Chapter I
Fluid Kinematics

I.1 Introduction

The study of fluid mechanics includes :

- Fluid statics: in this case, we study the fluid at rest (course S3) and the essential law is the fundamental relation of statics.
- Fluid kinematics is the analytical description of a system in motion. In other words, we are interested in the movements of fluids in relation to time, independently of the causes that provoke them, i.e. without taking into account the forces that are at their source.
- Fluid dynamics, in which fluid motion is studied in the context of interacting forces.

I.2 Mathematical concepts of fluid mechanics

I.2.1 Differential of a function

Consider the function f which depends on the variables x, y and z, f=f(x ,y, z)

The total differential **df is** written :

df $= \frac{\partial}{\partial} d \; + \frac{\partial}{\partial} d \; + \frac{\partial}{\partial} d$

$\frac{\partial}{\partial}, \frac{\partial}{\partial} e \; \frac{\partial}{\partial}$ Are the partial derivatives of f with respect to x, y and z

I.2.2 Vector analysis operators

- **Operator** nabla

$$\vec{} \; X \quad \begin{matrix} \frac{|}{|x} \\ \frac{|}{|y} \\ \frac{|}{|z} \end{matrix}$$

- **Gradient of a** scalar **field**

$$\overrightarrow{grad}(f) = \vec{\nabla} f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix}$$

- **Divergence of a vector field**

$$div(\vec{V}) = \vec{\nabla}.\vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

- **Rotational vector fields**

$$\overrightarrow{rot}(\vec{V}) = \vec{\nabla} \wedge \vec{V} = \begin{vmatrix} \frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} \\ \frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x} \\ \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \end{vmatrix}$$

- **Laplacian of a function**

$$\Delta\varphi = \nabla^2\varphi = div(\overrightarrow{grad}\varphi) = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}$$

I.3 <u>Description of a moving fluid</u>

I.3.1 The fluid particle

The fluid particle is chosen as the elementary entity for a complete description of flows: This is a "packet" of molecules surrounding a given point M, all assumed to have the same velocity at the same instant.

In the study of fluid motion, we generally define at each point M: the velocity $\vec{V}$, the density ρ and the pressure P (and possibly the temperature T). Describing the motion of a fluid calls on notions different from those developed in point or solid mechanics. Fluid motion is a **flow** in which there is continuous deformation of the fluid. In a similar way to solid mechanics, we can isolate (by thought or by finding a means of visualization, coloring for example) a restricted part of the fluid called a **particle** and "follow" it over time, i.e. know its position at each instant. This position will be known, for example, by its Cartesian coordinates

$x(t, x_0, y_0, z_0)$, $y(t, x_0, y_0, z_0)$ and $z(t, x_0, y_0, z_0)$ where x_0, y_0 and z_0 represent the coordinates of the chosen particle at time t_0.

The particle's velocity has the following components

$$u = \frac{\partial x}{\partial t}, \quad v = \frac{\partial y}{\partial t} \text{ et } \quad w = \frac{\partial z}{\partial t} \qquad (I.1)$$

The velocity of the fluid particle is then defined by :

$$\vec{V} = \begin{cases} u = \dfrac{\partial x}{\partial t} \\ v = \dfrac{\partial y}{\partial t} \\ w = \dfrac{\partial z}{\partial t} \end{cases} = \vec{V}(r_0, t) \qquad (I.2)$$

Different types of fluid flow regimes can be observed.

- **Permanent (or stationary) regime**: quantities do not depend on time. $\frac{\partial \vec{V}}{\partial t} = 0$, $\vec{V} = \vec{V}(M)$ (ditto for ρ and P)(this does not mean that the fluid has a constant velocity everywhere, but only that the fluid velocity at a given point is the same at every instant.
- **Uniform regime**: speed does not depend on the point considered $\vec{V} = \vec{V}(t)$
- **Laminar regime**: fluid layers slide relative to each other, speeds are continuous.
- **Turbulent regime**: velocities are discontinuous, fluid layers interpenetrate randomly.

I.3.2 Lagrange description - Euler description

The fluid in motion can be described in two equivalent ways. We can choose to follow the fluid particles as they move (**Lagrange point of view**) and the variables $r_0 = (x_0, y_0, z_0)$ and t are called **Lagrange variables**.

