University of M’sila

Science Faculty

Physics department

Second-year Master of Theoretical Physics

Classical and quantum mechanics of time-dependent systems

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Chapter 1 (Part :2)

PATH INTEGRALS IN NON-RELATIVISTIC QUANTUM MECHANICS

Perturbation theory and generalized Green functions

I_Perturbative expansion :

We consider a particle moves in potential $V(q,t)$ and the form Hamiltonian function $H(p,q)$ is :

نعتبر جسيم يتفاعل مع الكمون $V(q,t)$ و شكل التابع الهاميلتوني $H(p,q)$ هو:

$$H(p,q)=\frac{p^2}{2m}+V(q) \tag{1}$$

We have seen that the propagator, in this case, can be written in the following form :

رأينا أن الناشر في هاته الحالة يمكن ان يكتب كمايلي:

$$K(q,t,q_i,t_i)=N\int D_q \exp\left\{\frac{i}{\hbar}S(q(t'))\right\} \tag{2}$$

Where the classical action is given by :

حيث أن الفعل الكلاسيكي:

$$S(q(t'))=\int_{t_i}^{t} dt' L\left(\frac{dq(t')}{dt},q(t')\right) \tag{3}$$

And

$$L\left(\frac{dq(t')}{dt}, q(t')\right) = \frac{m}{2}\left(\frac{dq(t')}{dt}\right)^2 - V(q) \tag{4}$$

We can write the exponent function $\exp\left\{\frac{i}{\hbar}S(q(t'))\right\}$ as :

$$\exp\left\{\frac{i}{\hbar}S(q(t'))\right\} = \exp\left\{\frac{i}{\hbar}\int_{t_i}^{t}\frac{m}{2}\left(\frac{dq(t')}{dt}\right)^2 dt\right\}\exp\left\{-\frac{i}{\hbar}\int_{t_i}^{t}V(q,t)dt\right\} \tag{5}$$

It is possible to expand the second term as follows :

يمكن نشر الحد الثاني كميلي:

$$\exp\left\{-\frac{i}{\hbar}\int_{t_i}^{t}V(q,t)dt\right\} \cong 1 - \frac{i}{\hbar}\int_{t_i}^{t}V(q,t)dt - \frac{1}{2\hbar^2}\left(\int_{t_i}^{t}V(q,t)dt\right)^2 + ... \tag{6}$$

Which allows us to write the expanded propagator as follows :

مما يسمح بكتابة الناشر القابل للنشر على كمايلي:

$$K(q,t,q_i,t_i) = N\int D_q \exp\left\{\frac{i}{\hbar}S_0(q(t'))\right\}\left[1 - \frac{i}{\hbar}\int_{t_i}^{t}V(q,t)dt - \frac{1}{2\hbar^2}\left(\int_{t_i}^{t}V(q,t)dt\right)^2 + ...\right] \tag{7}$$

Of course, the first term for the Eq.(7) is just the free propagator, we introduce the following notations :

طبعا الحد الأول من المعادلة (7) يمثل الناشر الحر, لندخل الرموز التالية:

$$\begin{cases} K_1(q,t,q_i,t_i) = -\dfrac{iN}{\hbar}\int D_q \exp\left\{\dfrac{i}{\hbar}S_0(q(t'))\right\}\displaystyle\int_{t_i}^{t}V(q,t)dt \\ K_2(q,t,q_i,t_i) = -\dfrac{N}{2\hbar^2}\int D_q \exp\left\{\dfrac{i}{\hbar}S_0(q(t'))\right\}\left(\displaystyle\int_{t_i}^{t}V(q,t)dt\right)^2 \end{cases} \tag{8}$$

Where $K_1(q,t,q_i,t_i)$ and $K_2(q,t,q_i,t_i)$ are known by first-order propagator and second-order propagator, respectively. It is possible to rewrite the first order propagator as follows :

حيث أن $K_1(q,t,q_i,t_i)$ و $K_2(q,t,q_i,t_i)$ يمثلان الناشر من الرتبة الأولى و الثانية على التوالي, بالإمكان كتابة الناشر من الرتبة الاةولى كمايلي:

$$K_1(q,t,q_i,t_i) = \underset{N\to 0}{Lim}\left(\frac{m}{2\prod i\hbar\eta}\right)^{\frac{N+1}{2}} \int \prod_{k=1}^{N} dq_k \exp\left\{\frac{i}{\hbar}\eta\sum_{j=1}^{N}\frac{m}{2}\left(\frac{q_{j+1}-q_j}{\eta}\right)\right\}\int_{t_i}^{t} V(q,t)dt \tag{9}$$

