Ternary relations: Basic concepts and results

Matiere: Ensembles et relations flous

Cours pour les étudiants de 1 iere Année Master AMD.

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- Preliminaries
- The main results
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A pre-order fulfilling completeness is called weak order.

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Theorem (Theorem 1.1)

A binary relation $\lesssim$ on a non-empty domain X is a crisp pre-order if and only if there exists an ordered non-empty set $(Y, \preccurlyeq)$ and a mapping $f : X \to Y$ such that $\lesssim$ can be represented in the following way, for all $x, y \in X$:
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1. The binary relation $\lesssim$ is a weak order (i.e., Linear or complete pre-order) if and only if the order relation $\preccurlyeq$ is linear.
2. Analogously to the essential linearity axioms of orders (or partial orders), any pre-order can be linearized (Szpilrajn theorem for pre-orders): For any pre-order $\lesssim$, there exists a weak order $\precsim$ which extends $\lesssim$ in the sense that, for all $x, y \in X$, $x \lesssim y \Rightarrow x \precsim y$. Also, $\lesssim$ is uniquely characterized as intersection of weak orders and there is one-to-one correspondence between linearity and maximality of pre-orders.

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3. Pre-orders are also the basis for representing orders and hence other fundamental concepts in preference modeling theory.

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first concept is called T-antisymmetry or T-E-antisymmetry with E is the crisp equality (i.e., for all $x, y \in X : T(r(x, y), r(y, x)) = 0$ whenever $x \neq y$), in this case T-preorder in which the T-antisymmetry is fulfilled is called T-order.

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The second concept of antisymmetry considered in this paper is for all $x, y \in X$: $r(x, y) = r(y, x)) = 1$ implies $x = y$), in this case T-preorder in which the antisymmetry is fulfilled is called fuzzy order.

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Note that these fuzzy analogies of Theorem 1.1 are given by an alternative construction of Y and f.

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These fuzzy analogies of Theorem 1.1 allow us to answer the question given in [4] Whether there is any standard choice Y, E, R, f into which we can embed all weak T-orders (Complete fuzzy pre-orders in general).

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Also, these results let us generalize easily to the fuzzy pre-orders the linearity axioms of T-E-orders given by Bodenhofer and Klawonn , Georgescu , Gottwald , Höhle and Blanchard and Zadeh.

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In this section, we give some notations and definitions on which our work in this paper is based.

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Definition

Let X be a nonempty set. A fuzzy subset A of X is characterized by its membership function $A : X \to [0, 1]$ and $A(x)$ is interpreted as the degree of membership of the element x in the fuzzy subset A for each $x \in X$. The set of fuzzy sets on a domain X will be called fuzzy power set of X and denoted $\mathcal{F}(X)$.

2. Preliminaries (t-norms and t-conorms)

Definition

A function $T : [0,1] \times [0,1] \longrightarrow [0,1]$ is called a triangular norm (briefly, a t-norm) if the following conditions hold:
(i) for every $\alpha, \beta \in [0,1]$, we have $T(\alpha, \beta) = T(\beta, \alpha)$ (Commutativity) ;
(ii) for every $\alpha, \beta, \gamma \in [0,1]$ we have $T(\alpha, T(\beta, \gamma)) = T(T(\alpha, \beta), \gamma)$ (associativity) ;
(iii) for every $\alpha, \beta, \gamma, \lambda \in [0,1]$ if $\alpha \leq \gamma$ and $\beta \leq \lambda$, then $T(\alpha, \beta) \leq T(\gamma, \lambda)$ (order-preserving in both variables) ;
(iv) for every $\alpha \in [0,1]$, $T(\alpha, 1) = \alpha$, (neutral element).

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Definition

(1) A t-norm T is said to have zero divisors if and only if there exists a pair $(x, y) \in]0, 1[^2$ such that $T(x, y) = 0$ holds.
(2) A t-norm T_1 is said to dominate another t-norm T_2 if and only if, for any quadruple $(x, y, u, v) \in [0, 1]^4$, the following holds:
$T_1(T_2(x, y), T_2(u, v)) \geq T_2(T_1(x, u), T_1(y, v))$.

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Lemma (De Baets and Mesiar [7])

Any t-norm T dominates itself.

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(ii) The Zadeh's t-norm (resp., t-conorm) or the minimum (resp., the maximum): for every $\alpha, \beta \in [0, 1]$, we have
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(iii) The Lukasiewicz t-norm (resp., t-conorm): for every $\alpha, \beta \in [0,1]$, we have

$$T(\alpha,\beta) = \max\{\alpha + \beta - 1, 0\}.$$

2. Preliminaries (Fuzzy relations)

Given a nonempty set X, a binary fuzzy relation on X is a map $r : X \times X :\longrightarrow [0,1]$. For every $x, y \in X$, the value $r(x,y)$ is called the grade of membership of (x,y) in r and means how far x and y are related under r. Let T be a t-norm, S be a $t-$conorm and let $r : X \times X :\longrightarrow [0,1]$ be a fuzzy relation on X. We are interested in the following properties:

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- Reflexivity: if $r(x,x) = 1$ for all $x \in X$;
- Symmetry: if $r(x,y) = r(y,x)$ for all $x, y \in X$;
- T-transitivity: if $T(r(x,y), r(y,z)) \leq r(x,z)$ for all $x, y, z \in X$;
- Antisymmetry: if $r(x,y) = r(y,x)) = 1 \Rightarrow x = y$ for all $x, y \in X$;
- T-antisymmetry: if $T(r(x,y), r(y,x)) = 0$ whenever $x \neq y$ for all $x, y \in X$;
- $S-$complete (or Completeness): if $S(R(x,y); R(y,x)) = 1$.