Lagrange's approach focuses on the trajectory of fluid particles.

We can take a snapshot of the velocity field of all fluid particles at a given point in time (**Euler's point of view**). **Euler**'s point of view focuses on the evolution of fluid properties at different points and over time.

Lagrange's method proves tricky in most cases, since it's not easy to keep track of the particles: it's rarely used.

The **Euler method** consists in knowing the particle velocity over time t at a given location determined by its coordinates, for example Cartesian x,y and z. The three projections on a system of Cartesian axes of the velocity $\vec{V}(r,t)$ of the fluid particle passing through point M at time t are **called Euler variables. This** method is more widely used than **Lagrange's, as** knowledge of the velocity field is sufficient to describe the fluid in motion.

I.4 <u>Current paths and lines</u>

I.4.1 The trajectory :

The trajectory of a fluid particle is defined by the path followed by the particle over time, i.e. the set of successive positions of the particle during movement.

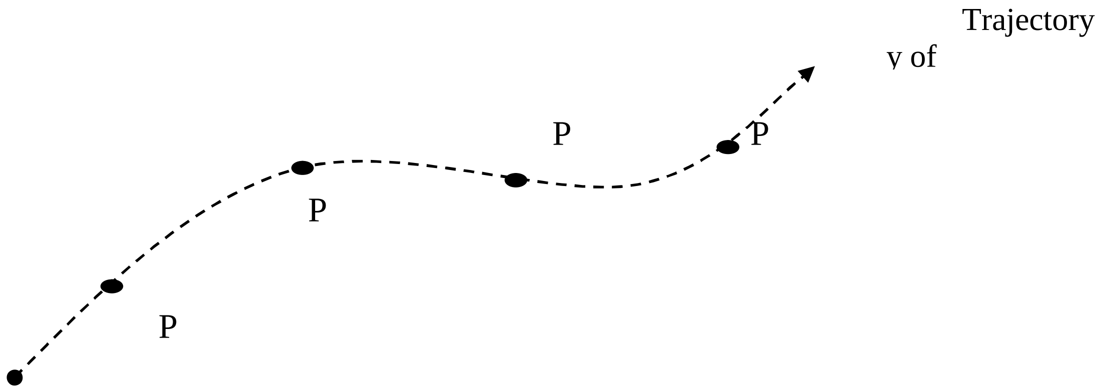


Figure I.1 Particle trajectory

The trajectory can be visualized by injecting a drop of dye and following its movement.

Trajectories are generally calculated by eliminating time from the expressions expressing the position of a fluid particle at each instant:

$$\overrightarrow{OM} = \vec{r}(t) = (x(t),y(t),z(t))$$

If we know the velocity in Eulerian description, we can determine the particle trajectories by integrating this velocity with respect to time.

Consider the given speed $\vec{V}(r,t) = \vec{V}(x,y,z,t) = \begin{Bmatrix} u(x,y,z,t) \\ v(x,y,z,t) \\ w(x,y,z,t) \end{Bmatrix}$ in Eulerian description.

$$\text{By definition } \vec{V} = \frac{d\vec{r}}{dt} = \begin{cases} \dot{x} = \dfrac{dx}{dt} \\ \dot{y} = \dfrac{dy}{dt} \\ \dot{z} = \dfrac{dz}{dt} \end{cases} \qquad (I.3)$$

$$\text{This gives us the differential system} \begin{cases} \dfrac{dx}{dt} = u(x,y,z,t) \\ \dfrac{dy}{dt} = v(x,y,z,t) \\ \dfrac{dz}{dt} = w(x,y,z,t) \end{cases} \qquad (I.4)$$

By integrating this system with the initial conditions r_0 =(x_0 =x(t_0), y_0 = y(t_0), z0=z(t_0)), we obtain the position at each instant:

$$\vec{r}(t) = (x(t), y(t), z(t)) = \vec{r}_0 + \int_0^t \vec{V}(r_0, t)dt$$

By eliminating time, we obtain a relationship between the variables (x, y, z) corresponding to the equation of the particle's trajectory.

