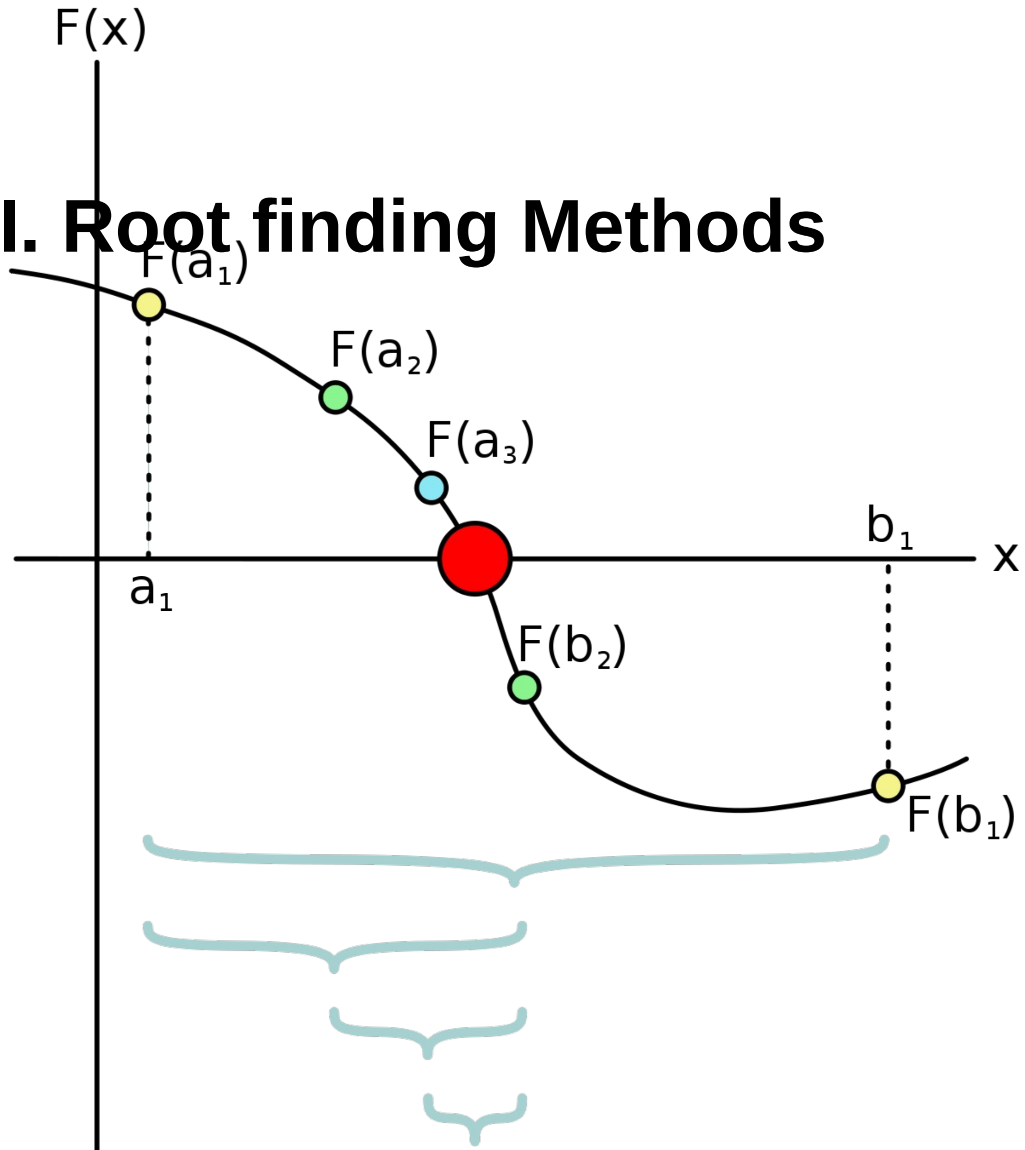
III. Root finding Methods



- Bisection method
- Newton's Method

If the function is defined as **y = f(x)**, we seek the value a such that $\mathbf{f(x_0)=0}$.