If we use the approximation :

و باستعمال التقريب:

$$\int_{t_i}^{t} V(q,t)dt \cong \sum_{\alpha=1}^{N} V(q_\alpha, t_\alpha) \tag{10}$$

The Eq. (9) will be to the following form :

المعادلة (9) تصبح على الشكل:

$$K_1(q,t,q_i,t_i) = -\frac{iN}{\hbar}\underset{N\to 0}{Lim}\left(\frac{m}{2\prod i\hbar\eta}\right)^{\frac{N+1}{2}} \int \prod_{k=1}^{N} dq_k \sum_{\alpha=1}^{N} V(q_\alpha, t_\alpha)\exp\left\{\frac{i}{\hbar}\eta\sum_{j=1}^{N}\frac{m}{2}\left(\frac{q_{j+1}-q_j}{\eta}\right)\right\} \tag{11}$$

Homework :

Show that the first order propagator as follows expressed in Eq. (11) can be rewritten as follow :

$$K_1(q,t,q_i,t_i) = -\frac{i}{\hbar}\underset{N\to 0}{Lim}\sum_{\alpha=1}^{N}\eta\int dq_k\left[K_0(q,t,q_\alpha,t_\alpha)V(q_\alpha,t_\alpha)K_0(q_\alpha,t_\alpha,q_i,t_i)\right] \tag{12}$$

Where

$$K_0(q_\alpha,t_\alpha,q_i,t_i)=N^{\frac{\alpha}{2}}\int dx_1...dx_{\alpha-1}\exp\left\{\frac{im}{2\hbar\eta}\sum_{j=1}^{N}(q_{j+1}-q_j)\right\}$$
$$K_0(q,t,q_\alpha,t_\alpha)=N^{\frac{N-\alpha+1}{2}}\int dx_{\alpha+1}...dx_N\exp\left\{\frac{im}{2\hbar\eta}\sum_{j=\alpha}^{N}(q_{j+1}-q_j)\right\} \quad (13)$$

Here $K_0(q_\alpha,t_\alpha,q_i,t_i)$ present the free propagator between initial coordinates (q_i,t_i) and intermediate coordinates (q_α,t_α) while $K_0(q,t,q_\alpha,t_\alpha)$ is the free propagator between intermediate coordinates (q_α,t_α) and finale coordinates (q,t), in the intermediate coordinates the reaction is done with the potential $V(q_\alpha,t_\alpha)$. Based on the Eqs. (12) and (13) one can be established the first order propagator as follows :

هنا $K_0(q_\alpha,t_\alpha,q_i,t_i)$ يمثل الناشر الحر بين الموضع الابتدائي (q_i,t_i) و الموضع الوسيطي (q_α,t_α) في حين $K_0(q,t,q_\alpha,t_\alpha)$ يمثل الناشر الحر بين الموضع الوسيطي و الموضع النهائي (q,t), في الموضع الوسيطي يحدث التفاعل, بالاعتماد على المعادلتين (12) و (13) يمكن أن نحدد الناشر من المرتبة الأولى كمايلي:

$$K_1(q_f,t_f,q_i,t_i)=-\frac{i}{\hbar}\int_{-\infty}^{+\infty}dt\int_{-\infty}^{+\infty}dqK_0(q_f,t_f,q,t)V(q,t)K_0(q,t,q_i,t_i) \quad (14)$$

Homework :

Similarly, show that the second-order propagator can be written as follow :

بشكل مماثل لما سبق رؤيته بخصوص الناشر من الرتبة الأولى, بين أن الناشر من الرتبة الثانية يمكن كتابته على النحو التالي:

$$K_2(q_f,t_f,q_i,t_i)=-\frac{1}{\hbar^2}\int_{-\infty}^{+\infty}dq_1dq_2dt_1dt_2K_0(q_f,t_f,q_1,t_1)V(q_1,t_1)K_0(q_1,t_1,q_2,t_2)V(q_2,t_2)K_0(q_2,t_2,q_i,t_i) \quad (15)$$

With condition $t_i \prec t_2 \prec t_1 \prec t_f$ or $t_i \to t_2 \to t_1 \to t_f$.