I.4.2 Current lines:

Let's adopt Euler's approach and assume that we know **at each instant** *t* the velocity vector of a fluid particle located at M. The velocity vector $\vec{V}(M,t)$ then designates a vector field.

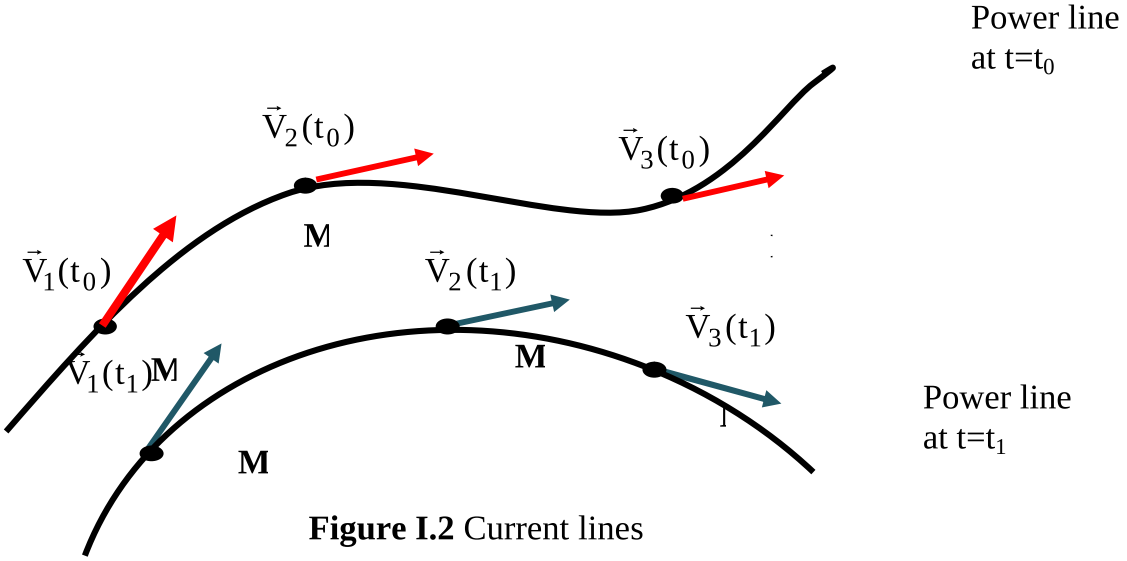


Figure I.2 Current lines

By definition, a **streamline** or **flowline is** a velocity vector field line, i.e. a curve C such that at a fixed instant *t*, for any point M $\in C, \vec{V}(M,t)$ is tangent to C at M. When the velocity field does not depend on time, flowlines do not change over time: the **flow regime is** said to be **stationary or permanent**.

Let $\overrightarrow{dM}$ be an element of the current line, , $\overrightarrow{dM} = (dx, dy, dz)$ $\overrightarrow{dM}$ is parallel in M to the speed $\vec{V}(M,t)$: $\overrightarrow{dM} \,//\, \vec{V} \quad \overrightarrow{dM} \wedge \vec{V} = 0$

Gold

$$\vec{V}(M,t) = \vec{V}(x,y,z,t) = \begin{Bmatrix} u(x,y,z,t) \\ v(x,y,z,t) \\ w(x,y,z,t) \end{Bmatrix} \Rightarrow d\overrightarrow{M} \wedge \vec{V} = \begin{Bmatrix} wdy - v\,dz \\ udz - wdx \\ vdx - udy \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

(I.5)

Finally, we obtain the relationships defining the current lines

$$\frac{dx}{u(x,y,z,t)} = \frac{dy}{v(x,y,z,t)} = \frac{dz}{w(x,y,z,t)} \qquad \text{(I.6)}$$

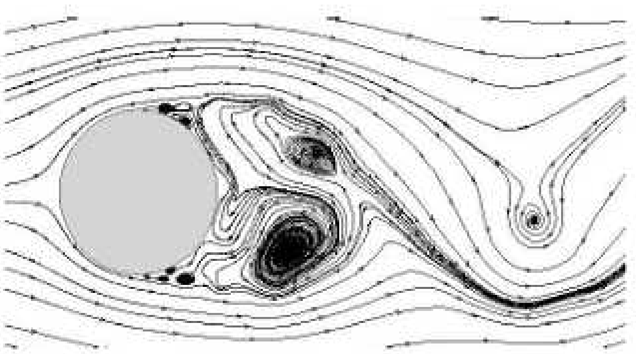


Figure I.3 Current lines around an obstacle

Note:

- Current lines are generally time-dependent, so they deform over time.
- In steady state (stationary flow), velocities no longer depend on time, and the two previous conditions coincide with :

$$\frac{dx}{u(x,y,z)} = \frac{dy}{v(x,y,z)} = \frac{dz}{w(x,y,z)} \qquad \text{(I.7)}$$

The particles continuously follow the same trajectories, generating the same streamlines. In this particular case, trajectory and current lines are one and the same.

Other quantities characterizing fluid motion can also be defined:

I.4.3 Current tube:

A **current tube is** the set of current lines supported by a closed contour.

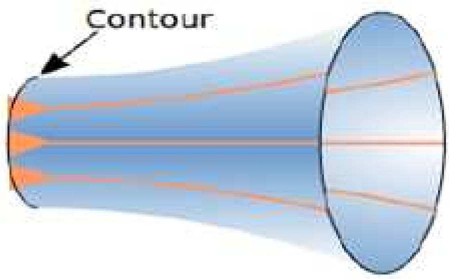


Figure I.4 Contour current tube

I.4.4 Emission lines:

Emission lines are the set of all particles that have coincided at an earlier instant with a fixed point E.

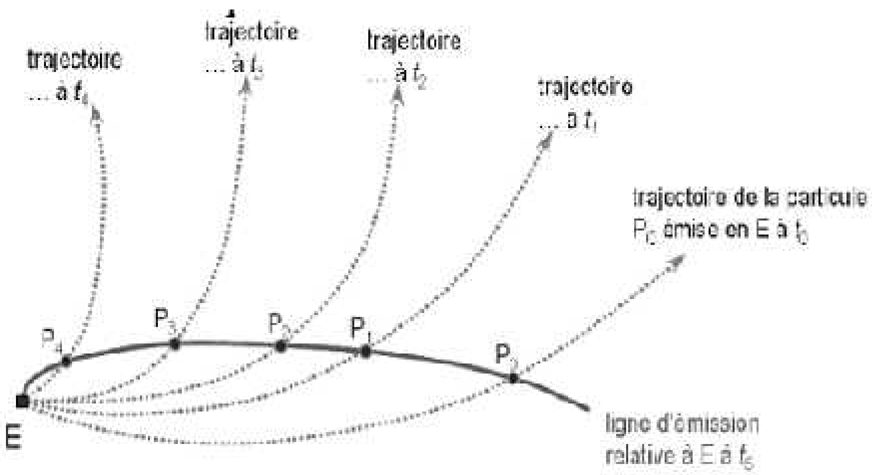


Figure I.5 Emission lines

To visualize emission lines, dye can be injected continuously at point E. The colored curves correspond to the emission lines.

I.5 <u>Particle derivative</u>

Consider a local physical quantity $G(\mathrm{M}, t)$ attached to a fluid particle located in M at time t . We can think of temperature, pressure, density and so on. Let's calculate the rate of change of this quantity **as we follow the particle**. This quantity is called the **particle derivative** and is denoted by $\frac{\mathbf{D}}{\mathbf{D}}$**.**

The fluid particle at time t+dt will be at the point with coordinates x+udt, y+vdt, z+wdt

The variation of the G function will therefore be equal to :

$$dG = G(x + udt, y + vdt, y + wdt) - G(x, y, z) = \frac{\partial G}{\partial x}udt + \frac{\partial G}{\partial y}vdt + \frac{\partial G}{\partial z}wdt + \frac{\partial G}{\partial t}dt$$

The derivative $\frac{\mathbf{d}}{\mathbf{d}}\frac{\mathbf{D}}{\mathbf{D}}$ and called the **particle derivative**, is equal to :

$$\frac{DG}{Dt} = \frac{dG}{dt} = \frac{\partial G}{\partial x}u + \frac{\partial G}{\partial y}v + \frac{\partial G}{\partial z}w + \frac{\partial G}{\partial t}dt = \vec{V}\,\overrightarrow{grad}G + \frac{\partial G}{\partial t} \qquad \text{(I.8)}$$