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$S-$ complete $T-$pre-orders are called complete fuzzy pre-orders with respect to T and S, short complete T-pre-orders.

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Given a T-equivalence $E : X^2 \to [0,1]$, a binary fuzzy relation $r : X^2 \to [0,1]$ is called a fuzzy order with respect to T and E, short $T - E-$order, if it is T-preorder and additionally has the following properties:

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Next, we shall give some examples of T-preorders in $\mathbb{R}$.

1. Let $x, y \in \mathbb{R}$ and $\lambda > 0$. Then, the fuzzy relation r_λ defined for all $x, y \in \mathbb{R}$ by:

$$r_\lambda(x,y) = \begin{cases} 1\ , & if \quad x = y; \\ \min(1, \frac{|y-x|}{\lambda}), & if \quad x \neq y \end{cases} ,$$

2. Preliminaries (Examples of fuzzy pre-orders)

2. Let $X = \mathbb{R}$. Then, the fuzzy relation r defined for all $x, y \in \mathbb{R}$ by:

$$r(x,y) = \begin{cases} 1 \ , \ if \quad x = y; \\ 0 \ , \ if \quad x > y; \\ 1 - \frac{x}{y} \ , \ if \quad 0 \leq x < y; \\ 1 - \frac{y}{x} \ , \ if \quad x < y \leq 0; \\ 1 \ , \ if \quad x < 0 \quad and \quad y > 0; \end{cases} ,$$

is a T-preorder on $\mathbb{R}$.

2. Preliminaries (Preorders extensions)

Definition

Consider two T-preorders r_1 and r_2. We say that r_1 extends r_2 if and only if, for all $x, y \in X$, $r_2(x, y) \leq r_1(x, y)$ holds. For brevity, we denote this $r_2 \subseteq r_1$.

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We call r_1 a non-trivial extension of r_2 if there exists at least one pair $(x, y) \in X^2$ for which $r_2(x, y) < r_1(x, y)$ holds, for brevity $r_2 \subsetneq r_1$.

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A T-preorder r is called maximal if and only if it does not have a non-trivial extension, equivalently, $ext(r) = r$ in which $ext(r)$ is the set of all extensions of r.

3. The main results

Theorem (Theorem 3.1)

A binary fuzzy relation $r : X^2 \to [0,1]$ is a T-preorder if and only if there exist a non-empty domain $Y \subseteq \mathcal{P}(X)$, a fuzzy order (antisymmetric $T-$preorder) $R : Y^2 \to [0,1]$, and a mapping $f : X \to Y$ such that the following equality holds for all $x, y \in X$:

$$r(x,y) = R(f(x), f(y)).$$

Moreover, r is $S-$complete if and only if R is $S-$complete.

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Theorem (Theorem 3.2)

Assume that T has no zero divisors. Then A binary fuzzy relation $r : X^2 \to [0,1]$ is a T-preorder if and only if there exist a non-empty domain $Y \subseteq \mathcal{P}(X)$, a T-order (T-antisymmetric $T-$preorder) $R : Y^2 \to [0,1]$, and a mapping $f : X \to Y$ such that the following equality holds for all $x, y \in X$:

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The following Theorem 3.3 (resp. Theorem 3.4) is the same as Theorem 3.1 (resp. Theorem 3.2) but the non-empty domain Y has been chosen as a non-empty family of fuzzy sets $Y \subseteq \mathcal{F}(X)$ the fuzzy power set of X.

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Assume that T has no zero divisors. Then A binary fuzzy relation $r : X^2 \rightarrow [0,1]$ is a T-preorder if and only if there exist a non-empty domain $Y \subseteq \mathcal{F}(X)$, a T-order (T-antisymmetric $T-$preorder) $R : Y^2 \rightarrow [0,1]$, and a mapping $f : X \rightarrow Y$ such that the following equality holds for all $x, y \in X$:

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Corollary (Theorem 3.1, Bodenhofer et al. ref.4)

A binary fuzzy relation $r : X^2 \to [0,1]$ is a complete T-preorder if and only if there exist a non-empty domain Y, a complete fuzzy order (or complete T-order) $R : Y^2 \to [0,1]$, and a mapping $f : X \to Y$ such that the following equality holds for all $x, y \in X$:

$$r(x,y) = R(f(x), f(y)).$$

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Corollary (Theorem 4.1, Bodenhofer et al. ref.4)

Consider a binary fuzzy relation $r : X^2 \to [0,1]$. *Then the following two statements are equivalent: (i)* r *is a weak T-order. (ii) There exists a non-empty family of fuzzy sets* $\mathcal{S} \subseteq \mathcal{F}(X)$ *that are linearly ordered with respect to the inclusion relation* $\subseteq$ *and a mapping* $\varphi : X \to \mathcal{S}$ *such that the following representation holds for all* $x, y \in X : r(x,y) = INCL_T(\varphi(x), \varphi(y))$.

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Also, these results let us generalize easily to the fuzzy pre-orders the linearity axioms of T-E-orders given by Bodenhofer and Klawonn , Georgescu, Gottwald , Höhle and Blanchard and Zadeh.

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Theorem (Theorem 3.7)

Let r be a T-preorder. Then r is uniquely characterized as intersection of Complete preorders.

4. References

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