بشرط $t_i \prec t_2 \prec t_1 \prec t_f$ أو $t_i \to t_2 \to t_1 \to t_f$,

Allowing to gets the expanded propagator as a sum of the free propagator and the first order propagator in addition to the second-order propagator and other terms which importances not interesting physically as follows:

مما يسمح بالحصول على الناشر القابل للنشر كمجموع للناشر الحر و الناشر من الرتبة الأولى و الناشر من الرتبة الثانية و حدود أخرى مساهمتها تقل كلما زادت رتبة التصحيح,

$$
\begin{aligned}
K(q_f,t_f,q_i,t_i) &= K_0(q_f,t_f,q_i,t_{ii}) - \frac{i}{\hbar}\int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} dq K_0(q_f,t_f,q,t) V(q,t) K_0(q,t,q_i,t_i) \\
&- \frac{1}{\hbar^2}\int_{-\infty}^{+\infty} dq_1 dq_2 dt_1 dt_2 K_0(q_f,t_f,q_1,t_1) V(q_1,t_1) K_0(q_1,t_1,q_2,t_2) V(q_2,t_2) K_0(q_2,t_2,q_i,t_i) + ...
\end{aligned}
\tag{16}
$$

It is possible to rewrite the expanded propagator as :

بالإمكان أعادة كتابة الناشر المنشور كمايلي:

$$
\begin{aligned}
K(q_f,t_f,q_i,t_i) &= K_0(q_f,t_f,q_i,t_i) + K_0(q_f,t_f,q,t) U(q,t) K_0(q,t,q_i,t_i) \\
&+ K_0(q_f,t_f,q_1,t_1) U(q_1,t_1) K_0(q_1,t_1,q_2,t_2) U(q_2,t_2) K_0(q_2,t_2,q_i,t_i) + ...
\end{aligned}
\tag{17}
$$

With $U \equiv -\frac{i}{\hbar}V$, we rewrite Eq. (17) as follows:

حيث $U \equiv -\frac{i}{\hbar}V$, نعيد صياغة المعادلة (17) كمايلي:

$$
K(q_f,t_f,q_i,t_i) = K_0(q_f,t_f,q_i,t_i) + K_0 U (K_0 + UK_0U + ...) \tag{18}
$$

We replace $(K_0 + UK_0U + ...)$ by K, allows us to gets the following results:

نعوض $(K_0 + UK_0U + ...)$ بـالناشر K , هذا يسمح بإيجاد النتيجة التالية:

$$
K(q_f,t_f,q_i,t_i) = K_0(q_f,t_f,q_i,t_i) + K_0(q_f,t_f,q,t) U(q,t) K(q,t,q_i,t_i) \tag{19}
$$

This is known by the Bethe-Salpeter equation. The integral form of the Bethe-Salpeter equation is given by :

تعرف هاته المعادلة بمعادلة Bethe-Salpeter. الشكل التكاملي لمعادلة Bethe-Salpeter يعطى بالشكل:

$$
K(q_f,t_f,q_i,t_i) = K_0(q_f,t_f,q_i,t_{ii}) - \frac{i}{\hbar}\int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} dq K_0(q_f,t_f,q,t) V(q,t) K(q,t,q_i,t_i) \tag{20}
$$

By using the ondulation equation which we have seen in the first chapter :

و باستعمال المعادلة الموجية التي رايناها في الفصل الأول:

$$\Psi(q_f,t_f)=\int K(q_f,t_f,q_i,t_i)\Psi(q_i,t_i)dq_i \tag{21}$$

The Bethe-Salpeter equation also can be rewritten in the following equivalent form :

معادلة Bethe-Salpeter يمكن كتابتها على الصيغة التكاملية المكافئة:

$$\Psi(q_f,t_f)=\int K_0(q_f,t_f,q_i,t_i)\Psi(q_i,t_i)dq_i-\frac{i}{\hbar}\int_{-\infty}^{+\infty}dt\int_{-\infty}^{+\infty}dqK_0(q_f,t_f,q,t)V(q,t)\Psi(q,t) \tag{22}$$

II_Generalized Green functions :

The ground state to ground state amplitude :

We have seen that in the first chapter, in Eq. (34), in the quadratic case, the propagator $K(q,t,q_i,t_i)$ was written as follows :