This derivative appears as the sum of two terms:

- the first, called **convective** or **advective**, is due to the non-uniformity of the flow,
- the second, called **temporal**, is due to the unsteady nature of the flow

I.6 Particle acceleration

Let's calculate the acceleration of a fluid particle from the Eulerian velocity field $\vec{V}(M,t)$. Acceleration is the rate of change of the velocity field as it follows a fluid particle. We therefore have :

$$\vec{a} = \frac{D\vec{V}}{Dt} = \frac{Du}{Dt}u + \frac{Dv}{Dt}v + \frac{Dw}{Dt}w \qquad \text{(I.9)}$$

$$\text{Speed}\ \vec{V} = \begin{cases} u = \dfrac{dx}{dt} \\ v = \dfrac{dy}{dt} \\ w = \dfrac{dz}{dt} \end{cases},$$

$$\vec{a} = \begin{cases} a_x = \dfrac{du}{dt} = \dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial x}.\dfrac{\partial x}{\partial t} + \dfrac{\partial u}{\partial y}.\dfrac{\partial y}{\partial t} + \dfrac{\partial u}{\partial z}.\dfrac{\partial z}{\partial t} \\ a_y = \dfrac{dv}{dt} = \dfrac{\partial v}{\partial t} + \dfrac{\partial v}{\partial x}.\dfrac{\partial x}{\partial t} + \dfrac{\partial v}{\partial y}.\dfrac{\partial y}{\partial t} + \dfrac{\partial v}{\partial z}.\dfrac{\partial z}{\partial t} \\ a_z = \dfrac{dw}{dt} = \dfrac{\partial w}{\partial t} + \dfrac{\partial w}{\partial x}.\dfrac{\partial x}{\partial t} + \dfrac{\partial w}{\partial y}.\dfrac{\partial y}{\partial t} + \dfrac{\partial w}{\partial z}.\dfrac{\partial z}{\partial t} \end{cases} \qquad \text{(I.10)}$$

$$\vec{a} = \begin{cases} a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \\ a_y = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \\ a_z = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \end{cases} \quad (I.11)$$

This gives :

$$\begin{cases} a_x = \frac{Du}{Dt} = \frac{\partial u}{\partial t} + (\vec{V}.\vec{\nabla})u \\ a_y = \frac{Dv}{Dt} = \frac{\partial v}{\partial t} + (\vec{V}.\vec{\nabla})v \\ a_z = \frac{Dw}{Dt} = \frac{\partial w}{\partial t} + (\vec{V}.\vec{\nabla})w \end{cases} \quad (I.12)$$

The acceleration breaks down as follows:

- The first term ($\frac{\partial}{\partial}$): is linked to the non-permanent nature of velocity. It's called the **local term.**
- The second term $\vec{V}.\vec{\nabla}$: the convective derivative indicates a non-uniform velocity. It's called the **convective term**.

I.7 Volume flow and mass flow

To solve fluid mechanics and hydraulics problems, we often use the concepts of flow rate and mean flow velocity.

Volume flow $\mathbf{q_v}$ measured in (m^3 /s) or (l/s)

Mass flow rate $\mathbf{q_m}$ measured in (Kg/s)

Volumetric flow is the volume of fluid $\boldsymbol{\delta v_{trav}}$ passing through a given surface per unit time (m^3 /s).

$$\delta v_{trav} = q_v \, dt \quad (I.13)$$

The total volume passing through the surface in question over a period of time $(t_2 - t_1)$ is given by :

$$v_{trav} = \int_{t_1}^{t_2} q_v dt \quad (I.14)$$

The flow rate for a constant velocity perpendicular to a given cross-section of a pipe or channel (perfect fluid) is :

$$q_v = V.S \qquad (I.15)$$

$$\mathbf{q_m = \rho V.S = \rho.q_v} \qquad (I.16)$$

(ρ is the density of the fluid)

- **Expression of q_v as a function of the velocity field on the surface**

Volume flow is **the flow of the vector** $\vec{V}$ through the surface in question.

