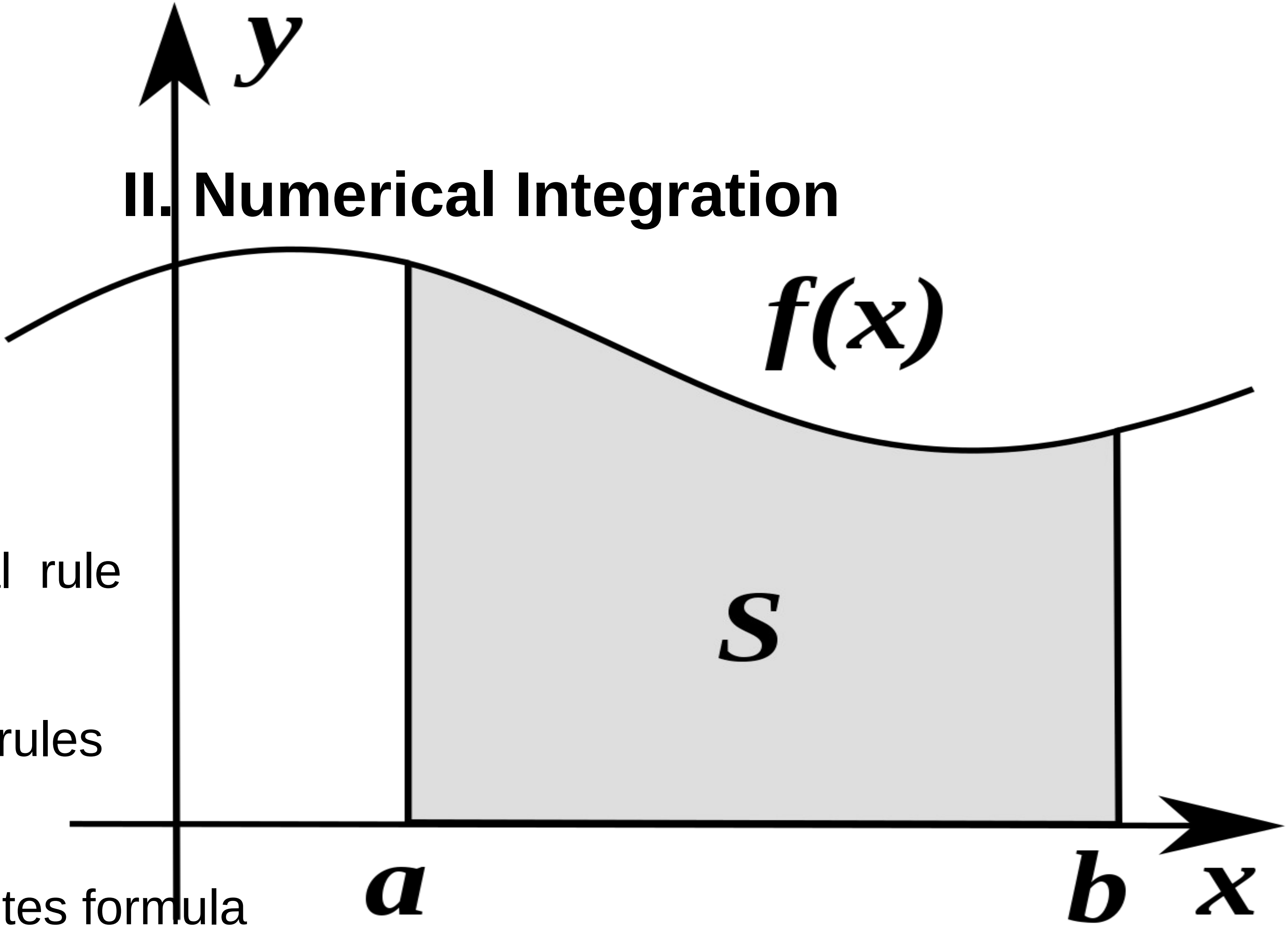
II. Numerical Integration

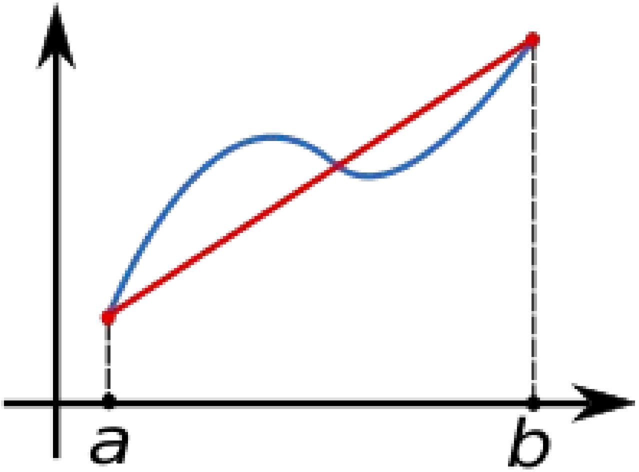


- Trapezoidal rule
- Simpson’s rules
- Newton-Cotes formula

Trapezoidal rule

the *trapezoidal rule* is the oldest[1] (*Babylon before 50 BCE*)technique for approximating the definite integral.

