Nonlinear geometry of spaces

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Contents

PREFACE

This course was given in the first semester of 2020-2021 at the university of M'sila. The text is intended for the students of M_2. The main theme of this course is to give an introduction to the non linear summing operators in the domain of "the non linear geomerty of Banach spaces". We treat and study in chapter I, the Lipschitz functions between metric spaces and the Lipschitz dual space of a metric space. This space is a conjugate Banach space. We study the predual and their properties. Chapter two is devote to the notion of Lipschitz p-summing functions introduced by Farmer and Johnson. We end this by giving the non linear Grothendieck's theorem. In chapter three, We introduce and studied some other classes of summability and their connections. I have tried to make this course fairly complete and comprehensive. For this, I recommend essentially the excellent book of Weaver and the papers of Farmer-Johnson and Godfroy-Kalton.

CHAPTER 1

The Space $\mathrm{Lip}_0(X)$

1. Lipschitz Functions

1.1. Metric Spaces. The notion of metric spaces was formalized by Maurice Fréchet in his thesis "Doctorat d'Etat" in 1906 (see, "Sur quelques points du calcul fonctionnel", Rendic. Circ. Mat. Palermo 22 (1906) 1–74) and was among the first who used the word space. A good reference for this is the book of weaver [**Wea99**].

Definition 1. *Let X be a non empty set. We say that* d *is a distance on X if d is an application from X^2 into $\mathbb{R}_+$ such that for all x, y, z in X, we have*

(*i*) $\mathrm{d}(x,y) = 0 \Longleftrightarrow x = y$ (*separation*),
(*ii*) $\mathrm{d}(x,y) = d(y,x)$ (*symmetry*),
(*iii*) $\mathrm{d}(x,z) \leq d(x,y) + d(y,z)$ (*triangular inequality*).

The space X equipped with d is called metric space (X, d).

Definition 2. *Let $(X, \mathrm{d}, \mathrm{e})$ be a pointed metric space, i.e., a metric space (X, d) with a distinguished or neutral element e (a fixed point in X which is taken to be the zero element if X is a normed space). We denote by $\mathcal{M}_0$ the class of complete pointed metric spaces.*

We now give some particular metric spaces

Definition 3. *Let (X, d) be a metric space. One say that* d

(1) *is ultrametric if it satisfies for all $(x, y, z) \in X^3$*

$$\mathrm{d}(x,y) \leq \max(\mathrm{d}(x,z), \mathrm{d}(y,z)) \tag{1.1}$$

We can see that any triangle in X is isosceles,

(2) *satisfies the four point condition $(4PC)$ or is additive or is 0-hyperbolic if, for any (x, y, u, v) in X^4 (not necessarily distinct) we have*

$$\mathrm{d}\,(x,y) + \mathrm{d}\,(u,v) \leq \max\left\{\mathrm{d}\,(x,u) + \mathrm{d}\,(y,v)\,, \mathrm{d}\,(x,v) + \mathrm{d}\,(y,u)\right\}, \tag{1.2}$$

Note that if d *satisfies the $(4PC)$ then one of the sums must be less or equal than the other which must be equal (argue by contradiction that one of the sums is strictly larger than the other two),*

(3) *satisfies Reshetnyak's inequality if, for any (x, y, u, v) in X we have*

$$\mathrm{d}^2(x,y)+\mathrm{d}^2(u,v)\le \mathrm{d}^2(x,u)+\mathrm{d}^2(y,v)+\mathrm{d}^2(x,v)+\mathrm{d}^2(y,u)\,. \tag{1.3}$$

The inequality (1.1) is called strong triangle inequality or ultrametric inequality. Sometimes the ultrametric is also called a super-metric. We observe that in the ultrametric space X all triangles are isosceles with the two equal sides at least as long as the third side. To see this, consider $x,y,z\in(X,\mathrm{d})$ with $\mathrm{d}(y,z)\ge \mathrm{d}(x,z)$ and suppose

$$\mathrm{d}(x,y)\le \max(\mathrm{d}(x,z),\mathrm{d}(y,z)).$$

Then $\mathrm{d}(x,z)=\mathrm{d}(y,z)$ because otherwise

$$\mathrm{d}(y,z)>\mathrm{d}(x,z)\Longrightarrow \mathrm{d}(y,z)>\max(\mathrm{d}(x,y),d(x,z)).$$

Remark 1. *Let (X,d) be a metric space.*

(1) *If (X,d) is ultrametric then $(Y,\mathrm{d}\diagup Y)$ is ultrametric for any $Y\subset X$.*

(2) *If $(X_1,\mathrm{d}_1),...,(X_n,\mathrm{d}_n)$ are ultrametric spaces then the cartesian product $X_1\times...\times X_n$ is ultrametric with respect to*

$$\mathrm{d}\left((x_1,...,x_n),(y_1,...,y_n)\right)=\max\left(\mathrm{d}_1(x_1,y_1),...,\mathrm{d}_n(x_n,y_n)\right).$$

(3) *Isosceles triangles. If a triangle in (X,d) has sides (distances between vertices) $a\le b\le c$, then $b=c$.*

(4) *Radius $\ge$ diameter. For any ball its radius is greater or equal to its diameter.*

Proposition 1. *Ultrametric $\Longrightarrow$ $(4PC)$ $\Longrightarrow$ Reshetnyak's inequality.*

Proof. Second implication [**AO10**] . Suppose the elements x_1,x_2,x_3,x_4 of a metric space (X,d) satisfy

$$d(x_1,x_2)+d(x_3,x_4)\le\max\{d(x_1,x_3)+d(x_2,x_4),d(x_1,x_4)+d(x_2,x_3)\},$$

and show that

$$d(x_1,x_2)^2+d(x_3,x_4)^2\le d(x_1,x_3)^2+d(x_2,x_4)^2+d(x_1,x_4)^2+d(x_2,x_3)^2.$$

By scaling and relabeling, we can assume that

$$d(x_1,x_2)+d(x_3,x_4)=1\le d(x_1,x_3)+d(x_2,x_4).$$

Let $a=d(x_1,x_2)$, $b=d(x_1,x_3)$. Then

$$d(x_3,x_4)=1-a,\qquad d(x_2,x_4)\ge 1-b.$$

And furthermore

$$d(x_1,x_4)\ge|d(x_1,x_3)-d(x_3,x_4)|=|a+b-1|,$$

and

$$d(x_2,x_3)\ge|d(x_1,x_2)-d(x_1,x_3)|=|a-b|.$$

Thus, it suffices to show that, for any $a \in [0,1]$ and $b > 0$,

$$a^2 + (1-a)^2 \le b^2 + (1-b)^2 + (a+b-1)^2 + (a-b)^2.$$

This inequality is easily verified because it is equivalent to $(2b-1)^2 \ge 0$. The reciprocal is false.
The 4PC is stronger than the triangle inequality (take $u = v$), but the ultrametric is stronger than the 4PC [**God07**]. In fact we have, $d(x,y) + d(u,v) = d(x,u) + d(y,v)$ or $d(x,v) + d(y,u)$. Indeed, suppose that

$$d(x,y) + d(u,v) < \max\{d(x,u) + d(y,v), d(x,v) + d(y,u)\}$$

and

$$d(x,u) + d(y,v) \le d(x,v) + d(y,u).$$

We have

$$d(x,u) + d(y,v) \le \max\{d(x,y) + d(u,v), d(x,v) + d(u,y)\}$$

and

$$\begin{array}{rcl} d(x,y) + d(u,v) & < & d(x,u) + d(y,v) \\ & \le & \max\{d(x,y) + d(u,v), d(x,v) + d(u,y)\} \\ & \le & d(x,y) + d(u,v) \end{array}$$

This implies

$$d(x,y) + d(u,v) < d(x,y) + d(u,v)$$

or

$$d(x,v) + d(y,u) > d(x,y) + d(u,v)$$

and hence $d(x,y) + d(u,v) = d(x,y) + d(u,v)$. □

1.2. Product of Metric Spaces.

We interest to $\mathcal{M}_o$.

Definition 4. *Let* $\{(X_i, d_i, \mathrm{e}_i), i \in I\}$ *be a family of metric spaces in* $\mathcal{M}_o$. *We can define by* $\left(\prod^\infty X_i, d, \mathrm{e}\right)$ *the set of elements* $x = (x_i)$ *such that* $\sup_{i\in I} d_{X_i}(x_i, \mathrm{e}_i) < \infty$, *with the metric*

$$d(x,y) = \sup_{i\in I} d_i(x_i, y_i)$$

and the distinguished point $\mathrm{e} = (\mathrm{e}_i)_{i\in I}$.
We have $\left(\prod^\infty X_i, d, \mathrm{e}\right) \in \mathcal{M}_o$.

Example 1. *The product* $\prod^\infty \mathbb{R}$ *is* $l^\infty(\mathbb{R})$.

1.3. Lipschitz functions. The natural morphism between metric spaces are Lipschitz functions like linear operators between Banach spaces. In mathematical analysis, Lipschitz continuity, named after Rudolf Lipschitz, is a strong form of uniform continuity for functions.

Definition 5. *A map $f : (X, \mathrm{d}_X) \longrightarrow (Y, \mathrm{d}_Y)$ between two metric spaces is called Lipschitz if there is a positive constant C such that*

$$\forall x, y \in X, \quad \mathrm{d}_Y(f(x), f(y)) \leq C\mathrm{d}_X(x, y).$$

If $C = 1$, the map is called nonexpansive (and contraction if $C < 1$).

For a Lipschitz map f, we define its Lipschitz constant by

$$\|f\|_{\mathrm{Lip}} = \mathrm{Lip}\,(f) := \sup_{x \neq y} \frac{\mathrm{d}_Y(f(x), f(y))}{\mathrm{d}_X(x, y)} = \inf\{C : C \text{ verifying the above inequality}\}$$

Let $(X, \mathrm{e}_X, \mathrm{d}_X), (Y, \mathrm{e}_Y, \mathrm{d}_Y)$ be pointed metric spaces. We say a map $f : (X, \mathrm{e}_X, \mathrm{d}_X) \longrightarrow (Y, \mathrm{e}_Y, \mathrm{d}_Y)$ preserves distinguished point if $f(\mathrm{e}_X) = \mathrm{e}_Y$.

Definition 6. *Let (X, d_X), (Y, d_Y) be two metric spaces. A map $f : (X, \mathrm{d}_X) \longrightarrow (Y, \mathrm{d}_Y)$ is called bi-Lipschitz or quasi-isometry, if f is bijective (one-to-one = injective, and onto = surjective) and both f, f^{-1} are Lipschitz.*

In this case X and Y are called
(1)- Lipschitz isomorphic or Lipschitz homeomorphic (Nigel Kalton)
or
(2)- Quasi-isometric (Nik Weaver).
A bi-Lipschitz function f is an isometry if

$$\forall x, y \in X, \quad \mathrm{d}_Y(f(x), f(y)) = d_X(x, y).$$

In the theory of the nonlinear geometry of Banach spaces, the linear isomorphisms are replaced by bi-Lipschitz maps, the isometric isomorphism correspond exactly isometric.

Proposition 2. *Let X, Y and Z be metric spaces and let $f : (X, \mathrm{d}_X) \longrightarrow (Y, \mathrm{d}_Y)$, $g : (Y, \mathrm{d}_Y) \longrightarrow (Z, \mathrm{d}_Z)$ be Lipschitz maps. Then $g \circ f : (X, \mathrm{d}_X) \longrightarrow (Z, \mathrm{d}_Z)$ is Lipschitz and* $\mathrm{Lip}\,(g \circ f) \leq \mathrm{Lip}\,(g)\,\mathrm{Lip}\,(f)$.

Proof. For x, y in X, we have

$$\begin{aligned} d_Z\left(g \circ f\left(x\right), g \circ f\left(y\right)\right) &\leq \mathrm{Lip}\left(g\right) \mathrm{d}_Y\left(f\left(x\right), f\left(y\right)\right) \\ &\leq \mathrm{Lip}\left(g\right) \mathrm{Lip}\left(f\right) \mathrm{d}_X\left(x, y\right) \end{aligned}$$

and this shows the proposition. □

Theorem 1. *Let X_0, Y_0 be metric spaces and let X, Y be their completions. Let $f_0 : X_0 \longrightarrow Y_0$ be Lipschitz. Then f has a unique Lipschitz extension $f : X \longrightarrow Y$ such that* $\mathrm{Lip}\,(f) = \mathrm{Lip}\,(f_0)$.

Proof. Since Lipschitz functions are continuous and X_0 is dense in X, there is at most one Lipschitz extension. Consider x in $X \diagdown X_0$ and put

$$f(x) = \lim f_0(x_n)$$

where x is a Cauchy sequence in X_0 such that $x_n \longrightarrow x$. We have $\mathrm{Lip}\,(f) = \mathrm{Lip}\,(f_0)$. Indeed

$$\begin{array}{rcl} \mathrm{d}_Y(f(x), f(y)) & = & \mathrm{d}_Y(\lim f_0(x_n), \lim f_0(y_n)) \\ & = & \lim_Y \mathrm{d}(f_0(x_n), f_0(y_n)) \\ & \leq & \lim \mathrm{Lip}\,(f_0)\, \mathrm{d}_X(x_n, y_n) \\ & \leq & \mathrm{Lip}\,(f_0)\, \mathrm{d}_X(x, y). \end{array}$$

This implies that $\mathrm{Lip}\,(f) \leq \mathrm{Lip}\,(f_0)$. For the converse, consider the following diagram

$$\begin{tikzcd} X_0 \arrow[r, "f_0"] \arrow[d, "i_X"'] \arrow[dr] & Y_0 \arrow[d, "i_Y"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

and we have in the first part

$$\begin{array}{rcl} \mathrm{Lip}\,(i_Y \circ f_0) & = & \sup\limits_{x \neq y} \dfrac{\mathrm{d}_Y\left(i_Y \circ f_0\left(x\right), i_Y \circ f_0\left(y\right)\right)}{\mathrm{d}_X(x, y)} \\ & = & \sup\limits_{x \neq y} \dfrac{\mathrm{d}_{Y_0}\left(f_0\left(x\right), f_0\left(y\right)\right)}{\mathrm{d}_X(x, y)} \\ & = & \mathrm{Lip}\,(f_0) \end{array}$$

and in the second part

$$\mathrm{Lip}\,(i_Y \circ f_0) = \mathrm{Lip}\,(f \circ i_X) \leq \mathrm{Lip}\,(f)\,.$$

This implies that $\mathrm{Lip}\,(f_0) \leq \mathrm{Lip}\,(f)$ and this completes the proof. $\square$

Proposition 3. *Let* (X, d) *be metric space. For Lipschitz functions* $f, g : (X, \mathrm{d}) \longrightarrow \mathbb{R}$ *and scalar* $a \in \mathbb{R}$, *the Lipschitz constant has the properties*

$$\begin{array}{lll} (a) \quad \mathrm{Lip}\,(f+g) & \leq & \mathrm{Lip}\,(f) + \mathrm{Lip}\,(g) \\ (b) \quad \mathrm{Lip}\,(af) & = & |a|\,\mathrm{Lip}\,(f) \\ (c) \quad \mathrm{Lip}\,(\min(f,g) \;\; \textit{or} \;\; \max(f,g)) & \leq & \max\left(\mathrm{Lip}\,(f), \mathrm{Lip}\,(g)\right) \end{array}$$

where $\min(f, g)$ (*resp.* $\max(f, g)$) *denotes the pointwise minimum* (*resp. maximum*) *of the functions* f *and* g.

Proof. (a) and (b) are obvious. For (c), let $h = \max(f, g)$ and fix x, y in X. Let $C = \max\left(\mathrm{Lip}\,(f), \mathrm{Lip}\,(g)\right)$. Without loss of generality suppose $h(x) \geq h(y)$ and $h(x) = f(x)$. Then

$$h(x) - h(y) \leq f(x) - f(y) \leq Cd\,(x, y)\,.$$

Taking the sup over x, y in X, we obtain $\mathrm{Lip}\,(g) \leq C$. From the formula $\min\,(f, g) = -\max\,(-f, -g)$, we get the second inequality. □

Proposition 4. *Let X, Y be metric spaces and let f and $\{f_n\}_{n\in\mathbb{N}}$ be Lipschitz functions from X to Y. Suppose that $f_n \longrightarrow f$ pointwise. Then*

$$\mathrm{Lip}\,(f) \leq \sup_n \mathrm{Lip}\,(f_n)\,.$$

Proof. Let x, y be in X. We have

$$\begin{aligned} d_Y\,(f\,(x)\,, f\,(y)) &= \lim_{n\longrightarrow\infty} d_Y\,(f_n\,(x)\,, f_n\,(y)) \\ \frac{d_Y\,(f\,(x)\,, f\,(y))}{d_X\,(x, y)} &= \lim_{n\longrightarrow\infty} \frac{d_Y\,(f_n\,(x)\,, f_n\,(y))}{d_X\,(x, y)} \\ \sup_{x\neq y} \frac{d_Y\,(f\,(x)\,, f\,(y))}{d_X\,(x, y)} &= \sup_{x\neq y} \lim_{n\longrightarrow\infty} \frac{d_Y\,(f_n\,(x)\,, f_n\,(y))}{d_X\,(x, y)} \\ &\leq \sup_{x\neq y} \sup_n \frac{d_Y\,(f_n\,(x)\,, f_n\,(y))}{d_X\,(x, y)} \end{aligned}$$

by permitting the sup, we obtain the result. □

Corollary 1. *If* $\sum_{n\geq 0} f_n$ *converges pointwise then* $\mathrm{Lip}\left(\sum_{n\geq 0} f_n\right) \leq \sum_{n\geq 0} \mathrm{Lip}\,(f_n)\,.$

Proof. Let $g_n = \sum_{i=1}^{n} f_i$ and $f = \sum_{n\geq 0} f_n$. then $g_n \longrightarrow f$ pointwise and $\mathrm{Lip}\,(g_n) \leq \sum_{i=1}^{n} \mathrm{Lip}\,(f_i)$. So By Proposition 4 we have

$$\begin{aligned} \mathrm{Lip}\,(f) &\leq \sup \mathrm{Lip}\,(g_n) \\ &\leq \sum_{i=1}^{\infty} \mathrm{Lip}\,(f_i) \end{aligned}$$

and this ends the proof. □

Proposition 5. *Let X be a metric space and let $f, g : X \longrightarrow \mathbb{R}$ be Lipschitz maps. Then*

(*a*) $\mathrm{Lip}\,(fg) \leq \|f\|_\infty \mathrm{Lip}\,(g) + \|g\|_\infty \mathrm{Lip}\,(f)\,,$

(*b*) $\mathrm{Lip}\left(\frac{1}{f}\right) \leq \frac{\mathrm{Lip}\,(f)}{\epsilon^2}$, *if* $|f\,(x)| \geq \epsilon > 0$ *for all* $x \in X$.

If diam$(X) < \infty$, then the product of any two scalar valued Lipschitz functions is Lipschitz.

Proof. (*a*) For all $x, y \in X$, we have

$$\begin{aligned} |fg\,(x) - fg\,(y)| &\leq |f\,(x)|\,|g\,(x) - g\,(y)| + |g\,(y)|\,|f\,(x) - f\,(y)| \\ &\leq \|f\|_\infty \mathrm{Lip}\,(g) + \|g\|_\infty \mathrm{Lip}\,(f)\,. \end{aligned}$$

(b) For all $x, y \in X$, we have

$$\begin{aligned}\left|\frac{1}{f(x)}-\frac{1}{f(y)}\right| &= \frac{|f(x)-f(y)|}{|f(x)\,f(y)|}\\ &\leq \frac{1}{\epsilon^2}\mathrm{Lip}\,(f)\,d(x,y).\end{aligned}$$

Then $\mathrm{Lip}\left(\frac{1}{f}\right) \leq \frac{\mathrm{Lip}\,(f)}{\epsilon^2}$. □

Proposition 6. *Let* $(X, d_X), (X_i, d_i) \quad (i \in I)$ *be metric spaces in* $\mathcal{M}_0$. *For each* i *in* I, *let* $f_i : X \longrightarrow X_i$ *be a Lipschitz map which preserves distinguished point. Suppose that* $\sup\limits_{i\in I}\mathrm{Lip}\,(f_i) < \infty$. *Then, the the product map* $f : X \longrightarrow \prod^{\infty} X_i$ *satisfies* $\mathrm{Lip}\,(f) := \sup\limits_{i\in I}\mathrm{Lip}\,(f_i)$.

Proof. Let x be in X. We prove that $(f_i(x)) \in \prod^{\infty} X_i$. We have

$$\begin{aligned}\sup_{i\in I}\mathrm{d}_{X_i}(f_i(x), e_i) &= \sup_{i\in I}\mathrm{d}_{X_i}(f_i(x), f_i(e))\\ (\mathrm{d} = \sup_{i\in I}\mathrm{d}_i) &\leq \sup_{i\in I}\mathrm{Lip}\,(f_i)\,\mathrm{d}(x,e)\\ &< \infty.\end{aligned}$$

For x, y in X. We have by definition

$$\frac{d(f(x), f(y))}{d(x,y)} = \sup_{i\in I}\frac{d_i(f_i(x), f_i(y))}{d(x,y)}$$

and hence

$$\begin{aligned}\sup_{x\neq y}\frac{\mathrm{d}(f(x), f(y))}{\mathrm{d}(x,y)} &= \sup_{x\neq y}\sup_{i\in I}\frac{\mathrm{d}_i(f_i(x), f_i(y))}{\mathrm{d}(x,y)}\\ &= \sup_{i\in I}\sup_{x\neq y}\frac{\mathrm{d}_i(f_i(x), f_i(y))}{\mathrm{d}(x,y)}\\ &= \sup_{i\in I}\mathrm{Lip}\,(f_i)\end{aligned}$$

This implies that $\mathrm{Lip}\,(f) := \sup\limits_{i\in I}\mathrm{Lip}\,(f_i)$; and we obtain the result. □

1.4. Extending Lipschitz maps. We give the nonlinear Hahn-Banach theorem.

Theorem 2 (Nonlinear Hahn-Banach theorem, McShane-Whitney extension theorem). *Let* E *be a subset of a metric space* (X, d) *and let* $f : E \longrightarrow l_\infty(I)$ *be a Lipschitz function. Then* f *can be extended to a Lipschitz function* $\widetilde{f} : X \longrightarrow l_\infty(I)$ *with the same Lipschitz constant* (*we say that* $l_\infty(I)$ *is* 1-*injective*).