$$q_v = \iint_s \vec{V}.\vec{n}ds \qquad (I.17)$$

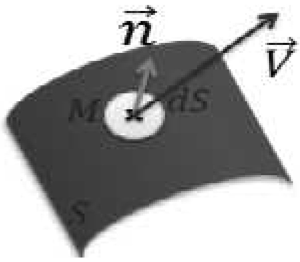


Figure I.6 Velocity vector flow across a surface

- If the flow is in the same direction as the surface normal vector: $q_v > 0$, otherwise $q_v < 0$

Flux is a synonym for **Débit** (a 'qtte' which passes through a surface per unit of time, unit: (**"qtte".s^{-1}**)

Current density means **surface flow** (or flow per unit area, unit: **("qtte".m^{-2} s^{-1}**)

- **Mass flow is** the mass of fluid passing through a given surface per unit time (Kg.S^{-1}).

$$\mathbf{\delta m_{trav} = q_m}\, dt \quad (I.18)$$

- The total mass passing through the surface in question over a period of time ($t_2 - t_1$) is given by :

$$m_{trav} = \int_{t_1}^{t_2} q_m dt \qquad (I.19)$$

Mass flow is **the flow of the vector** through the surface considered:

$$q_m = \oint_S \rho \vec{V}.\vec{n} ds \qquad \text{(I.20)}$$

- If the flow is in the same direction as the surface normal vector :

  $q_m > 0$ Otherwise $q_m < 0$

The $\rho\vec{V}$ field thus appears as the mass current density, or surface mass flow.

- In the particular case of a permanent conservative flow through a current tube, the mass d "bit is conserved: **$q_{m1} = q_{m2}$**
- If the fluid is also incompressible: **$q_{v1} = q_{v2}$**

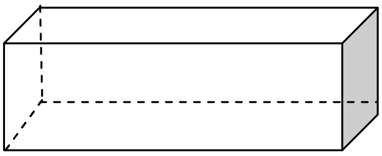
I.8 Continuity Equation

It translates the principle of conservation of mass:

The change in mass over time dt of a fluid volume element dv = dx dy dz must be equal to the sum of the masses of incoming fluid, minus that of outgoing fluid.

Consider a volume element of fluid *dv*

dv=*dx*.dy.dz



The mass $m=\iiint_v \rho$ of a portion of fluid volume bounded by a surface (S) that we follow in its motion remains constant, so its particle derivative is zero.

$$\frac{dm}{dt} = \frac{d}{dt}\iiint_v \rho dv = \iiint_v \frac{\partial\rho}{\partial t} dv + \iint_S \rho(\vec{V}.\vec{n}) dS = 0 \qquad \text{(I.21)}$$

Convective

Local derivative

I.8.1 Green-Ostrogradsky theorem or divergence theorem

The flux of a vector field $\vec{A}(M)$ through a closed surface (*S*) is equal to the integral over the volume (*v)* bounded by (*S) of the* divergence of the vector field.

$$\iint_S \vec{A}(M).\vec{n}.dS = \iiint_v div\vec{A}(M).dv$$

et $\quad div\vec{A} = \vec{\nabla}.\vec{A}$

(I.22)

So we can write :

$$\iint_S \rho(\vec{V}.\vec{n}).dS = \iiint_v \mathrm{div}(\rho\vec{V}).dv = \iiint_v (\vec{\nabla}.\rho\vec{V})dv \tag{I.23}$$

$$\frac{dm}{dt} = \iiint_v \frac{\partial\rho}{\partial t}dv + \iiint_v \mathrm{div}(\rho\vec{V})dv = 0 \tag{I.24}$$

Or still

$$\iiint_v (\frac{\partial\rho}{\partial t} + \vec{\nabla}.\rho\vec{V})dv = 0 \tag{I.25}$$

Then : On an arbitrary volume (the integral must be zero) this relationship becomes :

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}.\rho\vec{V} = \frac{\partial\rho}{\partial t} + \mathrm{div}\,\rho\vec{V} = 0 \tag{I.26}$$

$\frac{\partial\rho}{\partial t} + \mathrm{div}\rho\vec{V} = 0$ **Is the continuity equation** (I.27)

In Cartesian coordinates, this equation is written :

$$\frac{\partial\rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$$

This is the general continuity equation, applicable to all types of flow, and all types of compressible and incompressible fluids.