The precise terminology is that x_0 is a ***zero*** of the function **f** , or a ***root*** of the equation f(x) = 0

Bisection method

f(x),is a continuous function over **[a,b]** with **f(a)f(b)<0** then

1. take **c=(a+b)/2** and check if

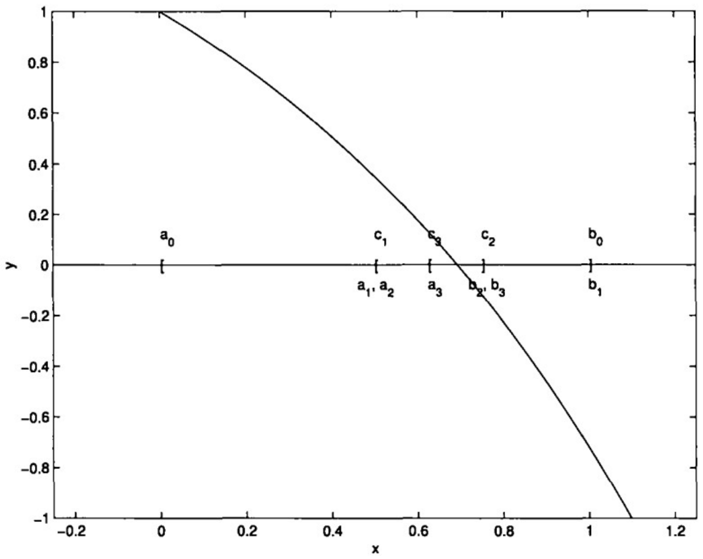
- **a. f(a)f(c)<0** the root exist between **a** and **c**, take **b=c** and go to **1**.

- **b. f(b)f(c)<0** the root exist between **b** and **c**, take **a=c** and go to **1**.

Continue till |a-b|<error

Error $|c_n - c| \le \frac{|b-a|}{2^n}.$

Bisection applied to $y = 2 - e^x$



The number n of iterations needed to achieve a required tolerance ε (that is, an error guaranteed to be at most ε), is bounded by

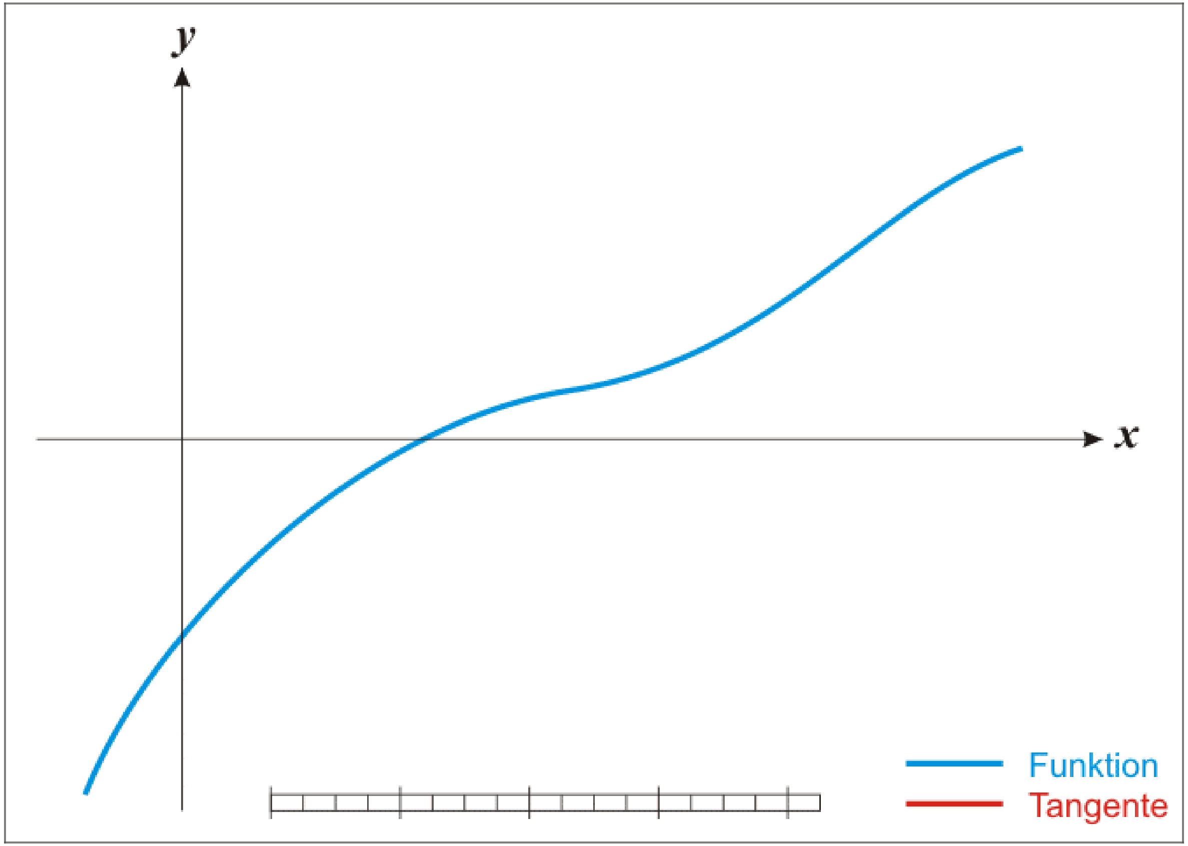
$$n \le \left\lceil \log_2\left(\frac{\epsilon_0}{\epsilon}\right) \right\rceil = \left\lceil \frac{\log \epsilon_0 - \log \epsilon}{\log 2} \right\rceil$$