Proof. By considering each coordinate separately, it suffices to prove that for $\mathbb{R}$ instead of $l_\infty(I)$. Fix z in $X - E$. We must find a value for $\widetilde{f}(z)$ such that for all x in E

$$\left|\widetilde{f}(z) - f(x)\right| \leq \mathrm{Lip}(f)\mathrm{d}(x,z), \quad \forall x \in E$$

or equivalently

$$f(y) - \mathrm{Lip}(f)\mathrm{d}(y,z) \leq \widetilde{f}(z) \leq f(x) + \mathrm{Lip}(f)\mathrm{d}(x,z), \quad \forall y \in E$$

hence

$$\sup_{y\in E}\left(f(y) - \mathrm{Lip}(f)\mathrm{d}(y,z)\right) \leq \widetilde{f}(z) \leq \inf_{x\in E}\left(f(x) + \mathrm{Lip}(f)\mathrm{d}(x,z)\right)$$

It is possible because for all x,y in E, we have

$$f(x) - f(y) \leq \mathrm{Lip}(f)\mathrm{d}(x,y) \leq \mathrm{Lip}(f)(\mathrm{d}(x,z) + \mathrm{d}(y,z)).$$

Define the function $\widetilde{f} : X \longrightarrow \mathbb{R}$ by the formula

$$\widetilde{f}(z) = \inf_{x\in E}\left(f(x) + \mathrm{Lip}(f)\mathrm{d}(x,z)\right),$$

To see that this function satisfies the results, fix an arbitrary $x_0 \in E$. Then, for any $x \in E$

$$\begin{aligned} f(x_0) - f(x) &\leq \mathrm{Lip}(f)\mathrm{d}(x_0,x), \\ &\leq \mathrm{Lip}(f)\left(\mathrm{d}(x_0,z) + \mathrm{d}(z,x)\right). \end{aligned}$$

This implies (that $f(x) + \mathrm{Lip}(f)\mathrm{d}(x,z)$ is bounded below)

$$f(x_0) - \mathrm{Lip}(f)\mathrm{d}(x_0,z) \leq f(x) + \mathrm{Lip}(f)\mathrm{d}(x,z).$$

So $\widetilde{f}(z)$ is well-defined. Also, if $z \in E$, the above shows that $\widetilde{f}(z) = f(z)$. Finally (by definition of the inf), for $z, y \in X$ and $\epsilon > 0$, choose $x_z \in E$ such that

$$\begin{aligned} \widetilde{f}(z) &\geq f(x_z) + \mathrm{Lip}(f)\mathrm{d}(z,x_z) - \epsilon \\ -\widetilde{f}(z) &\leq -f(x_z) - \mathrm{Lip}(f)\mathrm{d}(z,x_z) + \epsilon \end{aligned}$$

Then

$$\begin{aligned} \widetilde{f}(y) - \widetilde{f}(z) &\leq f(x_z) + \mathrm{Lip}(f)\mathrm{d}(y,x_z) - f(x_z) - \mathrm{Lip}(f)d(z,x_z) + \epsilon \\ &\leq \mathrm{Lip}(f)\mathrm{d}(y,z) + \epsilon. \end{aligned}$$

Thus, we see that $\widetilde{f}$ is indeed $\mathrm{Lip}(f)$-Lipschitz. □

THEOREM 3 (Kuratowski-Fréchet). *Every metric space (X, d) is isometric to a subset of $l_\infty(I)$ for some set I. If X is separable, then (X, d) is isometric to a subset of $l_\infty(\mathbb{N})$.*

Proof. Let X be in $\mathcal{M}_0$. Consider x_0 in X and define

$$f : X \longrightarrow l_\infty (X)$$

by

$$\begin{array}{lcl} f(x)(y) & = & \mathrm{d}(x,y) - \mathrm{d}(y,x_0) \\ f(x) & = & (\mathrm{d}(x,y) - \mathrm{d}(y,x_0))_{y\in X} \\ \|f(x)\|_{l_\infty(X)} & \leq & \mathrm{d}(x,x_0). \end{array}$$

We have

$$\begin{array}{lcl} \mathrm{d}(f(x_1), f(x_2)) & = & \sup\limits_{y\in X} |f(x_1)(y) - f(x_2)(y)| \\ & = & \sup\limits_{y\in X} |\mathrm{d}(x_1,y) - \mathrm{d}(x_2,y)| \\ & \leq & \mathrm{d}(x_1,x_2). \end{array}$$

In the other hand if we take $y = x_2$, we have

$$\mathrm{d}(f(x_1), f(x_2)) \geq \mathrm{d}(x_1,x_2).$$

This implies that $d(f(x_1), f(x_2)) = d(x_1,x_2)$ and hence f is an isometry. By Frechet's embedding, (X,d) is isometric to a subspace of $l_\infty(\mathbb{N})$. Fix x_0 in X

$$\begin{array}{rccl} f: & X & \longrightarrow & l_\infty(\mathbb{N}) \\ & x & \longmapsto & (\mathrm{d}(x,x_n) - \mathrm{d}(x_0,x_n))_{n\in\mathbb{N}} \end{array}$$

where (x_n) is the subset dense in X. We have in the first part

$$\begin{array}{lcl} \|f(x_1) - f(x_2)\|_{l_\infty(\mathbb{N})} & = & \sup\limits_{n\in\mathbb{N}} |\mathrm{d}(x_1,x_n) - \mathrm{d}(x_2,x_n)| \\ & \leq & \sup\limits_{n\in\mathbb{N}} \mathrm{d}(x_1,x_2) \\ & \leq & \mathrm{d}(x_1,x_2) \end{array}$$

and in the second part

$$\begin{array}{lcl} \|f(x_1) - f(x_2)\|_{l_\infty(\mathbb{N})} & = & \sup\limits_{n\in\mathbb{N}} |\mathrm{d}(x_1,x_n) - \mathrm{d}(x_2,x_n)| \\ & = & \sup\limits_{x\in X} |\mathrm{d}(x_1,x) - \mathrm{d}(x_2,x)| \\ \text{(we take } x = x_2\text{)} & \geq & d(x_1,x_2). \end{array}$$

The theorem is proved. □

Linear	Lipschitz
Banach space	metric space
isometric isomorphism	isometric
topological isomorphism	bi-Lipschitz or quasi-isometric

1.5. Retract spaces. The notion of Lipschitz retract in metric spaces is like the linear projection in Banach spaces.

Definition 7. *Let X be a metric space and let E be a subspace of X. A Lipschitz map $p : X \longrightarrow E$ is called a Lipschitz retraction if $p/E = \mathrm{Id}$. In this case, we say that E is a Lipschitz retract of X. A metric space E is called an absolute Lipschitz retract if it is a Lipschitz retract of every metric space containing it.*

Proposition 7. *Let Y be a metric space. Then, the following properties are equivalent.*

(i) The space Y is an absolute retract space.

(ii) For every metric space X, for every subset $E \subset X$ and for every Lipschitz function $f : E \longrightarrow Y$ can be extended to a Lipschitz function $\widetilde{f} : X \longrightarrow Y$.

$$\begin{tikzcd}
X \arrow[dr, "\widetilde{f}"] & \\
E \arrow[u, "i"] \arrow[r, "f"] & Y
\end{tikzcd}$$

(iii) For every metric space Z containing Y and for every metric space F, then every Lipschitz function $f : Y \longrightarrow F$ can be extended to a Lipschitz function $\widetilde{f} : Z \longrightarrow F$.

$$\begin{tikzcd}
Z \arrow[dr, "\widetilde{f}"] & \\
Y \arrow[u, "i"] \arrow[r, "f"] & F
\end{tikzcd}$$

Proof. (iii) or (ii) $\Longrightarrow$(i) We take $F = Y$ and $f = id_Y$ or $E = Y$ and $f = id_Y$.

(i)$\Longrightarrow$(iii) $\widetilde{f} = f \circ p$ is the extension by the following diagram

$$\begin{tikzcd}
Z \arrow[dr, "p"] \arrow[drr, "\widetilde{f}"] & & \\
Y \arrow[u, "i"] \arrow[r, "id"] & Y \arrow[r, "f"] & F
\end{tikzcd}$$

(i)$\Longrightarrow$(ii) By the last exercise Y can be regarded as a subspace of $l_\infty(Y)$. Hence there is a Lipschitz retraction $p : l_\infty(Y) \longrightarrow Y$. Let $k \circ f : E \longrightarrow l_\infty(Y)$ be a Lipschitz function. By Proposition 2, there is a Lipschitz extension $f' : X \longrightarrow l_\infty(Y)$. If we take $\widetilde{f} = p \circ f'$, we prove this implication

$$\begin{tikzcd}
X \arrow[drr, "f'"] \arrow[drrr, "p\circ f'"] & & & \\
E \arrow[u, "i"] \arrow[r, "f"] & Y \arrow[r, "k"] & l_\infty(Y) \arrow[r, "p"] & Y
\end{tikzcd}$$

and we end the proof of the proposition. □

2. Lipschitz Spaces

Definition 8. (*a*) *Let* (X, d) *be a metric space. Then* $\mathrm{Lip}\,(X)$ *is the space of all bounded scalar valued Lipschitz functions on* X *with the norm*

$$\|f\|_L = \max\{\|f\|_\infty, \mathrm{Lip}\,(f)\}.$$

Let now $(X, \mathrm{d}, \mathrm{e})$ *be a pointed metric space with a distinguished "base point"* e *which is fixed in advance. We denote by* $\mathrm{Lip}_0(X)$ *the space of all bounded scalar valued Lipschitz mappings on* X, *vanishing at* e *with the norm*

$$\mathrm{Lip}\,(f) := \sup_{x \neq y} \frac{\mathrm{d}_Y(f(x), f(y))}{\mathrm{d}_X(x, y)}.$$

The spaces $\mathrm{Lip}\,(X)$ and $\mathrm{Lip}_0(X, Y)$ become Banach spaces. We put

$$X^{\#} = \mathrm{Lip}_0(X) = \mathrm{Lip}_0(X, \mathbb{R}).$$

This Banach space of Lipschitz functions is called also Lipschitz dual. It has been used by various mathematicians as a framework to extend results from linear functional analysis to the nonlinear case. We denote by $\widetilde{X} = \left\{(x, y) \in X^2 : x \neq y\right\}$.

Proposition 8. *Let* (X, e, d) *be a pointed metric space. The space* $(\mathrm{Lip}_0(X), \mathrm{Lip}\,(.))$ *is a Banach space.*

Proof. 1. One verify that $\mathrm{Lip}\,(.)$ is a norm on $\mathrm{Lip}_0(X)$. Let f be in $\mathrm{Lip}_0(X)$, we have

$$\begin{aligned} & \mathrm{Lip}\,(f) = 0 \\ \iff & \forall\,(x, y) \in \widetilde{X},\ \frac{|f(x) - f(y)|}{\mathrm{d}(x, y)} = 0 \\ \iff & \forall\,(x, y) \in X,\ f(x) = f(y). \end{aligned}$$

This implies that f is constant, As $f(e) = 0$, thus $f \equiv 0$.
Consider f, g in $\mathrm{Lip}_0(X)$. We have

$$\begin{aligned} & \mathrm{Lip}\,(f + g) \\ = & \sup_{x \neq y} \frac{|f(x) + g(x) - (f(y) + g(y))|}{\mathrm{d}(x, y)} \\ \leq & \sup_{x \neq y} \frac{|f(x) - f(y)| + |g(x) - g(y)|}{\mathrm{d}(x, y)} \\ \leq & \sup_{x \neq y} \frac{|f(x) - f(y)|}{\mathrm{d}(x, y)} + \sup_{x \neq y} \frac{|g(x) - g(y)|}{\mathrm{d}(x, y)} \\ \leq & \mathrm{Lip}\,(f) + \mathrm{Lip}\,(g). \end{aligned}$$

Let f be in $\mathrm{Lip}_0(X)$and λ be in $\mathbb{R}$. One have

$$\begin{aligned}\mathrm{Lip}\,(\lambda f) &= \sup_{x\neq y}\frac{|\lambda f(x)-\lambda f(y)|}{\mathrm{d}(x,y)}\\ &= \sup_{x\neq y}\frac{|\lambda|\,|f(x)-f(y)|}{\mathrm{d}(x,y)}\\ &= \lambda \mathrm{Lip}\,(f)\,.\end{aligned}$$

This means that $(\mathrm{Lip}_0(X), \mathrm{Lip}\,(.))$ is a normed space.
We prove now that $(\mathrm{Lip}_0(X), \mathrm{Lip}\,(.))$ is a Banach space.
We use this: normed vector space is complete if, and only if, every absolutely convergent sequence ([1]) converges. Indeed, the forward direction of this is easy. To prove the reverse direction, let (g_n) be any Cauchy sequence; we must show that it converges. Passing to a subsequence, we may assume that $g_{n+1}-g_n \leq \frac{1}{2^{-n}}$ for all n. Then define $f_1 = g_1$ and, for $n > 1$, $f_n = g_n - g_{n-1}$. Evidently f_n is absolutely convergent, and since its n-th partial sum is just g_n, the implication "absolutely convergent implies convergent" now entails that (g_n) converges.

Let (f_n) be a sequence in $\mathrm{Lip}_0(X)$ such that $\sum\limits_{n=1}^{\infty}\mathrm{Lip}\,(f_n) < \infty$. For any $x \in X$ we have $|f_n(x)| \leq \mathrm{Lip}\,(f_n)\,\mathrm{d}(x,e) < \infty$. Thus (f_n) converges pointwise, and the sum f is Lipschitz by Proposition 4. Letting $g_n = \sum\limits_{k=1}^{n} f_k$ be the n-th partial sum, we have

$$\mathrm{Lip}\,(f-g_n) = \mathrm{Lip}\left(\sum_{k=n+1}^{\infty} f_k\right) \leq \sum_{k=n+1}^{\infty}\mathrm{Lip}\,(f_k) \to 0.$$

This shows that the series f_n converges to f in $\mathrm{Lip}_0(X)$. By the above, we conclude that $\mathrm{Lip}_0(X)$ is complete.
Let $(f_n)_{n\in\mathbb{N}}$ a Cauchy sequence in $\mathrm{Lip}_0(X)$. We have

$$\forall \epsilon > 0 \quad \exists n_0 \in \mathbb{N} : \forall m, n \geq n_0; \quad \mathrm{Lip}\,(f_m - f_n) \leq \epsilon$$

$$\mathrm{Lip}\,(f_m - f_n) = \sup_{x\neq y}\frac{|(f_m\,(x) - f_m\,(y)) - (f_n\,(x) - f_n\,(y))|}{\mathrm{d}(x,y)} \leq \epsilon.$$

So, for every $x \in X$ $\quad (f_m\,(x) - (f_n\,(x))$ is a Cauchy in $\mathbb{R}$ and hence converges. Let $f(x)$ be its limit. We have

a) $f(0) = \lim\limits_{n\longrightarrow\infty} f_n\,(0) = 0.$

b) Let x, y be in X. We have

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[1] A sequence (f_n) in a normed vector space is said to converge absolutely if $\sum \|f_n\|$ converges.

$$\begin{array}{rcl} |f(x)-f(y)| & = & \lim\limits_{n\longrightarrow\infty}|f_n(x)-f_n(y)| \\ & \leq & \lim\limits_{n\longrightarrow\infty}\mathrm{Lip}(f_n)\,\mathrm{d}(x,y) \\ & \leq & K\mathrm{d}(x,y) \end{array}$$

where $K = \mathrm{Lip}(f_n)$. Indeed, by

$$|\mathrm{Lip}(f_n) - \mathrm{Lip}(f_m)| \leq \mathrm{Lip}(f_n - f_m) \leq \epsilon.$$

Hence $(\mathrm{Lip}(f_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$ and thus converges to K. So $f \in \mathrm{Lip}_0(X)$.

c) (f_n) converges to f.

Consider $n \geq n_0$. We have $\mathrm{Lip}(f_n - f) = \lim\limits_{m\longrightarrow\infty}\mathrm{Lip}(f_n - f_m) \leq \epsilon$ and hence $(f_n)_{n\in\mathbb{N}}$ converges to f. $\square$

Example 2. *Let X be a set. We denote by*

$$l_\infty(X) = \left\{ f : X \longrightarrow \mathbb{K} \text{ such that } \sup_{x\in X}|f(x)| < \infty \right\}.$$

Let X be a pointed metric space of finite diameter, i.e., $\sup\limits_{x,y\in X}\mathrm{d}(x,y) < \infty$. Show that $\mathrm{Lip}_0(X) \subset l_\infty(X)$.

We have $\mathrm{Lip}(f) := \sup\limits_{x\neq y}\dfrac{|f(x)-f(y)|}{\mathrm{d}(x,y)}$. This implies that by taking $y = 0$, $|f(x)| \leq \mathrm{Lip}(f)\,\mathrm{d}(x,0)$. Consequently, $f \in l_\infty(X)$.

Remark 2. *Let X be a pointed metric space.*

(1) $\mathrm{Lip}(.)$ *is only a seminorm, not a norm on* $\mathrm{Lip}(X)$.

(2) *Consider the set of all real-valued Lipschitz functions modulo the set of constant functions.* $\mathrm{Lip}(.)$ *descends to a norm on this quotient space and it is not hard to see that the result is isometrically isomorphic to* $\mathrm{Lip}_0(X)$ (*regardless of the choice of base point*). *With this procedure there is no good way to define products or a partial order on the quotient.*

(3) *The space* $\mathrm{Lip}_0(X)$ *does not depend on the choice of base point. If* e_1 *and* e_2 *are two different distinguished elements, then the linear map*

$$\begin{array}{rcccc} \varphi & : & \mathrm{Lip}_0(X,\mathrm{e}_1) & \longrightarrow & \mathrm{Lip}_0(X,\mathrm{e}_2) \\ & & f & \longmapsto & f - f(\mathrm{e}_2) \end{array}$$

is a surjective isometry.

3. The predual of $\mathrm{Lip}_0(X)$

It was shown by Arens and Eells [**AE56**] (see also [**Wea99**]) that $\mathrm{Lip}_0(X)$ is even a dual Banach space (but not reflexive if X is infinite and does not have constant functions in general), i.e., there exists a Banach space Z such that $\mathrm{Lip}_0(X)$ is isometrically isomorphic to Z . This canonical space is known as the Arens-Eells space by Weaver and the Lipschitz-free space on X in [**GK03**]. It well be noted by $\mathcal{F}(X,\mathrm{d}_X)$.

3.1. Construction of this space. We show that the unit ball $\mathcal{B}_{X^\#}$ is compact.

Product Topology

Let $(X_i, \mathcal{T}_i)_{i\in I}$ be a net of topological spaces. We note by

$$X = \prod_{i\in I} X_i.$$

The product topology of X noted $\mathcal{T}$ is the least fine topology making projections continuous

$$\begin{array}{cccc} p_i & : & X & \longrightarrow & X_i \\ & & (x_i)_{i\in I} & \longmapsto & x_i \end{array}$$

The least fine, i.e., having the fewest openings. The elementary openings of the product topology are of the form

$$\bigcap_{j\in J} p_j^{-1}\mathcal{U}_j \qquad J \text{ (finite)} \subset I.$$

Remark 3. *Let $(Y, \mathcal{S})$ be a topological space.*

(1) The projection p_i is an open application.

(2) An application $f : (Y, \mathcal{S}) \longrightarrow (X, \mathcal{T})$ is continuous if, and only if, $p_i \circ f$ is continuous for every i in I.

3.1.1. *Tyckonov's theorem.* The celebrate theorem in the product topology is the theorem of Tychonov(ff).

Theorem 4 (Tychonov). *A product space product $X = \prod_{i\in I} X_i$ is compact if, and only if, X_i is compact for all i in I. In other words, the topological product of any family of compact spaces is a compact space.*

Pointwise convergence is the same as convergence in the product topology on the space Y^X, where X is the domain and Y is the codomain. If the codomain Y is compact, then, by Tychonov's theorem, the space Y^X is also compact.

Let (X, d, e) be a pointed metric space. The topology $\mathcal{T}_p$ of pointwise convergence is the topology induced by the product $\mathbb{R}^X$ and determinates by the condition

$$f_i \xrightarrow{\mathcal{T}_p} f \Longleftrightarrow \forall x \in X, \quad f_i(x) \longrightarrow f(x)$$

for any net $(f_i)_{i\in I}$ in $\mathbb{R}^X$ and $f \in \mathbb{R}^X$.

Let now giving the analog of the Aloaglu (1940 for every Banach spaces)-Banach (1932 for separable Banach spaces) theorem for the unit ball $\mathcal{B}_{X^\#}$ of $\mathrm{Lip}_0(X)$.

3.1.2. *Compactness of $\mathcal{B}_{X^\#}$ is compact.* We study the compacteness of the unit ball of $X^\#$.