كنا قد راينا في الفصل الأول في المعادلة (34) أن الناشر $K(q,t,q_i,t_i)$ تم كتابته كمايلي:

$$K(q,t,q_i,t_i)=\int D_q\int D_p\exp\left\{\frac{i}{\hbar}\int_{t_i}^{t}dt'\left[p(t')\frac{dq(t')}{dt'}-H(p(t'),q(t'))\right]\right\} \qquad \text{(Ch\_1:34)}$$

Here $D_q\equiv\prod_{k=1}^{N}dq_k$ and $D_p\equiv\prod_{l=1}^{N}\frac{dp_l}{2\prod\hbar}$. If an external source term, or deriving force, $-J(tq)$, is added to the Hamiltonian function $H(p,q)$ in (Ch_1:34) the transition amplitude $\langle q_i,t_i|q,t\rangle^J$ in the presence of deriving force is given by :

إذا أضيف مبع خارجي أو مولد قوة $-J(tq)$, لتابع الهاميلونيان في المعادلة (Ch_1:34) يصبح سعة الاحتمال $\langle q_i,t_i|q,t\rangle^J$ في حضور المنبع الخارجي:

$$\langle q_i,t_i|q,t\rangle^J=N\int D_q\int D_p\exp\left\{\frac{i}{\hbar}\int_{t_i}^{t}dt\left(p\frac{dq}{dt}-H(p,q)+Jq\right)\right\} \tag{23}$$

The ground state to ground state amplitude $W[J]$ is defined as :

سعة الحالة الأساسية $W[J]$ تعرف كمايلي:

$$W[J] = \underset{\substack{t_i \to -\infty \\ t \to +\infty}}{Lim} \langle q_i, t_i | q, t \rangle^J \tag{24}$$

Thus, the ground state to ground state amplitude $W[J]$ is given by :

و منه تصبح سعة الحالة الأساسية $W[J]$ كمايلي:

$$W[J] = N \int D_q \int D_p \exp\left\{ \frac{i}{\hbar} \int_{t_i}^{t} dt \left(p \frac{dq}{dt} - H(p,q) + Jq \right) \right\} \tag{25}$$

In the quadratic case $H(p,q) = \frac{p^2}{2m} + V(q)$, the ground state to ground state amplitude $W[J]$ in Eq . (25) reduced to the following form :

في حالة تابع الهاميلتونيان من الشكل $H(p,q) = \frac{p^2}{2m} + V(q)$ تختصر سعة الحالة الأساسية $W[J]$ للشكل التالي:

$$W[J] = N \int D_q \exp\left\{ \frac{i}{\hbar} \int_{t_i}^{t} dt \left(p \frac{dq}{dt} - H(p,q) + Jq \right) \right\} \tag{26}$$

The time ordering operation T is defined as :

مؤثر ترتيب المؤثرات زمنيا T يعرف كمايلي:

$$T\left(\hat{Q}_H(t_2)\hat{Q}_H(t_1)\right) = \begin{cases} \hat{Q}_H(t_1)\hat{Q}_H(t_2) & \text{for} \quad t_2 \succ t_1 \\ \hat{Q}_H(t_2)\hat{Q}_H(t_1) & \text{for} \quad t_1 \succ t_2 \end{cases} \tag{27}$$

The object $\langle q_i, t_i | T\left(\hat{Q}_H(t_2)\hat{Q}_H(t_1)\right) | q, t \rangle$ is given by :

يعطى الكائن $\langle q_i, t_i | T\left(\hat{Q}_H(t_2)\hat{Q}_H(t_1)\right) | q, t \rangle$ بالشكل:

$$\langle q_i, t_i | T\left(\hat{Q}_H(t_2)\hat{Q}_H(t_1)\right)| q, t\rangle = N\int D_q \int D_p q(t_2) q(t_1) \exp\left\{\frac{i}{\hbar}\int_{t_i}^{t} dt\left(p\frac{dq}{dt} - H(p,q)\right)\right\} \tag{23}$$

With

$$\hat{Q}_H(t)| q, t\rangle = q(t)| q, t\rangle \tag{24}$$

When we make the two simultaneously limits $t_i \to -\infty$ and $t \to +\infty$ of the object, $\langle q_i, t_i | T\left(\hat{Q}_H(t_2)\hat{Q}_H(t_1)\right)| q, t\rangle$ we obtain the expectation value in the ground state of the time-ordered product of two operators $\langle 0| T\left(\hat{Q}_H(t_2)\hat{Q}_H(t_1)\right)| 0\rangle$ as follows :