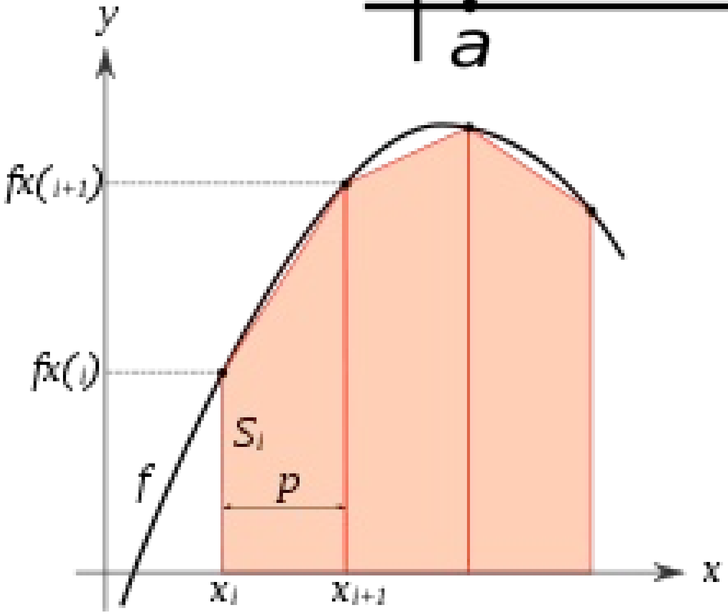
$$\int_a^b f(x)\,dx \approx (b-a)\cdot\frac{f(a)+f(b)}{2}.$$



The integral can be even better approximated by

partitioning the integration interval

$$\int_a^b f(x)\,dx \approx \sum_{k=1}^{N}\frac{f(x_{k-1})+f(x_k)}{2}\Delta x_k$$



1. Science. 351 (6272): 482–484. doi:10.1126/science.aad8085

The error of the composite trapezoidal rule is the difference between the value of the integral and the numerical result

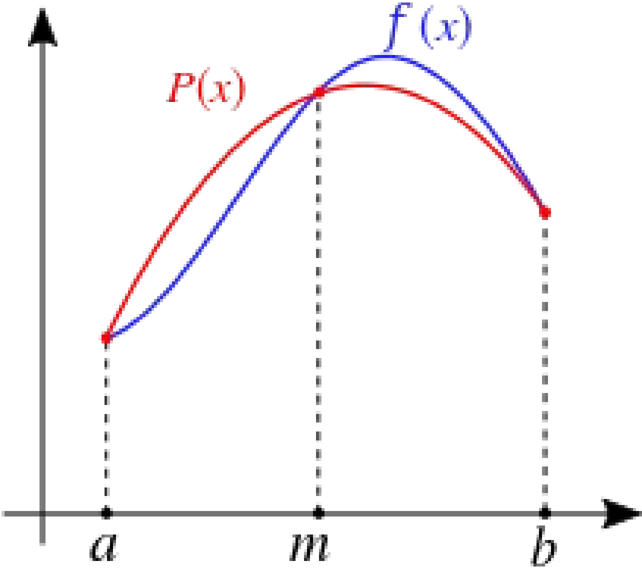
$$\text{error} = -\frac{(b-a)^3}{12N^2} f''(\xi)$$

Demo. Consider the function : $g_k(t) = \frac{1}{2}t[f(a_k) + f(a_k + t)] - \int_{a_k}^{a_k+t} f(x)dx$

Simpson's rule

Simpson's rule can be derived by approximating *f(x)* by a parabola

$$\int_a^b f(x)\,dx \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right].$$



Simpson’s rule

Error in Simpson's rule is $-\frac{1}{90}\left(\frac{b-a}{2}\right)^5 f^{(4)}(\xi)$

Composite Simpson's

$$\int_a^b f(x)\,dx \approx \frac{h}{3}\sum_{j=1}^{n/2}\Big[f(x_{2j-2}) + 4f(x_{2j-1}) + f(x_{2j})\Big]$$

$$= \frac{h}{3}\Big[f(x_0) + 2\sum_{j=1}^{n/2-1} f(x_{2j}) + 4\sum_{j=1}^{n/2} f(x_{2j-1}) + f(x_n)\Big]$$

With an error $\frac{h^4}{180}(b-a)\max_{\xi\in[a,b]}\left|f^{(4)}(\xi)\right|.$

Simpson's 3/8 rule

$$\int_a^b f(x)\,dx \approx \frac{3h}{8}\left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b)\right]$$

Where b-a=3h and the error is $-\frac{(b-a)^5}{6480} f^{(4)}(\xi)$

Composite Simpson's 3/8 rule

$$\int_a^b f(x)\,dx \approx \frac{3h}{8}\left[f(x_0) + 3\sum_{i\neq 3k}^{n-1} f(x_i) + 2\sum_{j=1}^{n/3-1} f(x_{3j}) + f(x_n)\right]$$

Where x_0=a, x_n=b and h=(b-a)/n and n is a multiple of three

Newton–Cotes formulas

Newton–Cotes rules, are a group of formulas for numerical integration (also called quadrature) based on evaluating the integrand at equally spaced points.

$$\int_a^b f(x)\,dx \approx \sum_{i=0}^{n} w_i\, f(x_i)$$

Closed Newton-Cotes Formulas

Degree n	Step size h	Common name	Formula	Error term
1	$b-a$	Trapezoid rule	$\frac{h}{2}(f_0+f_1)$	$-\frac{1}{12}h^3f^{(2)}(\xi)$
2	$\frac{b-a}{2}$	Simpson's rule	$\frac{h}{3}(f_0+4f_1+f_2)$	$-\frac{1}{90}h^5f^{(4)}(\xi)$
3	$\frac{b-a}{3}$	Simpson's 3/8 rule	$\frac{3h}{8}(f_0+3f_1+3f_2+f_3)$	$-\frac{3}{80}h^5f^{(4)}(\xi)$
4	$\frac{b-a}{4}$	Boole's rule	$\frac{2h}{45}(7f_0+32f_1+12f_2+32f_3+7f_4)$	$-\frac{8}{945}h^7f^{(6)}(\xi)$