Proposition 9. *The unit bull $\mathcal{B}_{X^\#}$ is compact for the topology $\mathcal{T}_p$.*

PROOF. Observe that $\mathcal{B}_{X^\#}$ is closed in $\mathbb{R}^X$ with respect the topology $\mathcal{T}_p$. Indeed, consider a net $(f_i)_{i\in I}$ in $\mathbb{R}^X$ such that

$$f_i \xrightarrow{\mathcal{T}_p} f.$$

For x, y in X , the inequality

$$|f_i\,(x) - f_i\,(y)| \leq d(x,y)$$

implies that

$$|f(x) - f(y)| \leq d(x,y)$$

and consequently $f \in \mathcal{B}_{X^\#}$. Let now $f \in \mathcal{B}$. We have

$$|f(x)| \leq d(x,e), \forall x \in X.$$

This shows that

$$f \in \prod_{x\in X} [0, d(x,e)]$$

and this implies

$$\mathcal{B}_{X^\#} \subset \prod_{x\in X} [0, d(x,e)].$$

The space $\prod\limits_{x\in X} [0, d(x,e)]$ is compact by Tychonov's theorem and $\mathcal{B}_{X^\#}$is closed so it is compact (closed of compact is compact). □

3.1.3. *Conjugate space.* Let E be a Banach space. We say that E is a conjugate space if there exists a Banach space B such that B^* is isometrically isomorphic to E (i.e., $B^* \equiv E$). We now give a simple sufficient condition to generate that space B exists.

Let us recall that a family of seminorms on a linear space generates a locally convex topology in the following sense.

THEOREM 5. *Let $\{p_i : i \in I\}$ be a family of seminorms on the linear space E . Let $\mathcal{U}$ be the class of all finite intersections of sets of the form*

$$\{x \in E : p_j(x) < r_j\}$$

where $j \in J$ (finite) $\subset I$, $r_j > 0$. Then $\mathcal{U}$ is a local base for a topology $\mathcal{J}$ that makes E a locally convex topological vector space. This topology is the weakest making all the p_i continuous, and for a net $\{x_\alpha\} \subset E$, $x_\alpha \to x$ in $\mathcal{J}$ if, and only if, $p_i(x_\alpha - x) \to 0$ for each $i \in I$.

THEOREM 6 (Dixmier-Ng theorem). *Let E be a Banach space. Suppose that there is a (Hausdorff) locally convex topology σ on E such that $\mathcal{B}_E$ is σ-compact. Then E is a conjugate space.*

Proof. Let $B = \left\{ \xi \in E' : \xi_{|\mathcal{B}_E} \text{ is } \sigma\text{-continuous} \right\}$ (E'= algebraic conjugate space of E). Then B is a closed linear subspace of E^* and is therefore a Banach space; (to see that $B \subset E^*$ observe that for any $\xi \in B$ the image $\xi(\mathcal{B}_E)$ is compact and hence bounded set of scalar; that is, $\|\xi\|$ is finite and so $\xi \in E^*$. Also B is closed in E^*; because convergence in E^* entails uniform convergence on $\mathcal{B}_E$. We now bring in the (canonical embedding) operator $J_{E,B} : E \longrightarrow B^*$ defined by

$$\langle \xi, J_{E,B}(x) \rangle = \xi(x).$$

This operator assigns to each $x \in X$ the functional "evaluation at x" in B^*, we clearly have $\|J_{E,B}(x)\| \leq 1$. The proof will be completed by showing that $J_{E,B}(x)$ is an isomorphic isometry between E and B^*. We do this by showing that $J_{E,B}(x)$ is injective and that it maps $\mathcal{B}_E$ onto $\mathcal{B}_{B^*}$. The first assertion follows because B is total. Indeed B contains the dual space E; which certainly separates the points of E. The second assertion follows from the fact (evident by definition of B) that $J_{E,B}$ is continuous from the σ-topology on E into the weak*-topology on B^*. This means in particular that $J_{E,B}(\mathcal{B}_E)$ is weak*-compact in B^*. But, by theGoldstine-Weston density lemma, this image is also weak*-dense in $\mathcal{B}_{B^*}$. □

Remark 4. *Any weak*-closed linear subspace F of a conjugate space E^* is itself a conjugate space. This follows from the observation that $\mathcal{B}_F$ is compact in the (relative) weak*-topology.*

We now give an example.

Example 3. *Consider the space* $\mathrm{Lip}(X, \mathrm{d}, \mathbb{R})$ *of bounded Lipschitz functions defined on the metric space* (X, d) *and normed by* $\|.\|_L = \max\{\|.\|_\infty, \mathrm{Lip}(.)\}$. *Let σ be the topology of pointwise convergence on X, which we denote by* $\sigma(\mathrm{Lip}(X, \mathrm{d}, \mathbb{R}), X)$. *Then $\mathcal{B}_X$ is certainly a* $\sigma(\mathrm{Lip}(X, \mathrm{d}, \mathbb{R}), X)$*-closed subset of X. We have*

$$\mathcal{B}_{\mathrm{Lip}\,(X,d,\mathbb{R})} \subset [-1,1]^X.$$

Since $[-1,1]$ is compact by Tychonov's theorem we have $[-1,1]^X$. Consequently, $\mathcal{B}_{\mathrm{Lip}\,(X,d,\mathbb{R})}$ is $\sigma(\mathrm{Lip}(X, d, \mathbb{R}), X)$*-compact and so X is a conjugate space.*

3.1.4. $\mathrm{Lip}_0(X)$ *is a dual space.* We have seen that the unit ball $\mathcal{B}_{X^\#}$ is $\mathcal{T}_p$-compact and according to "Dixmier-Ng theorem" $\mathrm{Lip}_0(X)$ is a dual space, for every $X \in \mathcal{M}_0$.

Theorem 7. *The space* $\mathrm{Lip}_0(X)$ *is a dual space, for every* $X \in \mathcal{M}_0$.

Proof. By Dixmier-Ng's theorem, it suffices to prove that $\mathcal{T}_p$ is Hausdorff locally convex.

(1) The topology $\mathcal{T}_p$ is locally convex.

(2) The topology $\mathcal{T}_p$ is separating.

(1) Define

$$p_x(f) = |f(x)|, \quad x \in E \text{ and } f \in \mathcal{B}_{X^\#}$$

and put $P = \{p_x\}_{x \in E}$. By the precedent theorem, the topology defined by P is locally convex and it is exactly the topology of pointwise convergence $\mathcal{T}_p$.

(2) The topology $\mathcal{T}_p$ is a Hausdorff topology if, and only if, the family $\{p_x\}_{x \in E}$ is separating, i.e., given $f \neq 0$, there exists $x \in E$ such that $p_x(f) \neq 0$. This is the case and this ends the proof. $\square$

Remark 5. *On bounded sets the weak*-topology agrees with the topology of pointwise convergence.*

3.2. Arens Eells space. Let $(X, \mathrm{e}, \mathrm{d})$ be pointed a metric space. A molecule on X is a real valued function m on X with finite support (i.e., the set where m has non-zero values) and satisfies

$$\sum_{x \in \mathrm{supp}(m)} m(x) = 0.$$

Denote by $\mathcal{M}(X)$ the real linear space of molecules on X. We can write

$$\begin{aligned} m &= \sum_{x \in \mathrm{supp}(m)} m(x) \mathbf{1}_{\{x\}} \\ &= \sum_{i=1}^{n} m(x_i) \mathbf{1}_{\{x_i\}}. \end{aligned}$$

where $\mathrm{supp}(m) = \{x_1, ..., x_n\}$ and $\mathbf{1}_{\{x\}}$ denotes the characteristic function of the set $\{x\}$. For $x, y \in X$ we define the basic molecule $m_{x_1 x_2} = \mathbf{1}_{\{x_1\}} - \mathbf{1}_{\{x_2\}}$ (with $x_1, x_2 \in X$ are called atoms). It is easy to to see that every molecule m can be written as a (non unique) finite linear combination of basic molecule (the condition $\sum_{i=1}^{n} m(x_i) = 0$ insures that such representations of m exist $m = \lambda_1 m_{x_1,x_2} + (\lambda_1 + \lambda_2) m_{x_2,x_3} + \cdots + (\lambda_1 + \cdots + \lambda_{n-1}) m_{x_{n-1},x_n}$). We have

$$\begin{aligned} m &= \sum_{j=1}^{l} a_j \left(\mathbf{1}_{\{x_j\}} - \mathbf{1}_{\{y_j\}} \right) \\ &= \sum_{j=1}^{l} a_j m_{x_j, y_j}. \end{aligned}$$

Example 4. *Consider $m : \mathbb{R} \longrightarrow \mathbb{R}$ such that*

$$\begin{cases} m(0) = -4, \\ m(1) = 1, \\ m(2) = 3, \\ 0 \textit{ otherwise.} \end{cases}$$

$$\begin{aligned} m &= -4.\mathbf{1}_{\{0\}} + 1.\mathbf{1}_{\{1\}} + 3.\mathbf{1}_{\{2\}}, \\ &= -3.\mathbf{1}_{\{0\}} - 1.\mathbf{1}_{\{0\}} + 1.\mathbf{1}_{\{1\}} + 3.\mathbf{1}_{\{2\}}, \\ &= 1.\left(\mathbf{1}_{\{1\}} - \mathbf{1}_{\{0\}}\right) + 3\left(\mathbf{1}_{\{2\}} - \mathbf{1}_{\{0\}}\right). \end{aligned}$$

Put now

$$\|m\|_{\mathcal{M}(X)} = \inf\left\{\sum_{j=1}^{l} |a_j|\, \mathrm{d}_X\left(x_j, y_j\right)\right\},$$

over all representation of $m = \sum_{j=1}^{l} \lambda_j \left(1_{\{x_j\}} - 1_{\{x'_j\}}\right)$.

It follows that $\|.\|_{\mathcal{M}(X)}$ is a norm on the vector space $\mathcal{M}(X)$. Denote by $\text{Æ}(X, \mathrm{d}_X)$ the completion of the normed space $(\mathcal{M}(X), \|.\|_{\mathcal{M}(X)})$. This space was first introduced by Arens and Eells [**AE56**].in 1956. Originally, the basic idea goes back to Kantorovich [**Kan42**]. The terminology Arens-Eells space $\text{Æ}(X, \mathrm{d})$ is due to Weaver [**Wea99**]. A different notation and appellation was used in [**GK03**] by Godefroy and Kalton. It is the Lipschitz-free space denoted by $\mathcal{F}(X, \mathrm{d})$ which we will introduce in the sequel.

REMARK 6. *Every molecule m is uniquely expressible in the form*

$$m = \sum_{j=1}^{l} a_j \left(\mathbf{1}_{\{x_j\}} - \mathbf{1}_{\{e\}}\right)$$

where the points x_j are all distinct and none equals to e.

We now prove that $(\text{Æ}(X))^* \overset{\text{isometrically}}{\equiv} \mathrm{Lip}_0(X)$.

THEOREM 8. $(\text{Æ}(X))^*$ *is isometrically isomorphic to* $\mathrm{Lip}_0(X)$.

PROOF. Define

$$S : \text{Æ}^*(X, \mathrm{d}) \longrightarrow \mathrm{Lip}_0(X)$$

by

$$(S\varphi)(x) = \varphi\left(\left(\mathbf{1}_{\{x\}} - \mathbf{1}_{\{\mathrm{e}\}}\right)\right).$$

Since $\left\|\mathbf{1}_{\{x\}} - \mathbf{1}_{\{x'\}}\right\|_{\text{Æ}(X,d)} = \mathrm{d}(x, x')$ for all $x, x' \in X$ ([2]), we have

$$\begin{aligned} |(S\varphi)(x) - (S\varphi)(x')| &= \left|\varphi\left(\left(\mathbf{1}_{\{x\}} - \mathbf{1}_{\{\mathrm{e}\}}\right)\right) - \varphi\left(\left(\mathbf{1}_{\{x'\}} - \mathbf{1}_{\{\mathrm{e}\}}\right)\right)\right| \\ &= \left|\varphi\left(\left(\mathbf{1}_{\{x\}} - \mathbf{1}_{\{x'\}}\right)\right)\right| \\ &\leq \|\varphi\|\, \mathrm{d}(x, x'). \end{aligned}$$

Also $(S\varphi)(e) = \varphi(0)$, so indeed $S\varphi \in \mathrm{Lip}_0(X)$. It follows that S is a nonexpansive linear mapping from $\text{Æ}^*(X, d)$ to $\mathrm{Lip}_0(X)$ i.e., $\mathrm{Lip}(S\varphi) \leq \|\varphi\|_{\text{Æ}^*}$.

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[2] Voir Proposition 10 below

Define now $R : \mathrm{Lip}_0(X) \longrightarrow Æ^*(X, \mathrm{d})$ by

$$(Rf)(m) = \sum_x m(x) f(x)$$

for $f \in \mathrm{Lip}_0(X)$ and m a molecule. If $m = \sum_{j=1}^{l} \lambda_j \left(\mathbf{1}_{\{x_j\}} - \mathbf{1}_{\{x'_j\}} \right)$, we have

$$\begin{aligned} |(Rf)(m)| &= \left| \left(\sum_x m(x) f(x) \right) \right| \\ &\leq \left| \sum_{j=1}^{l} \lambda_j f(x_j) - f\left(x'_j\right) \right| \\ &\leq \sum_{j=1}^{l} |\lambda_j| \left| f(x_j) - f\left(x'_j\right) \right| \\ &\leq \mathrm{Lip}(f) \sum_{j=1}^{l} |\lambda_j| \, \mathrm{d}\left((x_j, x'_j \right) . \end{aligned}$$

Hence $|(Rf)(m)| \leq \mathrm{Lip}(f) \, \|m\|_{M(X)}$, which uniquely extends to a continuous linear functional on the completion $Æ(X, d)$ of $\mathcal{M}(X)$, denoted by the same symbol Rf. Thus $Rf \in Æ^*(X, d)$ and $\|Rf\| \leq \mathrm{Lip}(f)$. Straightforward calculations show that R and S are inverses. Indeed, for all $x \in X$

$$\begin{aligned} (S \circ R)(f)(x) &= S(R(f))(x) \\ &= R(f)\left(\mathbf{1}_{\{x\}} - \mathbf{1}_{\{e\}}\right) \\ &= f(x) \end{aligned}$$

and for all $m \in \mathcal{M}(X)$

$$\begin{aligned} (R \circ S)(\varphi)(m) &= R(S(\varphi))(m) \\ &= \sum_x m(x) S(\varphi)(x) \\ &= \sum_{j=1}^{l} \lambda_j \left(S(\varphi)(x_j) - S(\varphi)\left(x'_j\right) \right) \\ &= \sum_{j=1}^{l} \lambda_j \varphi \left(1_{\{x_j\}} - 1_{\left\{x'_j\right\}} \right) \\ &= \varphi(m) . \end{aligned}$$

The operators R, S are nonexpansive and $R \circ S = S \circ R = \mathrm{Id}$, so S is isometric ($\|x\| = \|(R \circ S)(x)\| \leq \|R\| \, \|S(x)\| \leq \|S(x)\|$) and hence $\mathrm{Lip}_0(X)$ is isometrically isomorphic to $Æ^*(X, \mathrm{d}_X)$. $\square$

Proposition 10. *Let* $(X, \mathrm{e}, \mathrm{d})$ *be a pointed metric space.*
(1) *For any molecule* m *we have*

$$\|m\|_{Æ(X,\mathrm{d}_X)} = \sup\left\{ |\langle m, f\rangle| = \left| \sum_{x\in X} m(x)\, f(x) \right| : f \in \mathcal{B}_{X^\#} \right\}$$

and there exists $f \in \mathcal{B}_{X^\#}$ *such that* $\langle m, f\rangle = \|m\|_{Æ(X,\mathrm{d}_X)}$.

(2) $\|.\|_{Æ(X,d_X)}$ *is a norm on* $\mathcal{M}(X)$ *and* $\left\|\mathbf{1}_{\{x\}} - \mathbf{1}_{\{y\}}\right\|_{Æ} = \mathrm{d}(x,y)$ *for all* x, y *in* X.

(3) $\|.\|_{Æ(X,d_X)}$ *is the largest seminorm on* $\mathcal{M}(X)$ *which satisfies for all* x, y *in* X, $\left\|\mathbf{1}_{\{x\}} - \mathbf{1}_{\{y\}}\right\|_{Æ} = \mathrm{d}(x,y)$.

Proof. (1) This follows from the identification of $\mathrm{Lip}_0(X,d)$ with $Æ(X,d)^*$ and the Hahn-Banach theorem.

(2) The inequality $\left\|\mathbf{1}_{\{x\}} - \mathbf{1}_{\{y\}}\right\|_{Æ} \le \mathrm{d}(x,y)$ follows from the definition. Conversely, fix x in X and define

$$f_x(y) = \mathrm{d}(x,y) - \mathrm{d}(x,\mathrm{e}).$$

We have $f_x \in B_{\mathrm{Lip}_0(X,\mathrm{d})}$ because $f_x(\mathrm{e}) = 0$ and $\mathrm{Lip}(f_x) = 1$. Indeed,

$$\begin{aligned}
\mathrm{Lip}(f_x) &= \sup_{y_1\neq y_2} \frac{|f_x(y_1) - f_x(y_2)|}{\mathrm{d}(y_1,y_2)} \\
&\ge \sup_{x\neq y} \frac{|f_x(y) - f_x(x)|}{\mathrm{d}(x,y)} \\
&\ge \frac{\mathrm{d}(x,y)}{\mathrm{d}(x,y)} = 1.
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Lip}(f_x) &= \sup_{y_1\neq y_2} \frac{|f_x(y_1) - f_x(y_2)|}{\mathrm{d}(y_1,y_2)} \\
&\le \sup_{y_1\neq y_2} \frac{|\mathrm{d}(x,y_1) - \mathrm{d}(x,y_2)|}{\mathrm{d}(y_1,y_2)} \\
&\le \frac{\mathrm{d}(y_1,y_2)}{\mathrm{d}(y_1,y_2)} = 1.
\end{aligned}$$

By part (1), we have

$$\begin{aligned}
\left\|\mathbf{1}_{\{x\}} - \mathbf{1}_{\{y\}}\right\|_{Æ} &\ge |\langle m_{xy}, f_x\rangle| \\
&\ge |m_{xy}(x) f_x(x) + m_{xy}(y) f_x(y)| \\
&\ge |-m_{xy}(x)\mathrm{d}(x,\mathrm{e}) + m_{xy}(y)\mathrm{d}(x,y) + m_{xy}(y)\mathrm{d}(x,\mathrm{e})| \\
&\ge |m_{xy}(y)\mathrm{d}(x,y)| \\
&\ge \mathrm{d}(x,y).
\end{aligned}$$

(3) Let $\|.\|_0$ be any semi norm such that

$$\left\|\mathbf{1}_{\{x\}} - \mathbf{1}_{\{y\}}\right\|_0 \le d(x,y)$$

for all $x, y \in X$. Let $m = \sum_{i=1}^{n} a_i m_{x_i y_i}$ be a molecule. We have

$$\begin{aligned} \|m\|_0 &= \left\| \sum_{i=1}^{n} a_i m_{x_i y_i} \right\|_0 \\ &\leq \sum_{i=1}^{n} |a_i| \, \|m_{x_i y_i}\|_0 \\ &\leq \sum_{i=1}^{n} |a_i| \, d\left(x_i, y_i\right) \end{aligned}$$

Taking the infimum of all such representation of m yields $\|m\|_0 \leq \|m\|_{\text{Æ}}$. ☐

Corollary 2. *The application $i_X : X \longrightarrow \text{Æ}(X, d)$ defined by*

$$i_X(x) = \mathbf{1}_{\{x\}} - \mathbf{1}_{\{\mathrm{e}\}} = m_{x\mathrm{e}}$$

is an isometric embedding of X into $\text{Æ}(X, \mathrm{d}_X)$.

Proof. We have by Proposition 10

$$\|i_X(x) - i_X(y)\|_{\text{Æ}} = \left\|\mathbf{1}_{\{x\}} - \mathbf{1}_{\{y\}}\right\|_{\text{Æ}} = \mathrm{d}(x, y)$$

for all $x, y \in X$. So i_X is an isometry. ☐

The following theorem is known as the linearization of Lipschitz operators.

Theorem 9 ([**Wea99**, Theorem 2.2.4]). *Let $(X, \mathrm{d}, \mathrm{e})$ be a pointed metric space. Let E be a Banach space and let $T : X \longrightarrow E$ be a Lipschitz map which preserves base point (i.e., $T(\mathrm{e}) = 0$). Then there is a unique bounded linear operator $u : \text{Æ}(X) \longrightarrow E$ such that $T = u \circ i$ and $\|u\| = \mathrm{Lip}(T)$ $(i : X \longrightarrow \text{Æ}(X))$.*

$$\begin{tikzcd} \text{Æ}(X) \arrow[d, "i"'] \arrow[dr, "u"] & \\ X \arrow[r, "T"] & E \end{tikzcd}$$

Proof. Every molecule m is uniquely expressible in the form (3)

$$m = \sum_{j=1}^{l} \lambda_j \left(\mathbf{1}_{\{x_j\}} - \mathbf{1}_{\{\mathrm{e}\}}\right)$$

where the points x_j are all distinct and none equals to e. We then define u by

$$u(m) = \sum_{j=1}^{l} \lambda_j T\left(x_j\right)$$

---

[3] Voir Remark 6

Since u is essentially an extension of T that is $T = u \circ i$ and we automatically have $\|u\| \geq \mathrm{Lip}(T)$. For the rest it will suffice to show that $\|u\| \leq \mathrm{Lip}(T)$ (in particular, this implies that u is bounded and hence it extends to all $\text{Æ}(X, \mathrm{d}_X)$). Define a semi norm $\|.\|_0$ on the space of molecules by setting

$$\|m\|_0 = \frac{\|u(m)\|}{\mathrm{Lip}(T)}.$$

Then

$$\begin{array}{lcl} \left\|\mathbf{1}_{\{x\}} - \mathbf{1}_{\{y\}}\right\|_0 & = & (\mathrm{Lip}(T))^{-1} \left\|T(x) - T(y)\right\| \\ (m_{xy} = m_{xe} - m_{ye}) & \leq & \mathrm{d}(x, y) \end{array}$$

for all $x, y \in X$. This implies that $\|.\|_0 \leq \|.\|_{\text{Æ}}$ by Proposition 10. Thus $\|u(m)\| \leq \mathrm{Lip}(T). \|m\|_{\text{Æ}}$, which shows that $\|u\| \leq \mathrm{Lip}(T)$ as desired. The uniquess is simple. □

The operator u is denoted by T_L.