عندما ننجز النهايتين المتزامنتين $t_i \to -\infty$ و $t \to +\infty$ للكائن $\langle q_i, t_i | T\left(\hat{Q}_H(t_2)\hat{Q}_H(t_1)\right)| q, t\rangle$ نتحصل على القيمة المتقعة في الفراغ $\langle 0| T\left(\hat{Q}_H(t_2)\hat{Q}_H(t_1)\right)| 0\rangle$ كمايلي:

$$\lim_{\substack{t_i \to -\infty \\ t \to +\infty}} \langle q_i, t_i | T\left(\hat{Q}_H(t_2)\hat{Q}_H(t_1)\right)| q, t\rangle = \langle 0| T\left(\hat{Q}_H(t_2)\hat{Q}_H(t_1)\right)| 0\rangle \tag{25}$$

The connection between the ground state of the time-ordered product of two operators $\langle 0| T\left(\hat{Q}_H(t_2)\hat{Q}_H(t_1)\right)| 0\rangle$ and the ground state to ground state amplitude $W[J]$ is :

ترتبط سعة الاحتمال $W[J]$ بالقيمة المتوقعة $\langle 0| T\left(\hat{Q}_H(t_2)\hat{Q}_H(t_1)\right)| 0\rangle$ عن طريق العلاقة:

$$\langle 0| T\left(\hat{Q}_H(t_2)\hat{Q}_H(t_1)\right)| 0\rangle = (-i\hbar)^2 \left.\frac{\delta^2 W(J)}{\delta J(t_1)\delta J(t_2)}\right|_{J(t)=0} \tag{26}$$

The result obtained in Eq. (23) is easily generalized to a time-ordered product of any number n of operators as follows :

النتيجة المتحصل عليها في المعادلة (23) من السهل تعميمها لاي عدد n من المؤثرات بالشكل:

$$\langle q_i, t_i | T\left(\hat{Q}_H(t_n)...\hat{Q}_H(t_2)\hat{Q}_H(t_1)\right)| q, t\rangle = N\int D_q \int D_p q(t_2) q(t_n)...q(t_1) \exp\left\{\frac{i}{\hbar}\int_{t_i}^{t} dt\left(p\frac{dq}{dt} - H(p,q)\right)\right\} \tag{27}$$

In the quadratic case $H(p,q)=\frac{p^2}{2m}+V(q)$, the time-ordered product of any number n of operators $\langle q_i,t_i|T(\hat{Q}_H(t_n)...\hat{Q}_H(t_2)\hat{Q}_H(t_1))|q,t\rangle$ in Eq . (27) reduced to the following form :

في حالة تابع الهاميلتونيان من الشكل $H(p,q)=\frac{p^2}{2m}+V(q)$, مؤثر الترتيب الزمني لـ n من المؤثرات في المعادلة (27) يصبح بلشكل التالي:

$$\langle q_i,t_i|T(\hat{Q}_H(t_n)...\hat{Q}_H(t_2)\hat{Q}_H(t_1))|q,t\rangle = N\int D_q q(t_2)q(t_n)...q(t_1)\exp\left\{\frac{i}{\hbar}\int_{t_i}^{t} dt\left(p\frac{dq}{dt}-H(p,q)\right)\right\} \tag{28}$$

Now, $\langle 0|T(\hat{Q}_H(t_n)...\hat{Q}_H(t_2)\hat{Q}_H(t_1))|0\rangle$ is the expectation value in the ground state of the time-ordered product of n operators can be obtained by generalizing the formula (26) :

الان القيمة المتوقعة $\langle 0|T(\hat{Q}_H(t_n)...\hat{Q}_H(t_2)\hat{Q}_H(t_1))|0\rangle$ لـ n من المؤثرات نحصل عليها بتعميم المعادلة (26):

$$\langle 0|T(\hat{Q}_H(t_n)...\hat{Q}_H(t_2)\hat{Q}_H(t_1))|0\rangle = (-i\hbar)^n \left.\frac{\delta^{2n}W(J)}{\delta J(t_n)...\delta J(t_2)\delta J(t_1)}\right|_{J(t)=0} \tag{29}$$

The amplitude $W[J]$ is used to generate the Green functions.

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Wait for the third chapter soon

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