$$\epsilon_0 = |b - a|$$

Newton's method

Given a function, its derivative and an initial guess, the zero of a function can be found by iterative method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$



By Ralf Pfeifer - de:Image:NewtonIteration Ani.gif, CC BY-SA 3.0, https://commons.wikimedia.org/w/index.php?curid=2268473

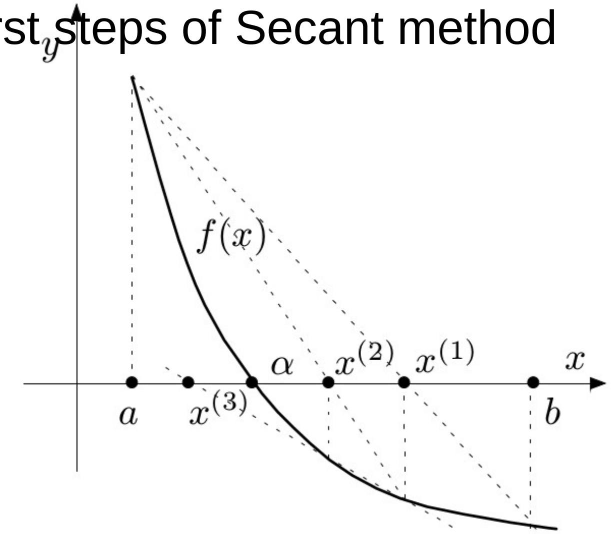
Secant method

Given a function, if the derivative is unknown one can substitute

f' by its numerical derivative and two initial points

$$x^{(k+1)} = x^{(k)} - f(x^{(k)})\frac{x^{(k)} - x^{(k-1)}}{f(x^{(k)}) - f(x^{(k-1)})}$$

First steps of Secant method



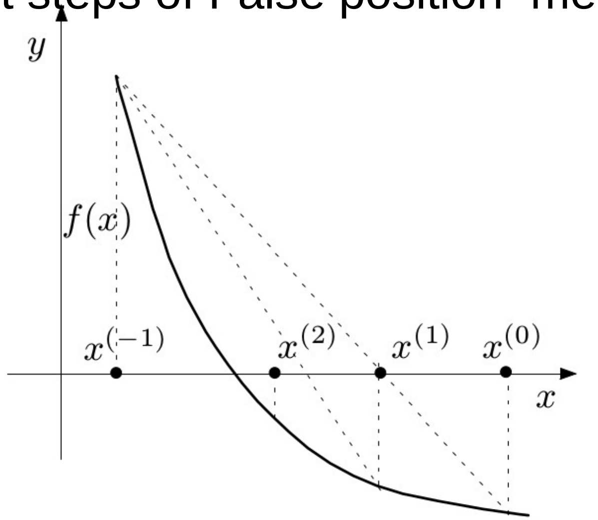
False position (regula falsi)