Proposition 11. *The weak*-topology* $\sigma(\mathrm{Lip}_0(X), \text{Æ}(X))$ *topology agrees with the topology of pointwise convergence on bounded subset of* $\mathrm{Lip}_0(X)$.

Proof. Let T_i, T be in $\mathrm{Lip}_0(X)$ such that

$$T_i \longrightarrow T, \quad \sigma\left(\mathrm{Lip}_0(X), \text{Æ}\left(X, d_X\right)\right).$$

Then, for all x in X we have

$$T_i\left(x\right) = \left(T_i\right)_L \left(\mathbf{1}_{\{x\}} - \mathbf{1}_{\{e\}}\right) \longrightarrow T_L \left(\mathbf{1}_{\{x\}} - \mathbf{1}_{\{e\}}\right) = T(x).$$

$$\begin{tikzcd} \text{Æ}\left(X\right) \arrow[d, "i_X"'] \arrow[dr, "T_L"] & \\ X \arrow[r, "T"] & E \end{tikzcd}$$

For the converse, it is a classical result. □

Let $T \in \mathrm{Lip}_0(X, Y)$ and let i_X, i_Y be the isometric embedding of X, Y into $\mathrm{Lip}_0(X)$, $\mathrm{Lip}_0(Y)$, respectively). Let $\Psi(T)$:$\text{Æ}(X, d_X) \longrightarrow Y$ be the bounded linear operator attached to T and let $\phi = i_Y \circ \Psi$. Let S, R be the linear isometrics between the spaces $\mathrm{Lip}_0(X)$ and $\text{Æ}(X, d_X)$, and $\mathrm{Lip}_0(Y)$ and $\text{Æ}(Y, d_Y)$.

Theorem 10 ([**Cob03**]). *We have* $T^{\#} = S_1 \circ \phi(T)^* \circ R_2$ *or equivalently* $\phi(T)^* = R_1 \circ T^{\#} \circ S_2$, *i.e., the following diagrams are commutative*

$$\begin{tikzcd} \text{Æ}\left(Y, d_Y\right)^* \arrow[r, "\phi(T)^*"] & \text{Æ}\left(X, d_X\right)^* \arrow[d, "S_1\left(= S^{-1}\right)"] \\ \mathrm{Lip}_0(Y) \arrow[u, "R_2\,(= R)"] \arrow[r, "T^{\#}"] & \mathrm{Lip}_0(X) \end{tikzcd}$$

or equivalently

$$\begin{tikzcd}
\text{Æ}\left(Y,d_Y\right)^* \arrow[r, "\phi(T)^*"] & \text{Æ}\left(X,d_X\right)^* \\
\mathrm{Lip}_0(Y) \arrow[r, "T^{\#}"] & \mathrm{Lip}_0(X)
\end{tikzcd}$$

$$S_2\left(=R^{-1}\right)\downarrow \qquad\qquad R_1\left(=S\right)\uparrow$$

Proof. We have

$$\phi(m_{x,0})=i_Y\left(\Psi\left(T\right)\right)\left(m_{x,0}\right)=i_Y\left(T(x)\right)=m_{T(x),0}. \tag{3.1}$$

Put

$$F=S_1\circ\phi(T)^*\circ R_2.$$

Therefore

$$\begin{array}{lcll}
(S_1\varphi)(x) & = & \varphi\left(M_{x,0}\right), & x\in X,\varphi\in \text{Æ}\left(X\right)^* \\
\phi(T)^*\left(\psi\right) & = & \psi\circ\phi(T), & \psi\in \text{Æ}\left(Y\right)^* \\
\left(R_2g\right)\left(m\right) & = & \displaystyle\sum_{y\in Y} m\left(y\right)g\left(y\right), & g\in Lip_0(Y), m\in M\left(Y\right).
\end{array}$$

Taking into account these formulas, the definitions of the operators R and S, and Formula 3.1, we obtain successively:

$$\begin{array}{lclcl}
(Fg)(x) & = & \left(S_1\circ\phi(T)^*\circ R_2\right)(g)(x) & = & S_1\left(\phi(T)^*\left(R_2(g)\right)\right)(x) \\
\phi(T)^*\left(\psi\right) & = & S_1\left(R_2(g)\circ\phi(T)\right)(x) & = & \\
\left(R_2g\right)(m) & = & S_1\left(R_2(g)\circ\phi(T)\right)\left(m_{x,0}\right) & = & \\
 & = & R_2(g)(m_{x,0}) & = & g\circ T\left(x\right)=T^{\#}\left(g\right)\left(x\right).
\end{array}$$

This proved the theorem. □

3.3. Banach free space. The following theorem was independently proved by Flood in [**Flo75**] and Pestov in [**Pes86**].

Theorem 11. *Let* $(X,\mathrm{d},\mathrm{e})$ *be a pointed metric space. then there exists a unique, up to an isometric isomorphism, Banach space* $\mathrm{B}\left(X\right)$ *over the field* $\mathbb{F}$ *and an isometric embedding* $i_X:X\longrightarrow\mathrm{B}\left(X\right)$ *such that*

1. *The linear span of* $i_X\left(X\right)$ *is dense in* $\mathrm{B}\left(X\right)$.
2. *Every map* T *in* $\mathrm{Lip}_0\left(X,E\right)$ *can be extended to a continuous linear operator* $T_L:\mathrm{B}\left(X\right)\longrightarrow E$ *such that* $\left\|T_L\right\|=\mathrm{Lip}\left(T\right)$ *for any arbitrary normed space.*

3.4. Lipschitz free space. J.-A. Johnson in [**Joh70**], proved without any reference to molecules that the closed linear subspace of $\left(X^{\#}\right)^*$ spanned by the evaluation functions $\delta_x:X^{\#}\longrightarrow\mathbb{K}$, given by

$$\delta_x\left(f\right)=f\left(x\right);\qquad x\in X$$

is a predual of $X^{\#}$ (we note that any weak*-closed linear subspace B of a conjugate space E^* is itself a conjugate space. This follows from the observation that $\mathcal{B}_B$ is compact in the (relative) weak*-topology). This space was called Lipschitz-free space and denoted $\mathcal{F}(X)$ by Godefroy and Kalton in [**GK03**].

Definition 9. *The Lipschitz free space on X is*

$$\mathcal{F}(X, \mathrm{d}_X) = \overline{\operatorname{span}\{\delta_x, \quad x \in X\}_,}^{\mathrm{Lip}_0(X)^*}.$$

We say that $\gamma \in \mathcal{F}(X, \mathrm{d}_X)$ is finitely supported if

$$\gamma \in \operatorname{span}\{\delta_x, \quad x \in X\}.$$

Then, the support of such a γ (denoted suppγ) is the smallest subset F of X which contains e and such that $\gamma \in \operatorname{span}\{\delta_x, \quad x \in F\}$.

Remark 7. *By applying the bipolar theorem, we give a precise description of $\mathcal{B}_{\mathcal{F}(X)}$ by means of the Lipschitz evaluation functional $\delta_{(x,y)} = \dfrac{\delta_x - \delta_y}{d(x,y)}$ defined on $X^{\#}$, where (x,y) runs through $\widetilde{X} = \{(x,y) \in X^2 : x \neq y\}$.*

(1) *The closed unit ball of $\mathcal{F}(X)$ is the closed, convex, balanced hull of the set $\{\delta_{(x,y)} : (x,y) \in \widetilde{X}\}$ in $(X^{\#})^*$.*

(2) *The space $\mathcal{F}(X)$ is the closed linear hull of the set $\{\delta_x : x \in X\}$ in $(X^{\#})^*$.*

(3) *From (1), we deduce that $\mathcal{F}(X)$ is the closed linear hull in $(X^{\#})^*$ of the set $\{\delta_{(x,y)} : (x,y) \in \widetilde{X}\}$.Then (2) follows since the linear hulls of this set and the set $\{\delta_x : x \in X\}$ coincide. Notice that $\delta_x = \delta_x - \delta_0 = d(x,0)\delta_{(x,0)}$ $(x \in X, x \neq 0)$.*

Proposition 12. *For any metric space X, $\mathcal{F}(X, \mathrm{d})^* \overset{isometrically}{\equiv} \mathrm{Lip}_0(X)$.*

Proof. We define a linear surjective isometry J on $\mathrm{Lip}_0(X)$ with values in $\mathcal{F}(X, \mathrm{d})^*$ by $J(f)(\delta_x) = f(x)$ and we extend by continuity to $\mathcal{F}(X, \mathrm{d})$. Consider f in $\mathrm{Lip}_0(X)$ and m in $\operatorname{span}\{\delta_x, \quad x \in X\}$ such that $m = \sum_{i=1}^{n} a_i \delta_{x_i}$. $J(f)(m) = \sum_{i=1}^{n} a_i f(x_i)$. We show that J is a surjective isometry.

a) Consider f in $\mathrm{Lip}_0(X)$ and m in $\mathcal{F}(X, \mathrm{d})$. We have

$$\begin{aligned} |J(f)(m)| &= \left|\langle f, m\rangle_{(\mathrm{Lip}_0(X), \mathcal{F}(X))}\right| \\ &= \left|\langle f, m\rangle_{(\mathrm{Lip}_0(X), \mathrm{Lip}_0(X)^*)}\right| \\ &\leq \mathrm{Lip}(f)\, \|m\|_{\mathcal{F}(X)} \end{aligned}$$

and we obtain $\|J(f)\| \leq \mathrm{Lip}(f)$.

b) Let (x,y) be in $\widetilde{X}$ and put $m = \dfrac{\delta_x - \delta_y}{\mathrm{d}(x,y)}$. We have $\|m\|_{\mathcal{F}(X,)} = 1$ because δ is an isometry see Proposition 13 below and

$$\begin{array}{rcl} \|J(f)\|_{\mathcal{F}(X,\mathrm{d})^*} & \geq & |J(f)\,(m)| \\ & \geq & \left|\dfrac{f\,(x)-f\,(y)}{\mathrm{d}\,(x,y)}\right| \\ \text{(we take the sup)} & \geq & \mathrm{Lip}\,(f)\,. \end{array}$$

c) Consider $\varphi \in \mathcal{F}\,(X,\mathrm{d})^*$. Then φ is determinate by δ_x for every x in X. We put for every x in X, $f\,(x)=\varphi\,(\delta_x)$ and we prove that f is Lipschitz and $J(f)=\varphi$.

(i) We show that $f \in \mathrm{Lip}_0(X)$.

- $f\,(0)=\varphi\,(\delta_0)=\varphi\,(0)=0$.

- Let x, y be in X

$$\begin{array}{rcl} |f\,(x)-f\,(y)| & = & |\varphi\,(\delta_x)-\varphi\,(\delta_y)| \\ & = & |\langle\varphi,\delta_x-\delta_y\rangle| \\ & \leq & \|\varphi\|_{\mathcal{F}(X,\mathrm{d})^*}\,\|\delta_x-\delta_y\|_{(\mathrm{Lip}_0(X))^*} \\ & \leq & \|\varphi\|_{\mathcal{F}(X,\mathrm{d})^*}\,\mathrm{d}\,(x,y)\,. \end{array}$$

(ii) Let $m=\sum_{i=1}^{n} a_i\delta_{x_i}$ be in span $\{\delta_x : x \in X\}$. Then, $\varphi\,(m)=\sum_{i=1}^{n} a_i f\,(x_i)=J(f)(m)$. □

Proposition 13. *Define*

$$\begin{array}{rcccc} \delta & : & X & \longrightarrow & \left(X^{\#}\right)^* \\ & & x & \longmapsto & \delta_x \end{array}$$

The application δ is an isometry, i.e., for every x_1, x_2 in X, one have $\|\delta_{x_1}-\delta_{x_2}\|=\mathrm{d}\,(x_1,x_2)$ (this implies that $\|\delta_x\|=\mathrm{d}\,(x,0)$).

Proof. For $x_1, x_2 \in X$, we have in the first part

$$\begin{array}{rcl} \|\delta_{x_1}-\delta_{x_2}\| & = & \sup\limits_{\mathrm{Lip}(f)=1} |\delta_{x_1}\,(f)-\delta_{x_2}\,(f)| \\ & = & \sup\limits_{\mathrm{Lip}(f)=1} |f\,(x_1)-f\,(x_2)| \\ & \leq & \mathrm{d}\,(x_1,x_2)\,. \end{array}$$

In the second part, for a fixed $x_0 \in X$, let $g \in \mathcal{B}_{X^{\#}}$ defined by

$$g\,(x)=\mathrm{d}\,(x,x_1)-\mathrm{d}\,(x_0,x_2)\,.$$

We have

$$\begin{array}{rcl} \|\delta_{x_1}-\delta_{x_2}\| & \geq & g\,(x_1)-g\,(x_2) \\ & \geq & d\,(x_1,x_2) \end{array}$$

and this ends the proof. □

Remark 8. *The subset* $\delta(X)$ *is linearly independent in* $\left(X^{\#}\right)^*$ *see* ([**Mic64**]. *Indeed, let* $x_1, ..., x_n, x_{n+1}$ *be distinct elements of* X*, then* $\delta_{x_{n+1}}$ *cannot be a linear combination of* $\delta_{x_1}, ..., \delta_{x_n}$*. So, if* $g(x) = d(x, \{x_1, ..., x_n\})$ *for* $x \in X$*, then* $g \in X^{\#}$ *and*

$$\begin{array}{lll} \delta_{x_i}(g) & = & g(x_i) \quad \text{for } 1 \leq i \leq n \\ \delta_{x_{n+1}}(g) & = & g(x_{n+1}) \quad . \end{array}$$

This implies that $\delta_{x_{n+1}}$ *cannot be a linear combination of* $\delta_{x_1}, ..., \delta_{x_n}$ *and consequently* $\delta(X)$ *is linearly independent in* $\left(X^{\#}\right)^*$.

The Banach space $\mathcal{F}(X)$ has some remarkable properties, from which we mention the following universal property; called "*universal linearization property*".

Theorem 12 ([**GK03**]). *Let* $(X, \mathrm{d}, \mathrm{e})$ *be a pointed metric space and let* E *be a Banach space. Let* $T : X \longrightarrow E$ *be a Lipschitz map such that* $T(\mathrm{e}) = 0$*. Then, there is a unique linear map* u *(noted* T_L*)* $: \mathcal{F}(X) \longrightarrow E$ *with* $\|T_L\| = \mathrm{Lip}(T)$ *and such that the following diagram commutes*

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, "\delta_X"'] & E \\ \mathcal{F}(X) \arrow[ur, "T_L"] & \end{tikzcd}$$

Moreover, the linear isometry $\varphi : \mathrm{Lip}_0(X, E) \longrightarrow \mathcal{B}(\mathcal{F}(X), E)$ *such that* $\varphi(T) = T_L$ *is onto.*

Proof. Extend linearly T from X onto $\mathrm{span}\{\delta_x : x \in X\}$ and denote this extension by u. We only need to check that $\|u\| = \mathrm{Lip}(T)$. Pick some $a \in \mathrm{span}\{\delta_x : x \in X\}$. Then $\|u(a)\| = f(u(a))$ for some $f \in \mathcal{B}_{X^*}$. However, $f \circ T$ then belongs to $\mathrm{Lip}_0(X)$ and $\mathrm{Lip}(f \circ T) \leq \mathrm{Lip}(T)$. It follows that $\|u(a)\| \leq \|u\| \, \mathrm{Lip}(T)$ which proves the claim. Then we can extend u to $\mathcal{F}(X)$, the closure of $\mathrm{span}\{\delta_x : x \in X\}$.
Let us fix a Lipschitz map $T \in \mathrm{Lip}_0(X, E)$. Let u be the linear map defined on $\mathrm{span}\{\delta_x : x \in M\}$ by $u(\sum_{i=1}^{n} a_i \delta_{x_i}) = \sum_{i=1}^{n} a_i T(x_i) \in E$. We have

$$\begin{aligned}
\left\|u(\sum_{i=1}^{n} a_i\delta_{x_i})\right\|_E &= \left\|\sum_{i=1}^{n} a_i T\left(x_i\right)\right\|_E \\
&= \sup\left|\left\langle\sum_{i=1}^{n} a_i T\left(x_i\right), e^*\right\rangle\right|, \quad e^* \in \mathcal{B}_{E^*} \\
&= \sup\left|\sum_{i=1}^{n} a_i \left\langle T\left(x_i\right), e^*\right\rangle\right|, \quad e^* \in \mathcal{B}_{E^*} \\
&\leq \sup\left|\sum_{i=1}^{n} a_i f\left(x_i\right), \quad e^* \in \mathcal{B}_{E*}\right|, \quad f \in \mathrm{Lip}(T)\mathcal{B}_{X^\#} \\
&\leq \mathrm{Lip}(T)\left\|\sum_{i=1}^{n} a_i\delta_{x_i})\right\|_{\mathcal{F}(X)}.
\end{aligned}$$

Thus $\|u\| \leq \mathrm{Lip}(T)$. Now we want to prove the reverse inequality. Fix $\epsilon > 0$ and consider $x \neq y$ such that $\|T(x) - T(y)\| \geq (\mathrm{Lip}(T) - \epsilon)d(x, y)$. We now define $m_{xy} := \dfrac{(\delta_x - \delta_y)}{d(x,y)}$. Clearly $\|m_{xy}\| = 1$ and $\|T(m_{xy})\| = \dfrac{\|T(x) - T(y)\|}{d(x,y)} \geq \mathrm{Lip}(T) - \epsilon$. We conclude that $\|u\| \geq \mathrm{Lip}(T)$. To finish, we extend u to $\mathcal{F}(X)$ and we denote T_L this unique continuous extension which has the same norm. It remains to show that the linear isometry $\varphi : \mathrm{Lip}_0(X, E) \longrightarrow \mathcal{B}(\mathcal{F}(X), E)$ is onto. Consider $u \in \mathcal{B}(\mathcal{F}(X), E)$. Then, define T on X by $T(x) = u\delta_x$ for every $x \in X$. The map T is clearly Lipschitz and satisfies $\varphi(T) = u$. □

Using this universal property of $\mathcal{F}(X)$), it is immediate to see that $\mathcal{F}(X)^* \equiv \mathrm{Lip}_0(X)$. Indeed, it is enough to consider $X = \mathbb{R}$ in the universal property mentioned above. Moreover, the weak* topology coincides with the topology of pointwise convergence on bounded sets of $\mathrm{Lip}_0(X)$. We also deduce the following variation of the universal property.

Remark 9. *By Theorem 11, the predual of $X^\#$ provided by the Dixmier-Ng theorem coincides with the Lipschitz-free space of X, i.e., $\mathcal{F}(X, \mathrm{d})$ is isometrically isomorphic to Æ(X, d).*

Corollary 3. *Let $(X_1, \mathrm{d}_1), (X_2, \mathrm{d}_2)$ be two pointed metric spaces. Let $T : X_1 \longrightarrow X_2$ be a Lipschitz map such that $T(0) = 0$. Then, there is a unique map $\widehat{T} : \mathcal{F}(X_1) \longrightarrow \mathcal{F}(X_2)$ such that $\widehat{T}\delta_{X_1} = \delta_{X_2}T$, i. e., the following diagram commutes.*

$$\begin{tikzcd}
X_1 \arrow[r, "T"] \arrow[d, "\delta_{X_1}"] & X_2 \arrow[d, "\delta_{X_2}"] \\
\mathcal{F}(X_1, \mathrm{d}_1) \arrow[r, "\widehat{T}"] & \mathcal{F}(X_2, \mathrm{d}_2)
\end{tikzcd}$$

and $\left\|\widehat{T}\right\| = \mathrm{Lip}(T)$.

