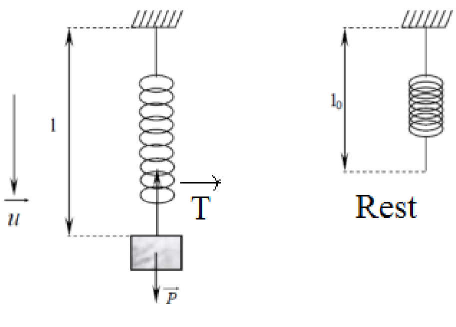


For high velocities: $f_f = k\,V^n$ ($n \geq 2$)

III.5- Tension forces

When a cord (or spring) is attached to a body and pulled, the cord is said to be in **tension**. The tension in the cord is defined as the force that the cord exerts on the body. This force is denoted usually by the symbol $\vec{T}$. A cord is considered to be *massless* (i.e., its mass is negligible compared to the body's mass) and *non-stretchable*.

Example: Spring



$$\vec{T} = -k\,(l - l_0)\,\vec{u} = -\,k.x\,\vec{u}$$

k: coefficient of elongation (spring stiffness coefficient).

IV- Fundamental principle of dynamics (PFD) in a non-galilean reference frame

We have seen that the PFD is applied in an absolute reference frame or galilean reference frame (R). The PFD can be applied in relative (non-galilean) reference frames (R') by adding pseudo-forces to the forces exerted on the body, the Coriolis force ($\vec{F}_c$) and the drag force ($\vec{F}_e$).

$$\sum \vec{F} = m\,\overrightarrow{a_{(M/_{R})}}\ \ (dans\ R)$$

$$\sum \vec{F} = m(\,\overrightarrow{a_{(R'/_{R})}} + \overrightarrow{a_c} + \overrightarrow{a_{(M/_{R'})}}\,)$$

$$\sum \vec{F} = m\ \overrightarrow{a_{(R'/_{R})}} + m\ \overrightarrow{a_c} + m\,\overrightarrow{a_{(M/_{R'})}}$$

$$\sum \vec{F} = \overrightarrow{F_e} + \overrightarrow{F_c} + \overrightarrow{a_{(M/_{R'})}}$$

$$\sum \vec{F} - \overrightarrow{F_e} + \overrightarrow{F_c} = \overrightarrow{a_{(M/_{R'})}}$$

V- Angular momentum $\vec{\mathbf{L}}$

V.1- Definition

The angular momentum at a point O of a material point M, of mass m and velocity $\vec{V}$ is the following vector product:

$$\vec{\mathbf{L}}_{\mathbf{M/O}} = \overrightarrow{\mathbf{OM}} \wedge \vec{\mathbf{P}}$$

(the unit of angular momentum is kg.m^2 .s^{-1}).

The angular momentum is a vector perpendicular to the plane formed by $\overrightarrow{\mathbf{OM}}$ and $\vec{\mathbf{V}}$

V.2- Curvilinear movement in the plane

$$\overrightarrow{OM} = \rho\vec{u}_\rho$$

$$\vec{V} = \dot{\rho}\vec{u}_\rho + \rho\dot{\theta}\vec{u}_\theta$$

$$\vec{\mathbf{L}}_{\mathbf{M/O}} = \overrightarrow{\mathbf{OM}} \wedge \vec{\mathbf{P}} = \rho\vec{u}_\rho \wedge (\dot{\rho}\vec{u}_\rho + \rho\dot{\theta}\vec{u}_\theta)$$

$$\vec{\mathbf{L}}_{\mathbf{M/O}} = \mathbf{m}\,\rho^2\dot{\theta}\,\vec{\mathbf{k}}$$

If the motion is uniformly circular (ρ= R et $\dot{\theta}$= w= constant) :

$$\vec{\mathbf{L}}_{\mathbf{M}/_{\mathbf{O}}} = \mathbf{m}\,R^2 w\,\vec{\mathbf{k}}$$