Instead of $\mathbf{x}^{(k-1)}$ we take any point $\mathbf{x}^{(k')}$ such

$$\mathbf{f(x^{(k)}).f(x^{(k')})<0}$$

$$x^{(k+1)} = x^{(k)} - \frac{x^{(k)} - x^{(k')}}{f(x^{(k)}) - f(x^{(k')})} f(x^{(k)}) \qquad \forall k \geq 0.$$

First steps of False position method

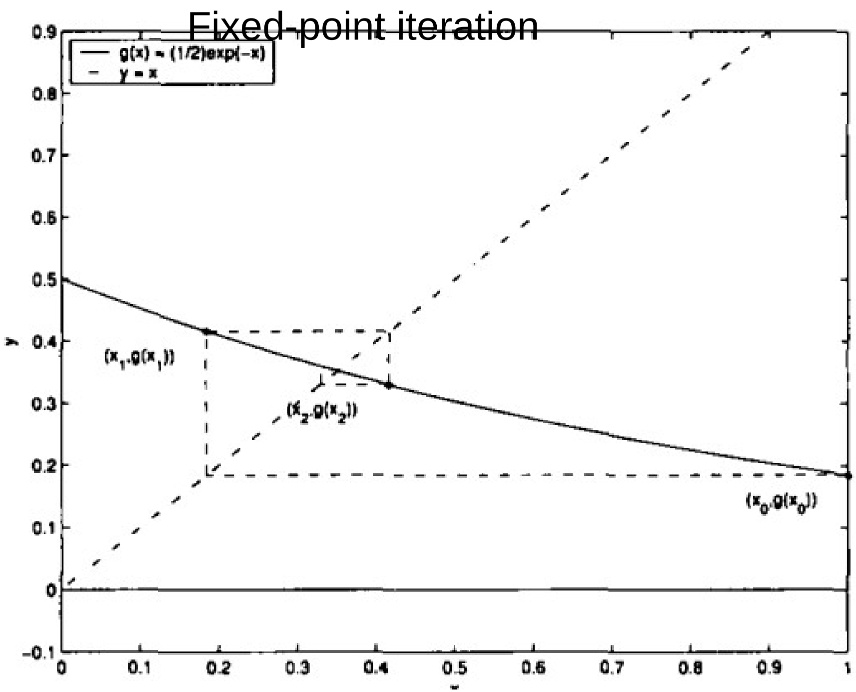


Fixed-point iteration

We can transform the equation **f(x)=0** to an equivalent one **x=g(x)** and apply

$$x_{n+1} = g(x_n)$$

Converges if $|g'(x)| < 1$



Theorem 3.5 (Fixed-point Existence and Iteration Convergence Theory) *Let* $g \in C([a,b])$ *with* $a \leq g(x) \leq b$ *for all* $x \in [a,b]$*; then:*

1. g *has at least one fixed point* $\alpha \in [a,b]$*;*
2. *If there exists a value* $\gamma < 1$ *such that*

$$|g(x) - g(y)| \leq \gamma |x - y|$$

   for all x *and* y *in* $[a,b]$*, then:*

   (a) α *is unique;*

   (b) *The iteration* $x_{n+1} = g(x_n)$ *converges to* α *for any initial guess* $x_0 \in [a,b]$*;*

   (c) *We have the error estimate*

$$|\alpha - x_n| \leq \frac{\gamma^n}{1-\gamma}|x_1 - x_0|.$$

3. *If* g *is continuously differentiable on* $[a,b]$ *with*

$$\max_{x \in [a,b]} |g'(x)| = \gamma < 1$$

then

(a) α *is unique;*

(b) *The iteration* $x_{n+1} = g(x_n)$ *converges to* α *for any initial guess* $x_0 \in [a,b]$*;*

(c) *We have the error estimate*

$$|\alpha - x_n| \leq \frac{\gamma^n}{1-\gamma}|x_1 - x_0|;$$

(d) *The limit*

$$\lim_{n \to \infty} \frac{\alpha - x_{n+1}}{\alpha - x_n} = g'(\alpha)$$

holds.