Remark 10. *If X_0 is a subspace of a metric space X, then $\mathcal{F}(X_0)$ is linearly isometric to a subspace of $\mathcal{F}(X)$. Indeed, denote* $I\mathrm{d} : X_0 \longrightarrow X$ *the identity map. Then the application* $\widehat{\mathrm{Id}}$ *given by Corollary 3 is the desired isometry. In order to prove this last claim, one uses "nonlinear Hahn Banach theorem" Furthermore, we also have the following.*

Remark 11. *Let X be a metric space and let $\widehat{X}$ be its completion. Then, the spaces $\mathcal{F}(X)$ and $\mathcal{F}\left(\widehat{X}\right)$ are linearly isometric. Indeed, the operator*

$$\begin{array}{rccc} T: & \mathrm{Lip}_0(\widehat{X}) & \longrightarrow & \mathrm{Lip}_0(X) \\ & f & \longmapsto & f\diagup X \end{array}$$

is a onto linear isometry which is weak-to-weak* continuous.*

Corollary 4. *Let $(X_1, \mathrm{d}_1), (X_2, \mathrm{d}_2)$ be two pointed metric spaces. If X_1 Lipschitz embeds into X_2, then $\mathcal{F}(X_1, \mathrm{d}_1)$ linearly embeds into $\mathcal{F}(X_2, \mathrm{d}_2)$. Moreover, if X_1 is Lipschitz equivalent to X_2, then $\mathcal{F}(X_1, \mathrm{d}_1)$ is linearly isomorphic to $\mathcal{F}(X_2, \mathrm{d}_2)$.*

Proof. Let $T : X_1 \longrightarrow X_2$ be a Lipschitz embedding map. Then, T is bi-bijective from X_1 into $T(X_2)$. We then consider the bounded linear operators $\widehat{T} : \mathcal{F}(X_1) \longrightarrow \mathcal{F}(T(X_2))$ and $\widehat{T^{-1}} : \mathcal{F}(T(X_2)) \longrightarrow \mathcal{F}(X_1)$ given by Corollary . It is easy to see that $\widehat{T} \circ \widehat{T^{-1}} = \mathrm{Id}_{\mathcal{F}(T(X_2))}$ and $\widehat{T^{-1}} \circ \widehat{T} = \mathrm{Id}_{\mathcal{F}(X_1)}$ so that $\widehat{T}$ is a linear isomorphism from $\mathcal{F}(X_1)$ to $\mathcal{F}(T(X_2))$. Since $\mathcal{F}(T(X_2))$ is isometric to a subspace of $\mathcal{F}(X_2)$ we get that $\mathcal{F}(X_1)$ is isomorphic to a subspace of $\mathcal{F}(X_2)$. The second part of the corollary is clear. □

Example 5. *1. We have $\mathcal{F}(\mathbb{R}) \equiv L^1(\mathbb{R})$.*

Indeed, define

$$\begin{array}{rccc} T: & \mathrm{Lip}_0(\mathbb{R}) & \longrightarrow & L^\infty(\mathbb{R}) \\ & f & \longmapsto & f' \end{array}$$

T is a surjective linear isometry. This implies that $(\mathcal{F}(\mathbb{R}))^* \equiv \left(L^1(\mathbb{R})\right)^*$.

Theorem (**Rademacher 1919-Lebesgue 1900**) Let X be a Banach space of finite dimension and $f : X \longrightarrow \mathbb{R}$ be a Lipschitz function. Then f is a. e. differential. Moreover

Example 6.

$$f(x) - f(0) = \int_0^x f'(t)dt.$$

Lebesgue for $f : \mathbb{R} \longrightarrow \mathbb{R}$ monotone and Rademacher for $\dim(X) < +\infty$. In infinite dimensional spaces there is no Lebesgue measure. If we want to extend Rademacher's theorem to infinite dimensional case, we have to extend the notion of a. e. to such spaces. This problem had been resolved independently by Christensen, Mankiewicz, Aronszajn and Phelps by introducing and used different notions of almost everywhere.

We have

$$\begin{aligned} |f(x)-f(y)| &= \left|\int_y^x f'(t)dt\right| \\ &\leq \int_y^x |f'(t)|\,dt \\ &\leq \sup_{t\in\mathbb{R}} |f'(t)|\,|x-y| \end{aligned}$$

This implies that $\dfrac{|f(x)-f(y)|}{|x-y|} \leq \|f'\|_\infty$ for all $(x,y)\in\widetilde{\mathbb{R}}$ and hence $\mathrm{Lip}_0(f)\leq \|f'\|_\infty$.
In the other hand, we have the inequality $|f'(x)| \leq \sup\limits_{(x,y)\in\widetilde{\mathbb{R}}} \left|\dfrac{f(x)-f(y)}{x-y}\right|$ and hence $\|f'\|_\infty \leq \mathrm{Lip}(f)$.Thus $\|f'\|_\infty = \mathrm{Lip}(f)$ and consequently T is an isometry..
The operator T is surjective. Indeed, Let g be in $L^\infty(\mathbb{R})$. We let $f(x) = \displaystyle\int_0^x g(t)dt$; which is lipschitzian because

$$\begin{aligned} |f(x)-f(y)| &= \left|\int_y^x g(t)dt\right| \\ &\leq \int_y^x |g(t)|\,dt \\ &\leq \sup_{t\in[x,y]} |g(t)|\,|x-y| \\ &\leq \|g\|_\infty\,|x-y| \end{aligned}$$

and this implies that $\mathrm{Lip}(f)\leq \|g\|_\infty$. Consequently, T is a surjective linear isometry.
We prove that the operator

EXAMPLE 7.

$$\begin{array}{rccc} S: & \mathcal{F}(\mathbb{R}) & \longrightarrow & L^1(\mathbb{R}) \\ & \delta_x & \longmapsto & \mathbf{1}_{[0,x]} \end{array}$$

extends to an isometry from $\mathcal{F}(\mathbb{R})$ onto $L^1(\mathbb{R})$. The operator S verifies $S^* = T^{-1}$. Indeed, $S^*: L^\infty(\mathbb{R}) \longrightarrow \mathrm{Lip}_0(\mathbb{R})$ is defined by

$$\begin{aligned} \langle S^*(f), \delta_x\rangle &= \langle f, S(\delta_x)\rangle \\ &= \left\langle f, \mathbf{1}_{[0,x]}\right\rangle \\ &= \int_0^x f(t)dt \\ &= T^{-1}(f)(x)\,. \end{aligned}$$

Let $S: X \longrightarrow Y$ be an operator between Banach spaces such that S^* is a surjective linear isometry, then S is a surjective isometry. Indeed, we have $\langle S(x), y^*\rangle = \langle x, S^*(y^*)\rangle$ and thus

$$\begin{aligned}\|S(x)\| &= \sup_{\|y^*\|} |\langle x, S^*(y^*)\rangle| \\ &\leq \sup_{\|y^*\|} \|x\| \, \|S^*(y^*)\| \\ &\leq \sup_{\|y^*\|} \|x\| \, \|y^*\| \\ &\leq \|x\| .\end{aligned}$$

In the other part, on have

Example 8.

$$\begin{aligned}\left|\left\langle S^* (S^*)^{-1}(x^*), x\right\rangle\right| &= |\langle x^*, x\rangle| \\ &= \left|\left\langle (S^*)^{-1}(x^*), S(x)\right\rangle\right| \\ &\leq \|x^*\| \, \|S(x)\|\end{aligned}$$

and this gives $\|x\| \leq \|S(x)\|$.
The surjectivity. Let (x, y) be in $\mathbb{R}$ $(x \leq y)$ and consider $g \in L_\infty(\mathbb{R})$. We have

Example 9.

$$\begin{aligned}\langle S(\delta_y - \delta_x), g\rangle &= \int_{\mathbb{R}} g(t) \left(\mathbf{1}_{[0,y]} - \mathbf{1}_{[0,x]}\right)(t)\, dt \\ &= \int_0^y g(t)\, dt - \int_0^x g(t)\, dt \\ &= \int_x^y g(t)\, dt \\ &= \int_{\mathbb{R}} g(t)\, \mathbf{1}_{[x,y]}(t)\, dt \\ &= \left\langle \mathbf{1}_{[x,y]}, g\right\rangle .\end{aligned}$$

Then $S(\mathrm{span}\{\delta_x : x \in X\})$ is dense in $L_1(\mathbb{R})$ by Remark 7. This implies that S is a surjective isometry

2. Let $X = \mathbb{N}$. The linear operator

$$\begin{aligned} T: \ \mathcal{F}(\mathbb{N}) &\longrightarrow l_1(\mathbb{R}) \\ \delta_n &\longmapsto \sum_{i=1}^{n} e_i \end{aligned}$$

is an onto isometry.

3. Let $X = [0, 1]$. The linear operator

$$\begin{aligned} S: \ \mathcal{F}([0,1]) &\longrightarrow L^1([0,1]) \\ \delta_x &\longmapsto \mathbf{1}_{[0,x]} \end{aligned}$$

is an onto isometry.

Example 10. *The space* $\mathrm{Lip}_0[0, 1]$ *of Lipschitz functions on* $[0, 1]$ *vanishing at* 0 *with the Lipschitz norm is isometrically isomorphic to the Banach space* $L^\infty[0, 1]$. *The isomorphism is given by the correspondence*

$$\begin{array}{rccl} S: & L^\infty[0,1] & \longrightarrow & \mathrm{Lip}_0[0,1] \\ & f & \longmapsto & T(f) = \displaystyle\int_0^x f(t)dt. \end{array}$$

The inverse mapping $T^{-1} : \mathrm{Lip}_0[0,1] \to L^\infty[0,1]$ *is given by* $T^{-1}(g) = g'$, *a.e..*

REMARK 12 (Godfroy). *We can see* $\mathcal{F}(X, \mathrm{d}_X)$ *as the completion of the set of all measures* μ *of finite support under the norm*

$$\|\mu\| = \sup\left\{\int f d\mu : \mathrm{Lip}\,(f) \leq 1\right\}.$$

The following theorems are due to Lindenstrauss [**Lin64**] when X is a Banach.

THEOREM 13 (Lindenstrauss, weak form). *If* X *is a Banach space then there is a norm one projection* p *from* $\mathrm{Lip}_0(X)$ *onto its subspace* X^*.

PROPOSITION 14. *If* X_0 *is a subset of a metric space* X *containing the base point, then* $Æ(X_0)$ *can be identified naturally and isometrically as a linear subspace of* $Æ(X)$.

PROOF. Consequence of Hahn-Banach Theorem. □

3.5. Adjoint of Lipschitz operators. The aim of this subsection is to show that the Lipschitz adjoint of a Lipschitz mapping T, defined by I. Sawashima, in [**Saw75**, Saw75], corresponds in a canonical way to the adjoint of a linear operator T_L associated to T.

DEFINITION 10. *Consider* X, Y *in* $\mathcal{M}_0$ *and let* $T : X \longrightarrow Y$ *be a Lipschitz map which preserves base point. We define* $T^{\#} : \mathrm{Lip}_0(Y) \longrightarrow \mathrm{Lip}_0(X)$ *by*

$$T^{\#}(g)(x) = (g \circ T)(x) = g(T(x)).$$

The definition make sense by the property of composition maps.

PROPOSITION 15. *Consider* X, Y *in* $\mathcal{M}_0$ *and let* $T : X \longrightarrow Y$ *be a Lipschitz map which preserves base point. Then* $T^{\#}$ *is a bounded linear map and* $\left\|T^{\#}\right\| = \mathrm{Lip}\,(T) = \left\|T^{\#}|_{Y^*}\right\|$ (*if* Y *is a Banach space*).

PROOF. We have

$$\mathrm{Lip}\left(T^{\#}(g)\right) = \mathrm{Lip}\,(g \circ T) \leq \mathrm{Lip}\,(g)\,\mathrm{Lip}\,(T)$$

so $\left\|T^{\#}\right\| \leq \mathrm{Lip}\,(T)$. For the converse inequality, fix $p, q \in Y$. Let $g_0 = d_Y(.,q) - d_Y(\mathrm{e}_Y, q)$, then $\mathrm{Lip}\,(g_0) = 1$. Indeed,

$$\begin{array}{rcl} |g_0(x) - g_0(y)| & = & |d_Y(x,q) - d_Y(y,q)| \\ & \leq & d_X(x,y). \end{array}$$

this implies that $\mathrm{Lip}\,(g_0) \leq 1$. We have also

$$\begin{aligned}
\mathrm{Lip}\,(g_0) &\geq \frac{|g_0\,(p) - g_0\,(q)|}{d_Y\,(p,q)} \\
&\geq \frac{d_Y\,(p,q)}{d_Y\,(p,q)} \\
&\geq 1.
\end{aligned}$$

And hence

$$\begin{aligned}
\left\|T^{\#}\right\| &\geq \mathrm{Lip}\left(T^{\#}\,(g)\right) \\
&\geq \frac{\left|T^{\#}\,(g)\,(x) - T^{\#}\,(g)\,(y)\right|}{d_X\,(x,y)} \\
&\geq \frac{|gT\,(x) - gT\,(y)|}{d_X\,(x,y)} \\
&\geq \frac{|gT\,(x) - gT\,(y)|}{d_Y\,(T\,(x)\,,T\,(y))} \frac{d_Y\,(T\,(x)\,,T\,(y))}{d_X\,(x,y)}.
\end{aligned}$$

Taking the supremum over x and y, we find $\left\|T^{\#}\right\| \geq \|T\|$. □

If $Y = E$ is a Banach space, we shall show that $T^{\#}$ corresponds in a canonical way to the usual adjoint of the linear operator attached to T by Theorem 9 of linearization, i.e., $T^{\#}\,|_{Y^*} = (T_L)^*$.

$$\begin{tikzcd}
\mathrm{Lip}_0(E) \arrow[r, "T^{\#}"] \arrow[d, "p"'] & \mathrm{Lip}_0(X) \\
E^* \arrow[ur, "T_L^*"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
E^* \arrow[r, "\mathrm{Id}"] \arrow[dr, "(T_L)^*"'] & \mathrm{Lip}_0(E) \arrow[r] \arrow[d, "(T_L)^{\#}"] & \mathrm{Lip}_0(X) \\
& \mathrm{Lip}_0(X) \arrow[ur, "\mathrm{Id}"'] &
\end{tikzcd}$$

The restriction of $T^{\#}$ to E^* is called the Lipschitz transpose map of T and is denoted here by T^t. The correspondence

$$T \longleftrightarrow T^t$$

establishes an isomorphism between the vector spaces $\mathrm{Lip}_0(X,E)$ and $\mathcal{L}((E^*,w^*),(X^{\#},w^*))$, where w^* denote the weak*-topology (see [**AJ13**, Theorem 3.1]).

CHAPTER 2

p-summing Lipschitz operators

1. Introduction

The nonlinear version of p-summing operators was introduced by J.-D. Farmer and W.-B. Johnson in [**FJ09**]. We consider now X a pointed metric space and E a Banach space.

Definition 11. *A Lipschitz map $T : X \longrightarrow E$ is called Lipschitz p-summing* $(1 \leq p < \infty)$*, if there is a positive constant C such that for all $\{x_i\}_{1\leq i\leq n}, \{y_i\}_{1\leq i\leq n}$ in X and all $\{a_i\}_{1\leq i\leq n} \subset \mathbb{R}^+$, we have*

$$\sum_{i=1}^{n} a_i \left\| T(x_i) - T(y_i) \right\|^p \leq C^p \sup_{f\in\mathcal{B}_{X^\#}} \sum_{i=1}^{n} a_i |f(x_i) - f(y_i)|^p \tag{1.1}$$

We denote by $\pi_p^L(T)$, the smallest constant C verifying inequality (1.1). The space $\Pi_p^L(X, E)$ of Lipschitz p-summing functions from any metric space into Y is a Banach space under the norm $\pi_p^L(.)$. If T is linear then $\pi_p^L(T) \leq \pi_p(T)$ (in fact we have $\pi_p^L(T) = \pi_p(T)$).

Notice that for any embedding $j : Y \to Z$, we have $\pi_p^L(T) = \pi_p^L(jT)$ and $\pi_p^L(T) = \sup_{X_0\subset X} \left\{\pi_p^L(T/X_0) : X_0 \text{ finite subset of } X\right\}$. Also, the definition stays the same if we restrict to $a_i = 1$, we can found it implicitly in [**FJ09**].

Proposition 16 (Ideal property). *Let X, Z be pointed metric spaces and E, F be Banach spaces. Let $R : Z \longrightarrow X$, $S : E \longrightarrow F$be Lipschitz functions and $T : X \longrightarrow E$ be a Lipschitz p-summing operator. Then STR is Lipschitz p-summing operator and $\pi_p^L(STR) \leq \operatorname{Lip}(S)\pi_p^L(T)\operatorname{Lip}(R)$.*

We have

$$\begin{aligned}
&\sum_{i=1}^{n}\|STR(z_i)-STR(z_i')\|^p\\
\leq\;& \mathrm{Lip}(S)^p\sum_{i=1}^{n}\|TR(z_i)-TR(z_i')\|^p\\
\leq\;& \mathrm{Lip}(S)^p\pi_p^L(T)^p\sup_{f\in\mathcal{B}_{X^\#}}\sum_{i=1}^{n}|f(R(z_i))-f(R(z_i'))|^p\\
\leq\;& \mathrm{Lip}(S)^p\pi_p^L(T)^p\,\mathrm{Lip}(R)^p\sup_{f\in\mathcal{B}_{X^\#}}\sum_{i=1}^{n}|\frac{f\circ R}{\mathrm{Lip}(R)}(z_i)-\frac{f\circ R}{\mathrm{Lip}(R)}(z_i')|^p\\
\leq\;& \mathrm{Lip}(S)^p\pi_p^L(T)^p\,\mathrm{Lip}(R)^p\sup_{g\in\mathcal{B}_{Z^\#}}\sum_{i=1}^{n}|g(z_i)-g(z_i')|^p.
\end{aligned}$$

Remark 13. *Every pointed metric space (X,d) is isometric to a subspace of $\mathcal{C}\left(\mathcal{B}_{X^\#}\right)$.*

Indeed, define

$$i_X : X \longrightarrow \mathcal{C}\left(\mathcal{B}_{X^\#}\right) \text{ by } i(x)(f)=f(x).$$

We have

$$\begin{aligned}
d(i_X(x_1),i_X(x_2)) &= \sup_{f\in\mathcal{B}_{X^\#}}|i_X(x_1)(f)-i_X(x_2)(f)|\\
&= \sup_{f\in\mathcal{B}_{X^\#}}|f(x_1)-f(x_2)|\\
&= \sup_{f\in\mathcal{B}_{X^\#}}\frac{|f(x_1)-f(x_2)|}{d(x_1,x_2)}d(x_1,x_2)\\
&= d(x_1,x_2)
\end{aligned}$$

because $|f(x_1)-f(x_2)|$ is at most $d(x_1,x_2)$ whenever $f\in\mathcal{B}_{X^\#}$ and this upper bound is in fact attained: given any two points $x,x'\in X$, the function $f : X\longrightarrow\mathbb{R}$ given by $f(.)=d(.,x_2)-d(x_2,0)$ is in $\mathsf{Lip}_0(X,\mathbb{R})$, has Lipschitz constant 1 and satisfies $|f(x)-f(x')|=d(x,x')$. This implies that $d(f(x_1),f(x_2))=d(x_1,x_2)$ and hence i_X is an isometry.

Proposition 17. *Let X be a metric space and E,F be Banach spaces. Consider two Lipschitz maps $T : X\longrightarrow E$ and $S : X\longrightarrow F$ such that $\|T(x_1)-T(x_2)\|\leq C\|S(x_1)-S(x_2)\|$ for a positive constant C. Suppose that S is injective. Then, There is $R:\overline{S(X)}\longrightarrow E$ lipschitzian such that $T=R\circ S$ and $\mathrm{Lip}(R)\leq C$.*

Proof. We let $R(z)=TS^{-1}(z)$ We have $R\circ S(x)=TS^{-1}(S(x))=T(x)$ and for all $z_1,z_2\in S(X)$

$$\begin{aligned}
\|R(z_1)-R(z_2)\| &= \left\|TS^{-1}(z_1)-TS^{-1}(z_2)\right\|\\
(x_i=S^{-1}(z_i)) &= \|T(x_1)-T(x_2)\|\\
&\leq C\|S(x_1)-S(x_2)\|\\
&\leq C\|z_1-z_2\|
\end{aligned}$$

We end the proof by extended R by density to $\overline{S(X)}$. □

2. Properties

We give now Pietsch domination-factorization theorem for Lipschitz p-summing operators.

THEOREM 14 ([**FJ09**]). *Let $1 \leq p < \infty$. The following properties are equivalent for a mapping $T : X \longrightarrow E$ and a positive constant C.*

(a) *The mapping T is Lipschitz p-summing and $\pi_p^L(T) \leq C$.*

(b) *There is a probability μ on $\mathcal{B}_{X^\#}$ such that*

$$\|T(x) - T(y)\| \leq C \left(\int_{B_{X^\#}} |f(x) - f(y)|^p \, d\mu(f) \right)^{\frac{1}{p}}.$$

(c) *For any isometric embedding j of Y into a 1-injective space Z, the following diagram commute*

$$\begin{tikzcd}
L_\infty(\mathcal{B}_{X^\#}, \mu) \arrow[rr, "i_p"] & & L_p(\mathcal{B}_{X^\#}, \mu) \arrow[d, "\widetilde{T}"] \\
X \arrow[u, "i"] \arrow[r, "T"] & Y \arrow[r, "j"] & Z
\end{tikzcd}$$

with $\mathrm{Lip}(\widetilde{T}) \leq C$.