V.3- The angular momentum theorem

The derivative with respect to time of the angular momentum at a fixed point O of a material point is equal to the momentum, at this point, of the sum of the applied forces, i.e. :

$$\frac{d\,\vec{\mathbf{L}}_{\mathbf{M/O}}}{dt} = \overrightarrow{\mathbf{OM}} \wedge \sum \vec{\mathbf{F}}_{\mathbf{ext}} = \boldsymbol{\mathcal{M}}_{\boldsymbol{M/O}}\left(\sum \vec{\mathbf{F}}_{\mathbf{ext}}\right)$$

$$\sum \vec{F} = m\,\vec{a}$$

$$\overrightarrow{OM} \wedge \left(\sum \vec{F}\right) = \overrightarrow{OM} \wedge m\,\vec{a}$$

$$\sum(\overrightarrow{OM} \wedge \vec{F}) = \overrightarrow{OM} \wedge \left(m\,\frac{d\vec{V}}{dt}\right)$$

$$\sum(\overrightarrow{OM} \wedge \vec{F}) = \frac{d}{dt}\left(\overrightarrow{OM} \wedge (m\,\vec{V})\right) - \frac{d\overrightarrow{OM}}{dt} \wedge m\,\vec{V}$$

$$\sum(\overrightarrow{OM} \wedge \vec{F}) = \frac{d}{dt}\left(\overrightarrow{OM} \wedge (m\,\vec{V})\right) - m\,\vec{V} \wedge \vec{V}$$

$$\sum(\overrightarrow{OM} \wedge \vec{F}) = \frac{d}{dt}\left(\overrightarrow{OM} \wedge (m\,\vec{V})\right)$$

$$\sum(\overrightarrow{OM} \wedge \vec{F}) = \frac{d\vec{L}}{dt}$$

V.4- Conservation of angular momentum

When the moment of the forces applied to a material point with respect to a point O is zero ($\frac{d\,\vec{\mathbf{L}}_{\mathbf{M/O}}}{dt} = \vec{\mathbf{0}}$ **and** $\vec{\mathbf{L}}_{\mathbf{M/O}} = \overrightarrow{\mathbf{cst}}$): either the system is isolated ($\sum \vec{F}_{ext} = \vec{0}$) or the forces are central ($\overrightarrow{OM} // \sum \vec{F}_{ext}$).

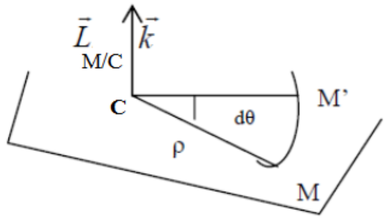
Central forces

A force is called a "central force" if it is always directed by a fixed point "c".

$\overrightarrow{\mathbf{CM}} \wedge \sum \vec{\mathbf{F}}_{\mathbf{ext}} = \vec{\mathbf{0}}$ ($\overrightarrow{CM} // \sum \vec{F}_{ext}$).

By definition, $\vec{\mathbf{L}}_{\mathbf{M/c}}$ is always perpendicular to the plane ($\overrightarrow{\mathbf{CM}}$, $\vec{\mathbf{V}}$) **and is** conserved. The trajectory of the material point is a curve in a plane ($\overrightarrow{\mathbf{CM}}$, $\vec{\mathbf{V}}$).

Now let's calculate the area swept between two instants t and t+dt.



From this figure, the area of the triangle of height ρ and base ρdθ gives:

$$\frac{dS}{dt} = \frac{1}{2}\rho^2 \frac{d\theta}{dt} = \frac{1}{2}\rho^2 \dot{\theta}$$

$$\overrightarrow{\mathbf{L}}_{\mathbf{M/C}} = \mathbf{m}\,\rho^2\dot{\theta}\,\overrightarrow{\mathbf{k}}$$

$$\frac{dS}{dt} = \frac{1}{2}\rho^2\,\dot{\theta} = \frac{1}{2}\frac{L_{M/C}}{m}$$

This result indicates that the radius vector scans equal areas for equal times. This is what happens with the planets: the force of attraction acting on the planets is always directed towards the sun (central force).

Their trajectory is therefore flat and their movement obeys the law of areas.