(d) *There is a probability μ on $K = \overline{ext(\mathcal{B}_{X^\#})}$ (for the topology of pointwise convergence on X), such that*

$$\|T(x) - T(y)\| \leq C \left(\int_K |f(x) - f(y)|^p \, d\mu(f) \right)^{\frac{1}{p}}. \tag{2.1}$$

PROOF. The property $(a) \Longrightarrow (b)$.
Let $\mathcal{C}$ be the convex cone in $C(\mathcal{B}_{X^\#})$ of the functions of the form

$$\varphi_{a_i,x_i,y_i}(f) = \left\{ \sum_{i=1}^{n} C^p a_i |f(x_i) - f(y_i)|^p - a_i \|T(x_i) - T(y_i)\|^p \right\}$$

where $n \in \mathbb{N}$, $a_i \in \mathbb{R}_+^*$ and $x_i, y_i \in X$.
The set $\mathcal{M}$ is a convex cone. Indeed, let φ_1, φ_2 be in $\mathcal{M}$ and $a \in [0,1]$ such that

$$\varphi_{1((a_{1i}),(x_{1i}),(y_{1i}))}(f) = \sum_{i=1}^{n_1} C^p a_{1i} |f(x_i) - f(y_{1i})|^p - a_{i1} \|T(x_{1i}) - T(y_{1i})\|^p$$

and

$$\varphi_{2((a_{2i}),(x_{2i}),(y_2))}(f) = \sum_{i=1}^{n_2} C^p a_{2i} |f(x_{2i}) - f(y_{2i})|^p - a_{2i} \|T(x_{2i}) - T(y_{2i})\|^p$$

It follows that for $a \in \mathbb{R}^+$

$$\begin{aligned} & a\varphi \\ = & \; a\varphi_{((a_{1i}),(x_{1i}),(y_{1i}))}(f) \\ = & \sum\nolimits_{i=1}^{n_1} C^p a a_{1i} |f(x_i) - f(y_{1i})|^p - a a_{i1} \left\| T(x_{1i}) - T(y_{1i} \right\|^p \\ = & \; \varphi_{((aa_{1i}),(x_{1i}),(y_{1i}))}(f) \end{aligned}$$

and

$$\begin{aligned} & \varphi_1 + \varphi_2 \\ = & \sum\nolimits_{i=1}^{n_1} C^p a_{1i} |f(x_{1i}) - f(y_{1i})|^p - a_{i1} \left\| T(x_{1i}) - T(y_{1i}) \right\|^p + \\ & \sum\nolimits_{i=1}^{n_2} C^p a_{2i} |f(x_{2i}) - f(y_{2i})|^p - a_{2i} \left\| T(x_{2i}) - T(y_{2i}) \right\|^p \\ = & \sum\nolimits_{i=1}^{n} C^p a_i |f(x_i) - f(y_i)|^p - a_i \left\| T(x_i - y_i) \right\|^p . \end{aligned}$$

Finally we have

$$\varphi_1 + \varphi_2 = \sum_{i=1}^{n} C^p a_i |f(x_i) - f(y_i)|^p - a_i \left\| T(x_i) - T(y_i) \right\|^p$$

with $n = n_1 + n_2$,
$a_i = \begin{cases} a_{1i} & \text{if } \; 1 \le i \le n_1, \\ a_{2i} & \text{if } \; n_1 + 1 \le i \le n \end{cases}$, $x_i = \begin{cases} x_{1i} & \text{if } \; 1 \le i \le n_1, \\ x_{2i} & \text{if } \; n_1 + 1 \le i \le n \end{cases}$ and $y_i = \begin{cases} y_{1i} & \text{if } \; 1 \le i \le n_1, \\ y_{2i} & \text{if } \; n_1 + 1 \le i \le n. \end{cases}$

By hypothesis, the convex cone $\mathcal{C}$ is disjoint from the negative cone

$$\mathcal{C}_- = \left\{ \psi \in C(\mathcal{B}_{X^\#}) : \psi(f) < 0, \forall f \in \mathcal{B}_{X^\#} \right\}.$$

which is an open convex subset of $C(\mathcal{B}_{X^\#})$. By Hahn-Banach theorem analytic form "large separation theorem" and Riesz "representation theorem", there is a finite signed Radon-Borel (a signed Radon-Borel measure on the compact is finite) measure $\mu \neq 0$ and a real α such that for all $\varphi \in \mathcal{C}$ and $\psi \in \mathcal{C}_-$, we have

$$\int_{\mathcal{B}_{X^\#}} \psi(f) \, d\mu(f) \le \alpha \le \int_{\mathcal{B}_{X^\#}} \varphi(f) \, d\mu(f).$$

Because $0 \in \mathcal{C}$ and the negative constants are in $\mathcal{C}_-$, than we can take $\alpha = 0$. Also, one has

$$\int_{\mathcal{B}_{X^\#}} \psi(f) \, d\mu(f) \le 0, \quad \forall \psi \in \mathcal{C}_- \Longleftrightarrow \mu \ge 0.$$

We can put $\mu(\mathcal{B}_{X^\#}) = 1$, if is not the case we divide by $\lambda(\mathcal{B}_{X^\#})$. In particular we take $\varphi(f) = C^p |f(x) - f(y)|^p - \|T(x) - T(y)\|^p$, we have

$$\begin{aligned} \int_{B_{X^\#}} \varphi(f) \, d\mu(f) &= \int_{\mathcal{B}_{X^\#}} C^p |f(x) - f(y)|^p - \|T(x) - T(y)\|^p \, d\mu(f) \\ &\ge 0 \end{aligned}$$

this implies

$$\|T(x) - T(y)\| \leq C \left(\int_{\mathcal{B}_{X^{\#}}} |f(x) - f(y)|^p \, d\mu\,(f) \right)^{\frac{1}{p}}.$$

The property $(b) \Longrightarrow (c)$.
Let $i : X \longrightarrow L_\infty\left(\mathcal{B}_{X^{\#}}, \mu\right)$ be the natural isometric embedding which is the the formal identity from $C(\mathcal{B}_{X^{\#}})$ into $L_\infty\left(\mathcal{B}_{X^{\#}}, \mu\right)$ composed with i_X. Then (b) says the the Lipschitz norm of $\widetilde{T}$ restricted to $i_p\left(i\left(X\right)\right)$ is bounded by C, which is (c).
The property $(c) \Longrightarrow (a)$.
By the above, we have

$$\begin{aligned} \pi_p^L\left(T\right) = \pi_p^L\left(jT\right) &\leq \operatorname{Lip}\left(\widetilde{T}\right) \pi_p^L\left(i_p\right) \operatorname{Lip}\left(i\right) \\ &\leq \operatorname{Lip}\left(\widetilde{T}\right) \pi_p\left(i_p\right) \operatorname{Lip}\left(i\right) \\ &\leq \operatorname{Lip}\left(\widetilde{T}\right) \\ &\leq C. \end{aligned}$$

The property $(a) \Longrightarrow (d)$ is the same as the proof of $(a) \Longrightarrow (b)$ since the supremum in the right part of inequality (2.1) is taken on K. This ends the proof. □

As an immediate consequence, we have

Proposition 18. *Let $1 \leq p < q < \infty$. If $T : X \longrightarrow Y$ is Lipschitz p-summing then, T is is Lipschitz q-summing and $\pi_q^L\left(T\right) \leq \pi_p^L\left(T\right)$.*

3. Nonlinear version of Grothendiek's theorem

We start by recalling the linear case of Grothendieck's theorem (G.T. in short). For more informations, we can consult [**?**]. We start by the little G.T. in the linear case which goes back to Grothendieck.

Theorem 15. *Let K be a compact set and let H be a Hilbert space.*
(a) Any bounded linear operator $u : H \longrightarrow L_1$ satisfies

$$\left\| \left(\sum_{i=1}^{n} \left|u\left(x_i\right)\right|^2 \right)^{\frac{1}{2}} \right\|_{L_1} \leq \sqrt{\frac{\pi}{2}} \, \|u\| \left(\sum_{i=1}^{n} \|x_i\|^2 \right)^{\frac{1}{2}}, \quad \text{for any } \left(x_i\right) \subset H.$$

(b) Any bounded linear operator $v : \mathcal{C}\left(K\right) \longrightarrow H$ (or any $v : L_\infty \longrightarrow H$) is 2summing and satisfies $\pi_2\left(v\right) \leq \sqrt{\frac{\pi}{2}} \, \|v\|$.

Let now the dual form. It appeared in [**GL75**]

Theorem 16. *Let H be a Hilbert space. Then any bounded linear operator $w : L_1 \longrightarrow H$ is 2-summing and satisfies $\pi_2\left(w\right) \leq \sqrt{\frac{\pi}{2}} \, \|w\|$.*

The following theorem known as Grotendieck's theorem is due to Lindenstrauss Pełczyński [**?**].

Theorem 17. *Let H be a Hilbert space. Then any bounded linear operator $w : L_1 \longrightarrow H$ is 2-summing and satisfies $\pi_2(w) \leq K \|w\|$ for some absolute constant K. The best constant K is noted by K_G, the Grothendiek constant for the real case and $K_G^{\mathbb{C}}$ for the complex case.*

We now give the nonlinear version of Grothendiek's theorem.

Theorem 18 ([**FJ09**], [**CZ11**] and [**Saa15**]). *Let X be pointed metric space such that X embeds isometrically into an R-tree. Then for any Hilbert space H, we have*

$$\pi_1^L(X, H) = \mathrm{Lip}_0(X, H)$$

and

$$\pi_1^L(T) \leq K_G \mathrm{Lip}(T) \textit{ for every } T \textit{ in } \mathrm{Lip}_0(X, H).$$

Proof. Consider the diagram as in Theorem 9

$$\begin{tikzcd}
\text{Æ}(X) \arrow[d, "i_X"'] \arrow[dr, "T_L"] & \\
X \arrow[r, "T"] & H
\end{tikzcd}$$

where $\text{Æ}(X)$ is isometrically isomorphic to $L_1(\mathbb{R})$. We have T^L is 1-summing and $\pi_1(T_L) \leq K_G \|T_L\| \leq K_G \mathrm{Lip}(T)$.

Other proof. In the category of metric space with Lipschitz maps as isomorphisms, weighted trees play a role analogous to that of L_1 in the linear theory. In particular, every finite weighted tree has the lifting property, which is to say that if X is a finite weighted tree, $T : X \to Y$ is a Lipschitz mapping from X into a metric space Y , and $q : Z \to Y$ is a 1-Lipschitz quotient mapping, then for each $\epsilon > 0$ there is a mapping $S : X \to Z$ so that $\mathrm{Lip}(S) \leq \mathrm{Lip}(T) + \epsilon$ and $T = qS$.

$$\begin{tikzcd}
 & Z \arrow[d, "q"] \\
X \arrow[ur, "S"] \arrow[r, "T"] & Y
\end{tikzcd}$$

Letting Y be a Hilbert space and Z an L_1 space, we can deduce from Grothendieck's theorem and the ideal property of π_1^L that if every finite subset of X is contained in a finite subset of X that is a weighted tree (in particular, if X is a tree or a metric tree), then $\pi_1^L(T) \leq K_G \mathrm{Lip}(T)$, where K_G is Grothendieck's constant. Here we use the obvious fact that $\pi_p(T : X \to Y)$ is the supremum of $\pi_p(T_{|K})$ as K ranges over finite subsets of X. $\square$

CHAPTER 3

Other notions of summability

1. Lipschitz $\tau(p)$-summing operators

The following definition was studied by X. Mujica in [**Muj08**] for multilinear operators, which generalizes absolutely τ-summing linear operators introduced by A. Pietsch in [**Pie80**].

Definition 12 ([**MT17**]). *Let T be in $\mathrm{Lip}_0(X,E)$ and consider $1 \leq q \leq p < \infty$. We say that T is Lipschitz $\tau(p,q)$-summing if there is a positive constant C such that, for all $n \in \mathbb{N}; (x_i), (x'_i) \subset X; (a^*_i) \subset E^*$ and $(\lambda_i)_{1 \leq i \leq n} \subset \mathbb{R}_+$, we have*

$$\begin{aligned} &\left(\sum_{i=1}^{n} \lambda_i \left| \langle T(x_i) - T(x'_i), a^*_i \rangle \right|^p \right)^{\frac{1}{p}} \\ \leq\ & C \sup_{\substack{\|f\| \leq 1 \\ \|a\| \leq 1}} \left(\sum_{i=1}^{n} \lambda_i \left| (f(x_i) - f(x'_i)) \langle a^*_i, a \rangle \right|^q \right)^{\frac{1}{q}} \end{aligned} \tag{1.1}$$

where $f \in X^{\#}$ and $a \in E$. We will denote this class of mappings by $\Pi^L_{\tau(p,q)}(X,E)$ and we equip it with the norm $\pi^L_{\tau(p,q)}(T) = \inf C$, for the constants that appear in the above expression, for which it becomes a Banach space. When $p = q$, we write $\Pi^L_{\tau(p)}$ and $\pi^L_{\tau(p)}$ instead of $\Pi^L_{\tau(p,p)}$ and $\pi^L_{\tau(p,p)}$ respectively and we say that T is Lipschitz $\tau(p)$-summing. If $p = q = 1$, we simply write Π^L_{τ} and π^L_{τ} and we say that T is Lipschitz τ-summing. Like the linear case, if $1 \leq s \leq r \leq q \leq p$, then $\Pi^L_{\tau(q,r)} \subset \Pi^L_{\tau(p,s)}$ and $\pi^L_{\tau(p,s)}(T) \leq \pi^L_{\tau(q,r)}(T)$ for all T in $\Pi^L_{\tau(q,r)}$. Moreover, it follows that

$$\Pi^L_{\tau(q,r)} \subset \Pi^L_{\tau(p,r)} \text{ and } \pi^L_{\tau(p,r)}(T) \leq \pi^L_{\tau(q,r)}(T) \text{ for all } T \text{ in } \Pi^L_{\tau(q,r)}$$

and

$$\Pi^L_{\tau(q,r)} \subset \Pi^L_{\tau(q,s)} \text{ and } \pi^L_{\tau(q,s)}(T) \leq \pi^L_{\tau(q,r)}(T) \text{ for all } T \text{ in } \Pi^L_{\tau(q,r)}.$$

Remark 14. 1- *The definition is the same if we restrict to $\lambda_i = 1$ (by the same argument cited implicitly in* [**FJ09**]).

2- *By Goldstine's theorem, we can replace a by $a^{**} \in E^{**}$ in the inequality (1.1).*

Remark 15. - *If T is linear then T is $\tau(p)$-summing implies that T is Lipschitz $\tau(p)$-summing and $\pi^L_{\tau(p)}(T) \leq \pi_{\tau(p)}(T)$. We do not know if* *the converse is true. Because there is no factorization theorem and $\mathcal{B}_{X^{\#}}$ is difficult to handle. Is it a good generalization?*

LEMMA 1. *Let $1 \leq p < \infty$. For $n \in \mathbb{N}$, $(x_i)_{1\leq i\leq n}, (x'_i)_{1\leq i\leq n} \subset X$, $(a_i^*)_{1\leq i\leq n} \subset E^*$ and $(\lambda_i)_{1\leq i\leq n} \in \mathbb{R}_+$; let $v : l_{p^*}^n \longrightarrow X \boxtimes_\varepsilon E^*$ be a linear operator such that $v(e_i) = \delta_{(x_i,x'_i)} \boxtimes \lambda_i^{\frac{1}{p}} a_i^*$; where (e_i) denotes the unit vector basis of $l_{p^*}^n$ and $\boxtimes$ denotes the Lipschitz tensor product as introduced in* [**CCJV15**]. *We have*

$$\|v\| = \sup_{\substack{\|f\|_{X^{\#}}=1 \\ \|a\|_E=1}} \left(\sum_{i=1}^n \lambda_i \left| \left(f(x_i) - f(x'_i)\right) \langle a_i^*, a\rangle \right|^p \right)^{\frac{1}{p}}.$$

PROOF. We have

$$\begin{aligned}
\|v\| &= \sup_{\|\alpha\|_{l_{p^*}^n}=1} \|v(\alpha)\|_{X\boxtimes_\varepsilon E^*} \\
&= \sup_{\|\alpha\|_{l_{p^*}^n}=1} \left\| \sum_{i=1}^n \alpha_i v(e_i) \right\|_{X\boxtimes_\varepsilon E^*} \quad (\alpha = \sum_{i=1}^n \alpha_i e_i) \\
&= \sup_{\|\alpha\|_{l_{p^*}^n}=1} \left\| \sum_{i=1}^n \alpha_i \delta_{(x_i,x'_i)} \boxtimes \lambda_i^{\frac{1}{p}} a_i^* \right\|_{X\boxtimes_\varepsilon E^*} \\
&= \sup_{\|\alpha\|_{l_p^n}=1} \sup_{\substack{\|f\|_{X^{\#}}=1 \\ \|a\|_E=1}} \left(\sum_{i=1}^n \alpha_i \lambda_i^{\frac{1}{p}} \left| (f(x_i) - f(x'_i)) \langle a_i^*, a\rangle \right| \right) \\
&= \sup_{\substack{\|f\|_{X^{\#}}=1 \\ \|a\|_E=1}} \left(\sum_{i=1}^n \lambda_i \left| (f(x_i) - f(x'_i)) \langle a_i^*, a\rangle \right|^p \right)^{\frac{1}{p}}.
\end{aligned}$$

This proves the Lemma. □

PROPOSITION 19. *Let T be in $\mathrm{Lip}_0(X,E)$. The operator T is Lipschitz $\tau(p)$-summing if, and only if, for all $n \in \mathbb{N}$, $(x_i)_{1\leq i\leq n}, (x'_i)_{1\leq i\leq n} \subset X$, $(a_i^*)_{1\leq i\leq n} \subset E^*$, $(\lambda_i)_{1\leq i\leq n} \subset \mathbb{R}_+$ and all linear operator $v : l_{p^*}^n \longrightarrow X \boxtimes_\varepsilon E^*$ such that $v(e_i) = \delta_{(x_i,x'_i)} \boxtimes \lambda_i^{\frac{1}{p}} a_i^*$, we have*

$$\left(\sum_{i=1}^n \lambda_i \left| \left\langle T(x_i) - T(x'_i), a_i^* \right\rangle \right|^p \right)^{\frac{1}{p}} \leq C \|v\|. \tag{1.2}$$

We now give the left ideal property in "Pietsch's sense".

PROPOSITION 20. *Consider T in $Lip_0(Y,E)$ and R in $Lip_0(X,Y)$. If T is Lipschitz $\tau(p)$-summing operator, then $T \circ R$ is Lipschitz $\tau(p)$-summing and $\pi_{\tau(p)}^L(T \circ R) \leq \pi_{\tau(p)}^L(T)\mathrm{Lip}(R)$.*

PROOF. Let $n \in \mathbb{N}$, $(x_i)_{1\leq i\leq n}, (x_i')_{1\leq i\leq n} \subset X$, $(a_i^*)_{1\leq i\leq n} \subset E^*$and $(\lambda_i)_{1\leq i\leq n} \subset \mathbb{R}_+$. It suffices by inequality (1.2) to show that

$$\left(\sum_{i=1}^{n} \lambda_i \left|\left\langle T\circ R(x_i) - T\circ R\left(x_i'\right), a_i^*\right\rangle\right|^p\right)^{\frac{1}{p}} \leq \pi_{\tau(p)}^L(T)\mathrm{Lip}(R)\left\|w\right\|$$

where $w : l_{p^*}^n \longrightarrow X \boxtimes_\varepsilon E^*$ such that $w(e_i) = \delta_{\left(x_i,x_i'\right)} \boxtimes \lambda_i^{\frac{1}{p}} a_i^*$.
Consider the following commutative diagram

$$\begin{tikzcd} l_{p^*}^n \arrow[r, "v"] \arrow[d, "w"'] & Y \boxtimes_\varepsilon E^* \\ X \boxtimes_\varepsilon E^* \arrow[ur, "R \boxtimes id_{E^*}"'] & \end{tikzcd}$$

where

$$v(e_i) = \delta_{\left(R(x_i),R\left(x_i'\right)\right)} \boxtimes \lambda_i^{\frac{1}{p}} a_i^*$$

and

$$R \boxtimes id_{E^*}\left(\delta_{(x_i,x_i')} \boxtimes \lambda_i^{\frac{1}{p}} a_i^*\right) = \delta_{\left(R(x_i),R\left(x_i'\right)\right)} \boxtimes \lambda_i^{\frac{1}{p}} a_i^*.$$

The Lipschitz injective norm ε is uniform by [**CCJV15**, Theorem 7.1] and by [**CCJV15**, Proposition 4.2], we have

$$\begin{aligned} \left(\sum_{i=1}^{n} \lambda_i \left|\left\langle T\circ R(x_i) - T\circ R(x_i'), a_i^*\right\rangle\right|^p\right)^{\frac{1}{p}} &\leq \pi_{\tau(p)}^L(T)\left\|v\right\| \\ &\leq \pi_{\tau(p)}^L(T)\left\|w\right\|\left\|R \boxtimes id_{E^*}\right\| \\ &\leq \pi_{\tau(p)}^L(T)\mathrm{Lip}(R)\left\|w\right\|. \end{aligned}$$

This implies by inequality (1.2) that $T \circ R$ is Lipschitz $\tau(p)$-summing and $\pi_{\tau(p)}^L(T\circ R) \leq \pi_{\tau(p)}^L(T)\mathrm{Lip}(R)$ and this ends the proof. □

PROPOSITION 21. *Consider T in $Lip_0(Y,E)$ and S in $Lip_0(E,F)$. If T is Lipschitz $\tau(p)$-summing operator, then $S\circ T$ is Lipschitz $\tau(p)$-summing and $\pi_{\tau(p)}^L(S\circ T) \leq \mathrm{Lip}(S)\,\pi_{\tau(p)}^L(T)$.*

PROOF. Let $(y_i)_{1\leq i\leq n}, (y_i')_{1\leq i\leq n} \subset Y$, $(b_i^*)_{1\leq i\leq n} \subset F^*$and $(\lambda_i)_{1\leq i\leq n} \subset \mathbb{R}_+$, we have

$$\begin{aligned} & \left(\sum_{i=1}^{n} \lambda_i \left|\left\langle S\circ T(y_i) - S\circ T(y_i'), b_i^*\right\rangle\right|^p\right)^{\frac{1}{p}} \\ = & \left(\sum_{i=1}^{n} \lambda_i \left|\left\langle T(y_i), S^{\#}(b_i^*)\right\rangle - \left\langle T(y_i'), S^{\#}(b_i^*)\right\rangle\right|^p\right)^{\frac{1}{p}} \\ = & \left(\sum_{i=1}^{n} \lambda_i \left|\left\langle T(y_i), S^t(b_i^*)\right\rangle - \left\langle T(y_i'), S^t(b_i^*)\right\rangle\right|^p\right)^{\frac{1}{p}} \end{aligned}$$

(S^t is the transposed of the linear operator attached to S)

$$\begin{aligned}
&= \left(\sum_{i=1}^{n} \lambda_i \left|\left\langle T\left(y_i\right) - T\left(y_i'\right), S^t\left(b_i^*\right)\right\rangle\right|^p\right)^{\frac{1}{p}} \\
&\leq \pi_{\tau(p)}^L(T) \sup_{\substack{\|f\|_{Y^\#}\leq 1 \\ \|a\|_E=1}} \left(\sum_{i=1}^{n} \lambda_i \left|\left(f(y_i) - f(y_i')\right)\left\langle S^t\left(b_i^*\right), a\right\rangle\right|^p\right)^{\frac{1}{p}} \\
&\leq \pi_{\tau(p)}^L(T)\left\|S^t\right\| \sup_{\substack{\|f\|_{Y^\#}\leq 1 \\ \|a\|_E=1}} \left(\sum_{i=1}^{n} \lambda_i \left|\left(f(y_i) - f(y_i')\right)\left\langle \frac{S^t\left(b_i^*\right)}{\|S^t\|}, a\right\rangle\right|^p\right)^{\frac{1}{p}} \\
&\leq \pi_{\tau(p)}^L(T)\left\|S^t\right\| \sup_{\substack{\|f\|_{Y^\#}\leq 1 \\ \|a\|_E=1}} \left(\sum_{i=1}^{n} \lambda_i \left|\left(f(y_i) - f(y_i')\right)\left\langle b_i^*, \frac{S^L(a)}{\|S^t\|}\right\rangle\right|^p\right)^{\frac{1}{p}} \\
&\leq \operatorname{Lip}(S)\pi_{\tau(p)}^L(T) \sup_{\substack{\|f\|_{B_{Y^\#}}\leq 1 \\ \|b\|_F=1}} \left(\sum_{i=1}^{n} \lambda_i \left|\left(f(y_i) - f(y_i')\right)\left\langle b_i^*, b\right\rangle\right|^p\right)^{\frac{1}{p}}.
\end{aligned}$$

Therefore, $S \circ T$ is Lipschitz $\tau(p)$-summing operator and $\pi_{\tau(p)}^L(S \circ T) \leq \pi_{\tau(p)}^L(T)\operatorname{Lip}(S)$. ☐

We will present the following characterization (*Pietsch's domination theorem*) concerning this class of Lipschitz operators. For the proof, we use the same idea as used for example in [**AMS09**] and [**Muj08**, Theorem 3.6]. Before this, we first announce the Ky Fan's lemma. The proof can be consulted in [**DJT95**, p. 190].

LEMMA 2. *Let K be a Hausdorff topological vector space and let C be a compact convex subset of K. Let $\mathcal{M}$ be a set of functions on C with values in $(-\infty, \infty]$ having the following properties.*

(*a*) *each $f \in \mathcal{M}$ is convex and lower semicontinuous;*

(*b*) *if $g \in conv(\mathcal{M})$, there is an $f \in \mathcal{M}$ with $g(x) \leq f(x), \forall x \in C$;*

(*c*) *there is an $r \in \mathbb{R}$ such that each $f \in \mathcal{M}$ has a value $\leq r$.*

Then there is an $x_0 \in C$ such that $f(x_0) \leq r$ for all $f \in \mathcal{M}$.

THEOREM 19. *Consider $T \in \operatorname{Lip}_0(X, E)$ and C a positive constant.*

(1) *The operator T is Lipschitz $\tau(p)$-summing and $\pi_{\tau(p)}^L(T) \leq C$.*

(2) *There exist Radon probability measures μ_1 on $\mathcal{B}_{X^\#}$ and μ_2 on $\mathcal{B}_{E^{**}}$, such that for all x, x' in X and a^* in E^*, we have*

$$\begin{aligned}
& \left|\left\langle T(x) - T(x'), a^*\right\rangle\right| \\
\leq\ & C\left(\int_{\mathcal{B}_{X^\#}} \int_{\mathcal{B}_{E^{**}}} \left|\left(f(x) - f(x')\left\langle a^*, a^{**}\right\rangle\right|^p d\mu_1(f) d\mu_2(a^{**})\right)^{\frac{1}{p}}\right).
\end{aligned} \tag{1.3}$$

Moreover, in this case

$$\pi^L_{\tau(p)}(T) = \inf\left\{C > 0 : \text{ for all } C \text{ verifying the inequality } (1.3)\right\}.$$

Proof. We are interested only to the first affirmation because using inequality (1.3), one easily shows that T is Lipschitz $\tau(p)$-summing and $\pi^L_{\tau(p)}(T) \leq C$. Consider the sets $\mathcal{P}(\mathcal{B}_{X^{\#}})$ and $\mathcal{P}(\mathcal{B}_{E^{**}})$ of probability measures in $\mathcal{C}(\mathcal{B}_{X^{\#}})^*$ and $\mathcal{C}(\mathcal{B}_{E^{**}})^*$ respectively endowed with their weak*-topologies. These sets are compact and convex. We are going now to apply Ky Fan's Lemma with $K = \mathcal{C}(\mathcal{B}_{X^{\#}})^* \times \mathcal{C}(\mathcal{B}_{E^{**}})^*$ and $\mathsf{C}= \mathcal{P}(\mathcal{B}_{X^{\#}}) \times \mathcal{P}(B_{E^{**}})$ which is convex and compact.

Let $\mathcal{M}$ be the set of all functions φ from C with values in $\mathbb{R}$ of the form

$$\begin{aligned} & \varphi_{\left((x_i),(x_i'),(a_i^*),(\lambda_i)\right)}(\mu_1,\mu_2) \\ = \; & \sum_{i=1}^{n} \lambda_i \left|\langle T(x_i) - T(x_i'), a_i^*\rangle\right|^p - C \int_{B_{X^{\#}}} \int_{B_{E^{**}}} \\ & \lambda_i \left|(f(x_i) - f(x_i'))\,\langle a_i^*, a^{**}\rangle\right|^p d\mu_1(f)\, d\mu_2(a^{**}) \end{aligned}$$

where $(x_i)_{1\leq i\leq n}, (x_i')_{1\leq i\leq n} \subset X$; $(a_i^*)_{1\leq i\leq n} \subset E^*$ and $(\lambda_i)_{1\leq i\leq n} \subset \mathbb{R}_+$. These functions are continuous and convex. The set $\mathcal{M}$ is a convex cone. We now apply Key Fan's Lemma (the conditions (a) and (b) are satisfied). For the condition (c), since $\mathcal{B}_{X^{\#}}$ is a compact Hausdorff space in the topology of pointwise convergence on X and $\mathcal{B}_{E^{**}}$ are weak * compact and "norming" sets, using the fact that X is isometrically embedding into $\mathcal{B}_{X^{\#}}$ and by the classical Goldstine's theorem there exist for $\varphi \in \mathcal{M}$ two elements, f_0 in $\mathcal{B}_{X^{\#}}$ and a_0^{**} in $\mathcal{B}_{E^{**}}$ such that

$$\begin{aligned} & \sup_{\substack{\|a^{**}\|_{E^{**}}=1 \\ \|f\|_{X^{\#}}=1}} \left\| \left(\lambda_i^{\frac{1}{p}} \left(f(x_i) - f(x_i')\right) \langle a_i^*, a^{**}\rangle \right) \right\|^p_{l_p^n} \\ = \; & \sum_{i=1}^{n} \lambda_i \left|f_0(x_i) - f_0(x_i')\,\langle a_i^*, a_0^{**}\rangle\right|^p . \end{aligned}$$

If δ_{f_0} and $\delta_{a_0^{**}}$ denote the Dirac's measures supported by f_0 and a_0^{**} respectively, we have

$$\begin{aligned} & \varphi_{\left((x_i),(x_i'),(a_i^*),(\lambda_i)\right)}(\delta_{f_0}, \delta_{a_0^{**}}) = \\ & \sum_{i=1}^{n} \lambda_i \left|\langle T(x_i) - T(x_i'), a_i^*\rangle\right|^p - C^p \sum_{i=1}^{n} \lambda_i \left|f_0(x_i) - f_0(x_i')\,\langle a_i^*, a_0^{**}\rangle\right|^p \leq \\ & 0. \end{aligned}$$

Hypothesis (1) yields

$$\sup\left\{\varphi_{\left((x_i),(x_i'),(a_i^*),(\lambda_i)\right)}(\mu_1,\mu_2) : (\mu_1,\mu_2) \in K\right\} \leq 0.$$

By the conclusion of Key Fan's Lemma, there is $\mu = (\mu_1, \mu_2) \in \mathrm{C}$ such that $\mu(\varphi) \leq 0$ for all φ in $\mathcal{M}$. If φ is generated by the simple elements $x, x' \in X$, $a^* \in E^*$ and $\lambda = 1$, we find

$$\begin{aligned} & \varphi_{(x,x',a^*,1)}(\mu_1,\mu_2) \\ = \quad & |\langle T(x)-T(x'),a^*\rangle|^p - C^p \\ & \int_{\mathcal{B}_{X^\#}}\int_{\mathcal{B}_{E^{**}}} |(f(x)-f(x')\langle a^*,a^{**}\rangle)|^p \, d\mu_1(f)\, d\mu_2(a^{**}) \leq 0. \end{aligned}$$

It follows that

$$\begin{aligned} & |\langle T(x)-T(x'),a^*\rangle| \\ \leq \quad & C\left(\int_{\mathcal{B}_{X^\#}}\int_{\mathcal{B}_{E^{**}}} |(f(x)-f(x')\langle a^*,a^{**}\rangle)|^p \, d\mu_1(f)\, d\mu_2(a^{**})\right)^{\frac{1}{p}} \end{aligned}$$

and this completes the proof. □

As corollary, we get.

Corollary 5. $\Pi^L_{\tau(p)} \subseteq \Pi^L_{\tau(q)}$, *when* $1 \leq p \leq q < \infty$ *and* $\Pi^L_{\tau(p)} \subseteq \Pi^L_p$, *for all* $1 \leq p < \infty$.

2. Lipschitz strongly p-summing operators

The following notion was introduced independently by [**Saa15**] and [**YAR16**]. For our convenience, we will adopt the notation of [**YAR16**].

Definition 13. *A Lipschitz map* $T : X \to E$ *is Lipschitz strongly* p*-summing* $(1 < p \leq \infty)$ *if there is a constant* $C > 0$*, such that for all* $n \in \mathbb{N}$, $(x_i)_{1\leq i\leq n}$, $(x'_i)_{1\leq i\leq n}$ *in* X, $(a^*_i)_{1\leq i\leq n}$ *in* E^* *and* $(\lambda_i)_{1\leq i\leq n}$ *in* $\mathbb{R}_+$, *we have*

$$\sum_{i=1}^{n}\lambda_i\left|\left\langle T(x_i)-T(x'_i),a^*_i\right\rangle\right| \leq C\left(\sum_{i=1}^{n}\lambda_i d_X(x_i,x'_i)^p\right)^{\frac{1}{p}}\omega_{p^*}((a^*_i)_i)\,. \tag{2.1}$$

We denote by $\mathcal{D}^L_{st,p}(X,E)$ the class of all Lipschitz strongly p-summing operators from X into E and $d^L_{st,p}(T)$ the smallest C such that inequality (2.1) holds. This generalizes the definition introduced by [**Coh73**] in the linear case. If T is linear, then in the absence of $\mathcal{B}_{X^\#}$ we have $\mathcal{D}^L_{st,p}(X,E) = \mathcal{D}_p(X,E)$.

Let $T \in \mathrm{Lip}_0(X;E)$ and $v : l^n_p \to E^*$ be a bounded linear operator. The Lipschitz operator is a strongly Lipschitz p-summing if, and only if,

$$\sum_{i=1}^{n}\lambda_i\left|\left\langle T(x_i)-T(x'_i),v(e_i)\right\rangle\right| \leq C\left(\sum_{i=1}^{n}\lambda_i d_X(x_i,x'_i)^p\right)^{\frac{1}{p}}\|v\| \tag{2.2}$$

Remark 16. *Let* u *be a bounded linear operator from* E *into* F *and* $1 \leq p \leq \infty$*. Then* $d_p(u) = d^L_{st,p}(u)$ *because* $\mathcal{B}_{X^\#}$ *is not involving.*

Now, we give the domination theorem of the strongly Lipschitz p-summing (see [**Saa15**] and [**YAR16**]).

Theorem 20. *A Lipschitz operator T from X into E is Lipschitz strongly p-summing $(1 < p < \infty)$ if, and only if, there exist a positive constant C and Radon probability measure μ on $B_{E^{**}}$ such that for all $x, x' \in X$, we have*

$$\left|\left\langle T(x) - T(x'), a^* \right\rangle\right| \leq C d_X\left(x, x'\right) \left(\int\limits_{\mathcal{B}_{E^{**}}} \left|a^*\left(a^{**}\right)\right|^{p^*} d\mu\left(a^{**}\right) \right)^{\frac{1}{p^*}} . \tag{2.3}$$

Moreover, in this case

$$d^L_{st,p}(T) = \inf\left\{C > 0 : \textit{ for all } C \textit{ verifying the inequality (2.3) } \right\}.$$

Proposition 22. *The following properties are equivalent.*

(1) *The mapping T belongs to $\mathcal{D}^L_{st,p}(X, E)$.*

(2) *The linear operator T_L belongs to $D_p(\text{Æ}(X), E)$.*

Even more, $\mathcal{D}^L_{st,p}(X, E) = D_p(\text{Æ}(X), E)$ holds isometrically.

Proof. See [**Saa15**, Proposition 3.1.]. □

3. Cohen Lipschitz p-nuclear operators

We introduce the following generalization to Lipschitz operators of the class of Cohen p-nuclear operators studied in [**Coh73**]. It is a particular case from that defined by J. A. Chàvez-Domènguez in [**Cha11**] which called the Lipschitz (r, p, q)-summing operators if we take $(r, p, q) = (1, p, p^*)$ and $k_i = 1$ for all i. The notion of p-nuclear operators was introduced in [**PP69**] by A. Person and A. Pietsch. Initially the definition of nuclear operators for Banach spaces, was given by Grothendieck in [**?**]. J. S. Cohen has initiated another concept of p-nuclear operators in [**Coh73**] which is not the same as the precedent notion and was generalized to (p, q)-nuclear operators $(1 \leq q \leq \infty)$ by H. Apiola in [**Api76**]. In [**CZ12**], D. Chen and B. Zheng has generalized this notion to Lipschitz operators. For distinguish these two notions, we say Cohen p-nuclear operators for that investigated by J. S. Cohen and we try to generalize this notion to Lipschitz operators.

Definition 14. *A Lipschitz operator $T : X \longrightarrow E$ is Cohen Lipschitz p-nuclear $(1 < p < \infty)$, if there is a positive constant C such that for any n in $\mathbb{N}$; $(x_i)_{1 \leq i \leq n}, (x'_i)_{1 \leq i \leq n}$ in X; $(a^*_i)_{1 \leq i \leq n}$ in E^* and $(\lambda_i)_{1 \leq i \leq n}$ in $\mathbb{R}_+$, we have*

$$\begin{aligned} &\left| \sum_{i=1}^{n} \lambda_i \left\langle T(x_i) - T(x'_i), a^*_i \right\rangle \right| \\ \leq \quad & C \sup_{f \in \mathcal{B}_{X^\#}} \left(\sum_{i=1}^{n} \lambda_i \left| f(x_i) - f(x'_i) \right|^p \right)^{\frac{1}{p}} \sup_{\|a\|_E \leq 1} \left(\sum_{i=1}^{n} \left| \langle a, a^*_i \rangle \right|^{p^*} \right)^{\frac{1}{p^*}} . \end{aligned} \tag{3.1}$$

The smallest constant C which is noted by $\eta_p^L(T)$, such that the above inequality (3.1) holds, is called the Cohen Lipschitz p-nuclear norm on the space $\mathcal{N}_p^L(X,E)$ of all Cohen Lipschitz p-nuclear operators from X into E which is a Banach space. For $p=1$ and $p=\infty$ we have like the linear case $\mathcal{N}_1^L(X,E)=\Pi_1^L(X,E)$ and $\mathcal{N}_\infty^L(X,E)=\mathcal{D}_{st,\infty}^L(X,E)$ (see below). The definition remains the same if we restrict to $\lambda_i=1$, like that in [**FJ09**]. We use this definition with the λ_i only in the proof of " *Pietsch's domination theorem* ".

We know (see [**DJT95**]) that $l_p(E)\equiv l_p^\omega(E)$ (the symbol $\equiv$ indicates that two Banach spaces are isometrically isomorphic) for some $1\leq p<\infty$ if, and only if, $\dim(E)$ is finite. If $p=\infty$, we have $l_\infty(E)\equiv l_\infty^\omega(E)$. We have also if $1<p\leq\infty$, $l_p^\omega(E)\equiv\mathcal{L}(l_{p^*},E)$ isometrically. In other words, let $v:l_{p^*}\longrightarrow E$ be a linear operator such that $v(e_i)=a_i$ (namely, $v=\sum\limits_{i=1}^{\infty}e_i\otimes a_i$, e_i denotes the unit vector basis of l_p) then,

$$\|v\|=\|(x_i)\|_{l_p^\omega(E)}\,. \tag{3.2}$$

Let T be a Lipschitz operator between X,E and $v:l_{p^*}^n\longrightarrow E^*$ be a bounded linear operator. By (3.2), the Lipschitz operator T is Cohen Lipschitz p-nuclear if, and only if,

$$\begin{aligned}&\left|\sum_{i=1}^{n}\lambda_i\left\langle T(x_i)-T(x_i'),v(e_i)\right\rangle\right|\\ \leq\ & C\sup_{f\in B_{X^\#}}\left(\sum_{i=1}^{n}\lambda_i\left|f(x_i)-f(x_i')\right|^p\right)^{\frac{1}{p}}\|v\|\,.\end{aligned} \tag{3.3}$$

Proposition 23. *Consider T in $\mathrm{Lip}_0(X,E)$, R in $\mathrm{Lip}_0(E,F)$ and S in $\mathrm{Lip}_0(Z,X)$. If T is Cohen Lipschitz p-nuclear operator, then $R\circ T\circ S$ is Cohen Lipschitz p-nuclear operator and $\eta_p^L(R\circ T\circ S)\leq\mathrm{Lip}(R)\eta_p^L(T)\mathrm{Lip}(S)$.*

Proof. (a) Let $n\in\mathbb{N}$; $(z_i)_{1\leq i\leq n},(z_i')_{1\leq i\leq n}\subset Z$ and $(a_i^*)_{1\leq i\leq n}\subset E^*$. By (3.3), it suffices to prove that

$$\begin{aligned}&\left|\sum_{i=1}^{n}\left\langle TS\left(z_i\right)-TS\left(z_i'\right),a_i^*\right\rangle\right|\\ \leq\ & \eta_p^L(T)\mathrm{Lip}(S)\sup_{f\in\mathcal{B}_{Z^\#}}\left(\sum_{i=1}^{n}\left|f(z_i)-f(z_i')\right|^p\right)^{\frac{1}{p}}\|v\|\end{aligned}$$

where $v:E\longrightarrow l_{p^*}^n$ defined by $v(a)=\sum\limits_{i=1}^{n}a_i^*(a)\,e_i$. We have

$$\begin{aligned}
&\left|\sum_{i=1}^{n}\left\langle TS\left(z_i\right)-TS\left(z_i'\right),a_i^*\right\rangle\right|\\
\leq\ & \eta_p^L(T)\sup_{f\in\mathcal{B}_{X^\#}}\left(\sum_{i=1}^{n}\left|f(S\left(z_i\right))-f(S\left(z_i'\right))\right|^p\right)^{\frac{1}{p}}\|v\|,\\
\leq\ & \eta_p^L(T)\mathrm{Lip}(S)\sup_{f\in\mathcal{B}_{X^\#}}\left(\sum_{i=1}^{n}\left|\frac{f(S\left(z_i\right))}{\mathrm{Lip}(S)}-\frac{f(S\left(z_i'\right))}{\mathrm{Lip}(S)}\right|^p\right)^{\frac{1}{p}}\|v\|,\\
\leq\ & \eta_p^L(T)\mathrm{Lip}(S)\sup_{g\in\mathcal{B}_{Z^\#}}\left(\sum_{i=1}^{n}\left|g(z_i)-g(z_i')\right|^p\right)^{\frac{1}{p}}\|v\|.
\end{aligned}$$

This implies that

$\eta_p^L(T\circ S)\leq\eta_p^L(T)\mathrm{Lip}(S)$.

(b) Let $n\in\mathbb{N}$; $\left(x_i\right)_{1\leq i\leq n},\left(x_i'\right)_{1\leq i\leq n}\subset X$; $\left(b_i^*\right)_{1\leq i\leq n}\subset F^*$. It suffices by (3.3) to prove that

$$\begin{aligned}
&\left|\sum_{i=1}^{n}\left\langle RT\left(x_i\right)-RT\left(x_i'\right),b_i^*\right\rangle\right|\\
\leq\ & \eta_p^L(T)\mathrm{Lip}(R)\sup_{f\in\mathcal{B}_{X^\#}}\left(\sum_{i=1}^{n}\left|f(x_i)-f(x_i')\right|^p\right)^{\frac{1}{p}}\|w\|
\end{aligned}$$

where $w:F\longrightarrow l_{p^*}^n$ defined by $w\left(b\right)=\sum_{i=1}^{n}b_i^*\left(b\right)e_i$. We have

$$\begin{aligned}
&\left|\sum_{i=1}^{n}\left\langle RT\left(x_i\right)-RT\left(x_i'\right),b_i^*\right\rangle\right|\\
=\ & \left|\sum_{i=1}^{n}\left\langle T\left(x_i\right)-T\left(x_i'\right),R^\#\left(b_i^*\right)\right\rangle\right|,\\
\leq\ & \eta_p^L(T)\sup_{f\in\mathcal{B}_{E^\#}}\left(\sum_{i=1}^{n}\left|f(x_i)-f(x_i')\right|^p\right)^{\frac{1}{p}}\|u\|,\\
\leq\ & \eta_p^L(T)\mathrm{Lip}(R)\sup_{f\in\mathcal{B}_{E^\#}}\left(\sum_{i=1}^{n}\left|f(e_i)-f(e_i')\right|^p\right)^{\frac{1}{p}}\|w\|.
\end{aligned}$$

Where $u\left(y\right)=\sum_{i=1}^{n}\left\langle R^\#\left(b_i^*\right),a\right\rangle e_i=\sum_{i=1}^{n}\left\langle b_i^*,R\left(a\right)\right\rangle e_i$.

This implies that T is Cohen Lipschitz p-nuclear and $\eta_p^L(T\circ R)\leq\|R\|\,\eta_p^L(T)$.

□

Let us present the "*Pietsch's domination theorem*" concerning this class of Lipschitz operators. The proof is like that used in [**AMS09**]. In [**Cha11**], J. A. Chávez-Domínguez gives domination theorem for r,p,q such that $1/r+1/p+1/q=1$ and T in $\mathrm{Lip}_0\left(X,E^*\right)$.

THEOREM 21. *Consider $T\in\mathrm{Lip}_0(X,E)$ and C a positive constant. Then the following assertions are equivalent.*

(1) *The operator T is Cohen Lipschitz p-nuclear and $\eta_p^L(T) \leq C$.*

(2) *For any n in $\mathbb{N}$; $(x_i)_{1\leq i\leq n}, (x_i')_{1\leq i\leq n}$ in X; $(a_i^*)_{1\leq i\leq n}$ in E^* and $(\lambda_i)_{1\leq i\leq n}$ in $\mathbb{R}_+$, we have*

$$\begin{aligned} & \sum_{i=1}^{n} \lambda_i \left|\langle T(x_i) - T(x_i'), a_i^*\rangle\right| \\ \leq \quad & C \sup_{f\in\mathcal{B}_{X^\#}} \left(\sum_{i=1}^{n} \lambda_i \left|f(x_i) - f(x_i')\right|^p\right)^{\frac{1}{p}} \sup_{\|a\|_E\leq 1} \left(\sum_{i=1}^{n} \left|\langle a, a_i^*\rangle\right|^{p^*}\right)^{\frac{1}{p^*}}. \end{aligned} \tag{3.4}$$

(3) *There exist Radon probability measures μ_1 on $\mathcal{B}_{X^\#}$ and μ_2 on $\mathcal{B}_{E^{**}}$, such that for all x, x' in X and a^* in E^*, we have*

$$\begin{aligned} & \left|\langle T(x) - T(x'), a^*\rangle\right| \\ \leq \quad & C(\textstyle\int_{\mathcal{B}_{X^\#}} \left|f(x) - f(x')\right|^p d\mu_1(f))^{\frac{1}{p}} (\textstyle\int_{\mathcal{B}_{E^{**}}} \left|a^*(a^{**})\right|^{p^*} d\mu_2(a^{**}))^{\frac{1}{p^*}}. \end{aligned} \tag{3.5}$$

Moreover, in this case

$$\eta_p^L(T) = \inf\left\{C > 0 : \textit{ for all } C \textit{ verifying the above inequality (3.5) }\right\}.$$

4. Relationships between $\Pi_p^L(X,E), \mathcal{D}_{st,p}^L(X,E), \Pi_{\tau(p)}^L(X,E)$ and $\mathcal{N}_p^L(X,E)$.

In this section, we investigate the relationships between the various classes of Lipschitz operators.

THEOREM 22. *We have for a Lipschitz operator $T : X \to E$.*
(1) *$\mathcal{N}_p^L(X,E) \subseteq \mathcal{D}_{st,p}^L(X,E)$ and $d_{st,p}^L(T) \leq \eta_p^L(T)$ for $1 < p \leq \infty$.*
(2) *$\mathcal{N}_p^L(X,E) \subseteq \Pi_p^L(X,E)$ and $\pi_p^L(T) \leq \eta_p^L(T)$ for $1 \leq p < \infty$.*
(3) *$\Pi_{\tau(p)}^L(X,E) \subseteq \mathcal{D}_{st,p^*}^L(X,E)$ and $d_{st,p^*}^L(T) \leq \pi_{\tau(p)}^L(T)$ for $1 \leq p < \infty$.*
(4) *$\Pi_\tau^L(X,E) \subset \mathcal{N}_p^L(X,E)$ and $\eta_p^L(T) \leq \pi_\tau^L(T)$ for $1 \leq p \leq \infty$.*

PROOF. (1) Let $T \in \mathcal{N}_p^L(X,E)$. Consider x, x' in X and $a^* \in E^*$. We have by inequality (3.1)

$$\begin{aligned} & \left|\langle T(x) - T(x'), a^*\rangle\right| \\ \leq \quad & \eta_p^L(T)(\textstyle\int_{\mathcal{B}_{X^\#}} \left|f(x) - f(x')\right|^p d\mu_1(f))^{\frac{1}{p}} (\textstyle\int_{\mathcal{B}_{E^{**}}} \left|a^*(a^{**})\right|^{p^*} d\mu_2(a^{**}))^{\frac{1}{p^*}} \\ \leq \quad & \eta_p^L(T)(\textstyle\int_{\mathcal{B}_{X^\#}} d^p(x,x')\, d\mu_1(f))^{\frac{1}{p}} (\textstyle\int_{\mathcal{B}_{E^{**}}} \left|a^*(a^{**})\right|^{p^*} d\mu_2(a^{**}))^{\frac{1}{p^*}} \\ \leq \quad & \eta_p^L(T) d(x,x') (\textstyle\int_{\mathcal{B}_{E^{**}}} \left|a^*(a^{**})\right|^{p^*} d\mu_2(a^{**}))^{\frac{1}{p^*}}. \end{aligned}$$

Hence by , T is Lipschitz strongly p-summing and $d_{st,p}^L(T) \leq \eta_p^L(T)$.
(2) Let T be an operator in $\mathcal{N}_p^L(X,E)$. We have by inequality (3.1)

$$\begin{aligned} & \|T(x)-T(x')\| \\ = & \sup_{a^*\in B_{E^*}} \left|\langle T(x)-T(x'),a^*\rangle\right| \\ \leq & \sup_{a^*\in B_{E^*}} \eta_p^L(T)(\textstyle\int_{B_{X^\#}} |f(x)-f(x')|^p \, d\mu_1)^{\frac{1}{p}}(\textstyle\int_{B_{E^{**}}} |a^*(a^{**})|^{p^*} \, d\mu_2)^{\frac{1}{p^*}} \\ \leq & \eta_p^L(T)(\textstyle\int_{\mathcal{B}_{X^\#}} |f(x)-f(x')|^p \, d\mu_1(f))^{\frac{1}{p}}. \end{aligned}$$

By Pietsch domination theorem [**FJ09**], T is Lipschitz p-summing and $\pi_p^L(T)\leq\eta_p^L(T)$.

(3) Let T be in $\Pi_{\tau(p)}^L(X,E)$. Consider $x,x'\in E$ and $a^*\in E^*$. We have by (1.3)

$$\begin{aligned} & \left|\langle T(x)-T(x'),a^*\rangle\right| \\ \leq & \pi_{\tau(p)}^L(T)(\textstyle\int_{B_{X^\#}}\int_{\mathcal{B}_{E^{**}}} |(f(x)-f(x')\langle a^*,a^{**}\rangle|^p \, d\mu_1(f)d\mu_2(a^{**}))^{\frac{1}{p}}) \\ \leq & \pi_{\tau(p)}^L(T)d(x,x')(\textstyle\int_{\mathcal{B}_{X^\#}}\int_{\mathcal{B}_{E^{**}}} \left|\frac{(f(x)-f(x')}{d(x,x')}\langle a^*,a^{**}\rangle\right|^p d\mu_1(f)d\mu_2(a^{**}))^{\frac{1}{p}}) \\ \leq & \pi_{\tau(p)}^L(T)d(x,x')(\textstyle\int_{\mathcal{B}_{X^\#}}\int_{\mathcal{B}_{E^{**}}} \sup\limits_{x\neq x'} \left|\frac{(f(x)-f(x')}{d(x,x')}\langle a^*,a^{**}\rangle\right|^p d\mu_1(f)d\mu_2)^{\frac{1}{p}}) \\ \leq & \pi_{\tau(p)}^L(T)d(x,x')(\textstyle\int_{\mathcal{B}_{E^{**}}} |\langle a^*,a^{**}\rangle|^p \, d\mu_2(a^{**}))^{\frac{1}{p}}). \end{aligned}$$

This implies by (2.3) that T is Lipschitz strongly p^*-summing and $d_{st,p^*}^L(T)\leq\pi_{\tau(p)}^L(T)$.

(4) Let $T\in\Pi_\tau^L(X,E)$. For n in $\mathbb{N}$, $(x_i)_{1\leq i\leq n}, (x_i')_{1\leq i\leq n}$ in X and $(a_i^*)_{1\leq i\leq n}$ in E^*, we have

$$\begin{aligned} & \sum_{i=1}^n \left|\langle T(x_i)-T(x_i'),a_i^*\rangle\right| \\ \leq & \pi_\tau^L(T) \sup_{\substack{\|f\|\leq 1 \\ \|a^{**}\|\leq 1}} \left(\sum_{i=1}^n \left|(f(x_i)-f(x_i'))\langle a^{**},a_i^*\rangle\right|\right) \\ \leq & \pi_\tau^L(T)\,\omega_p^L(1,(x_i),(x_i'))\,\omega_{p^*}\left((a_i^*)_i\right) \text{ by Hölder inequality.} \end{aligned}$$

This proves that $T\in\mathcal{N}_p^L(X,E)$ and $\eta_p^L(T)\leq\pi_\tau^L(T)$. □

From the results obtained above we get.

THEOREM 23. *Consider* $1\leq p\leq\infty$. *Let* $T\in\mathrm{Lip}_0(X,E)$ *and* $L\in\mathrm{Lip}_0(E,F)$, *If* L *is Lipschitz strongly* p*-summing operator, and* T *is Lipschitz* p*-summing operator, then* $L\circ T$ *is Cohen Lipschitz* p*-nuclear operator and* $\eta_p^L(L\circ T)\leq d_{st,p}^L(L)\pi_p^L(T)$.

PROOF. Let $x,x'\in E$ and $b^*\in F^*$. By (2.3) we have

$$\begin{aligned} & \left|\left\langle L\circ T\left(x\right)-L\circ T\left(x\right),b^{*}\right\rangle\right| \\ = \ & \left|\left\langle L\left(T\left(x\right)\right)-L\left(T\left(x'\right)\right),b^{*}\right\rangle\right| \\ \leq \ & d_{st,p}^{L}(L)\left\|T\left(x\right)-T\left(x'\right)\right\| \textstyle\int_{\mathcal{B}_{F^{**}}}\left|b^{*}(b^{**})\right|^{p^{*}}d\mu_{2}(b^{**}))^{\frac{1}{p^{*}}}, \end{aligned}$$

and by Pietsch domination theorem in [**FJ09**]

$$\begin{aligned} & \left|\left\langle L\circ T\left(x\right)-L\circ T\left(x\right),b^{*}\right\rangle\right| \\ \leq \ & d_{st,p}^{L}(L)\pi_{p}^{L}(T)(\textstyle\int_{\mathcal{B}_{X^{\#}}}\left|f(x)-f(x')\right|^{p}d\mu_{1})^{\frac{1}{p}}\int_{\mathcal{B}_{F^{**}}}\left|b^{*}(b^{**})\right|^{p^{*}}d\mu_{2})^{\frac{1}{p^{*}}}. \end{aligned}$$

This gives that $L\circ T\in\mathcal{N}_{p}^{L}(X,F)$ and $\eta_{p}^{L}(L\circ T)\leq d_{st,p}^{L}(L)\pi_{p}^{L}(T)$. □

COROLLARY 6. *If $p\geq 2$. Then $\pi_{\tau(p)}^{L}(L\circ T)\leq d_{st,p}^{L}(L)\pi_{p}^{L}(T)$.*

THEOREM 24. *Let $1\leq r,p,q<\infty$ and $\frac{1}{r}=\frac{1}{p}+\frac{1}{q}$. Let $T\in\operatorname{Lip}_{0}(X,E)$ and $L\in\operatorname{Lip}_{0}(E,F)$. If L is Lipschitz $\tau\left(r\right)$-summing and T is Lipschitz p-summing, then $L\circ T$ is Lipschitz (r,p,q)-summing operator and $\pi_{(r,p,q)}^{L}(L\circ T)\leq\pi_{\tau(r)}^{L}(L)\pi_{p}^{L}(T)$.*

PROOF. Let $x,x'\in X$ and $b^{*}\in F^{*}$. We have by (1.3)

$$\begin{aligned} & \left|\left\langle L\circ T\left(x\right)-L\circ T\left(x'\right),b^{*}\right\rangle\right| \\ \leq \ & \pi_{\tau(r)}^{L}\left(L\right)(\textstyle\int_{\mathcal{B}_{E^{\#}}}\int_{\mathcal{B}_{F^{**}}}\left|(f(T\left(x\right))-f(T\left(x'\right))\left\langle b^{*},b^{**}\right\rangle\right|^{r}d\mu_{1}(f)d\mu_{2})^{\frac{1}{r}}). \end{aligned}$$

Using general Hölder's inequality and the fact that T is Lipschitz p-summing, we get

$$\begin{aligned} & \left|\left\langle L\circ T\left(x\right)-L\circ T\left(x'\right),b^{*}\right\rangle\right| \\ \leq \ & \pi_{\tau(r)}^{L}(L)(\textstyle\int_{\mathcal{B}_{E^{\#}}}\left|f(T\left(x\right))-f(T\left(x'\right))\right|^{p}d\mu_{1})^{\frac{1}{p}}(\int_{\mathcal{B}_{F^{**}}}\left|\left\langle b^{*},b^{**}\right\rangle\right|^{q}d\mu_{2})^{\frac{1}{q}} \\ \leq \ & \pi_{\tau(r)}^{L}(L)\left\|T\left(x\right)-T\left(x'\right)\right\|(\textstyle\int_{\mathcal{B}_{F^{**}}}\left|\left\langle b^{*},b^{**}\right\rangle\right|^{q}d\mu_{2}(b^{**}))^{\frac{1}{q}} \\ \leq \ & \pi_{\tau(r)}^{L}(L)\pi_{p}^{L}(T)(\textstyle\int_{\mathcal{B}_{X^{\#}}}\left|f(x)-f(x')\right|^{p}d\mu)^{\frac{1}{p}}(\int_{\mathcal{B}_{F^{**}}}\left|\left\langle b^{*},b^{**}\right\rangle\right|^{q}d\mu_{2})^{\frac{1}{q}}. \end{aligned}$$

This implies that $L\circ T\in\Pi_{(r,p,q)}^{L}(X,F)$ and $\pi_{(r,p,q)}^{L}(L\circ T)\leq\pi_{\tau(r)}^{L}(L)\pi_{p}^{L}(T)$. □

COROLLARY 7. *Let $1<p<\infty$. Let $T\in\operatorname{Lip}_{0}(X,E)$ and $L\in\operatorname{Lip}_{0}(E,F)$. If L is Lipschitz τ-summing and T is Lipschitz p-summing , then $L\circ T$ is Cohen Lipschitz p-nuclear operator and $\eta_{p}^{L}(L\circ T)\leq\pi_{\tau}^{L}(L)\pi_{p}^{L}(T)$.*

Bibliography

[AE56] R.-F. Arens and J. Eells, *On embedding uniform and topological spaces*, Pacific J. Math **6** (1956), 397-403.

[AJ13] A.-G. AKSOY and S. JIN, *The Apple Doesn't Fall Far From the (Metric) Tree: The Equivalence of Definitions,* Proceedings of the First Conference "Classical and Functional Analysis"Azuga – România, september 28-29, 2013.

[AMS09] D. Achour, L. Mezrag and K. Saadi, *On the Cohen p-nuclear sublinear operators*. J. Inequal. Pure and App. Math. **10(**2**)** **(**2009**),** 1-13**.**

[AO10] A.-G. Aksoy and T. OIkhberg, *Some results on metric trees*, Banach Center Pub. **91** (2010), 9-34.

[Api76] H. Apiola, *Duality between spaces of p-summable sequences, (p,q)-summing operators and characterizations of nuclearity*, Math. Ann. **219** (1976), 53–64.

[CCJV15] M. G. Cabrera-Padilla, J. A. Chávez-Domínguez, A. Jiménez-Vargas and M. Villgas-Vallecillos, *Lipschitz tensor product*, Khayyam J. Math. **1** (2) (2015), 185-218.

[Cha11] J.-A. Chávez-Domínguez, *Duality for Lipschitz p-summing operators*, J. Funct. Anal. **261** (2) (2011), 387–407.

[Cob03] Ş. Cobzaş, *Adjoints of Lipschitz mappings*, Studia Univ. "Babeş–Bolyai", Mathematica, **XLVIII** (1), (2003), 49-54.

[Coh73] J. S. Cohen *Absolutely p-summing, p-nuclear operators and their conjugates*. Math. Ann. **201** (1973), 177-200.

[CZ11] D. Chen and B. Zheng, *Remarks on Lipschitz p-summing operators*, Proc. Amer. Math. Soc. **139**(8), (2011), 2891-2898.

[CZ12] D. Chen and B. Zheng, *Lipschitz p-integral operators and Lipschitz p-nuclear operators*, Nonlinear Analysis **75** (2012), 5270-5282.

[DJT95] J. Diestel, H. Jarchow and A. Tonge. *Absolutely summing operators"* Cambridge Studies in Advanced mathematics **43** Cambridge University Press, Cambridge, 1995.

[Flo75] J. Flood, *Free topological vector spaces*, Ph.D. Thesis, Australian National University, Candberra, 1975, 109 p.

[FJ09] J.-D. Farmer and W.B. Johnson, *Lipschitz p-summing operators*, Proc. Amer. Math. Soc. **137**(9) (2009), 2989-2995.

[GK03] N.-J. Kalton and G. Godefroy, *Lipschitz-free Banach spaces*, Studia Math. **159** (2003), 121-141.

[GL75] Y. Gordon and D. Lewis, *Banach ideals on Hilbert space*, Studia Math. **54** (1975), 161-172.

[God07] A. Godard, *Espaces Lipschitz-libres, propriété (M) et lissité assympthotique*, Thesis, Université Paris6, 2007.

[Gro56] A. Grothendieck. *Résumé de la théorie métrique des produits tensoriels topologiques*" Bol. Soc. Mat. São Paulo **8** (1956), 1-79.

[Joh70] J.-A. Johnson, *Banach spaces of Lipschitz functions and vector-valued Lipschitz functions*, Trans. Amer. Math. Soc. **148** (1970), 147-169.

[Kan42] L.-V. Kantorovich, *On the translocation of masses*, Dokl. Akad. Nauk SSSR **37** (1942), 227-229.

[Lin64] J. Lindenstrauss, *On nonlinear projections in Banach spaces*, Michigan Math J. **11** (1964), 263-287.

[Mic64] E. Michael. *A short proof of the Arens Eells embedding theorem*, Proc. Amer. Math. Soc. **15** (1964), 415-416.

[MT17] L. Mezrag and A.Tallab, *On Lipschitz $\tau(p)$-summing operators*, Colloq. Math.**147**(1) (2017), 95-114.

[Muj08] X. Mujica, *$\tau(p;q)$-summing mappings and the domination theorem*, Portugal. Math. (N. S.) **65**(2) (2008), 211-226.

[Pes86] V.-G. Pestov, *Free Banach spaces and representation of topological groups*, Funct. Anal. Appl. **20**, (1986), 70-72.

[Pie80] A. Pietsch, *Operator ideals*. North-Holland Math. Library 20, North Holland Publishing Co., Amsterdam, 1980.

[Pis86] G. Pisier, *Factorisation of linear operators and the geometry of Banach spaces.* Regional conference of the A.M.S CBMS **60** (1986).

[PP69] A. Person and A. Pietsch, *p-nukleare and p-integrale Abbildungen in Banachräumen*, Studia Math. **33** (1969), 213-222.

[Saa15] K. Saadi, *Some properties of Lipschitz strongly p-summing operators*, J. Math. Anal. Appl. **423** (2015), 1410-1426.

[Saw75] I. Sawashima, *Methods of Lipschitz duals*, in Lecture Notes Ec. Math. Sust., **419**, Springer Verlag (1975), 247-259.

[Wea99] N. Weaver, *Lipschitz Algebras*, World Scientific, Singapore 1999.

[YAR16] R. Yahi, D. Achour and P. Rueda, *Absolutely summing Lipschitz conjugates*, Mediterr. J. Math. **13** (2016), 1949–1961.