If the fluid is in permanent motion, the density is independent of time, $\frac{\partial\rho}{\partial t} = 0$ and this becomes :

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0 \quad \text{ou} \quad \mathrm{div}(\rho\vec{V}) = 0$$

The equation obtained indicates that the $\rho\vec{V}$ flow through the closed surface is zero (**conservation of mass flow**).

$$\iint_S = \iiint_v \mathrm{div}(\rho\vec{V}).dv = \iint \rho(\vec{V}.\vec{n}).dS = 0$$

For a two-dimensional plane flow we write :

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

For one-dimensional flow in the x direction

$$\frac{\partial u}{\partial x}=0 \Rightarrow u=cte \Rightarrow q_v=uS=cte$$

(S flow cross-section)

Special case of an incompressible fluid :

In this case the density ρ **=cte** $\Rightarrow \frac{\partial \rho}{\partial t}=0 \Rightarrow div\vec{V}=0$

So the continuity equation reduces to : $div\vec{V}=0$ (I.28)

I.8.2 Divergence of a velocity field

I.8.2.1 Definition :

Velocity field divergence **(**$div\vec{V}$) is a differential operator with scalar values that measures changes in the volume of a continuous medium. A positive (resp. negative) value is associated with expansion (resp. compression). In Cartesian coordinates, it is written :

$$div\vec{V}=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=\sum_{i=1}^{3}\frac{\partial v_i}{\partial x_i}=\frac{\partial v_i}{\partial x_i}$$

In cylindrical coordinates, it is written :

$$div\vec{V}=\frac{1}{r}\frac{\partial (rv_r)}{\partial r}+\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{\partial v_z}{\partial z}$$

We can say that the divergence of the velocity field gives us information about the change in volume of a fluid element we're following as it moves. If this element maintains a constant volume, the divergence is zero. If this is true at any point in the fluid, then the volume of all fluid elements will remain constant throughout the flow: such a flow is said to be **incompressible.**

I.9 Some flow examples

I.9.1 Uniform flow

In the absence of deformation and rotation, the flow is said to be uniform. This movement corresponds to solid translational motion.

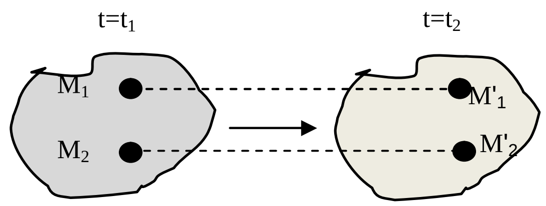
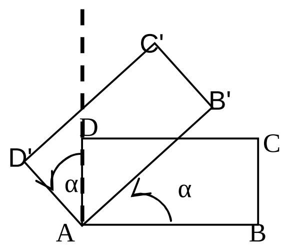


Figure I.7 Uniform flow without deformation or rotation

The pure rotational movement takes place without deformation and is therefore comparable to solid rotation, as shown in the following figure.



FigureI.8 Deformation-free rotational movement of a volume of deformation-free fluid

I.9.2 <u>Rotational</u> flow

The rotational velocity field of a flow $\overrightarrow{rot}\vec{V}$ is a vector-valued differential operator that measures twice the rate of rotation of fluid particles on themselves.

In Cartesian coordinates, the vortex vector is written as :

$$\vec{\Omega} = \overrightarrow{rot}\vec{V} = \begin{pmatrix} \dfrac{\partial w}{\partial y} - \dfrac{\partial v}{\partial z} \\ \dfrac{\partial u}{\partial z} - \dfrac{\partial w}{\partial x} \\ \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y} \end{pmatrix} = \vec{\nabla} \wedge \vec{V} \qquad \text{(I.29)}$$

A rotational flow is characterized by the vortex vector Ω such that :

$$\vec{\Omega} = 2\vec{\omega} = \vec{\nabla} \wedge \vec{V} \qquad \text{(I.30)}$$

And ω is the turnover rate.

In cylindrical coordinates with $\vec{V}(u_r, u_\theta, u_z)$,we have :

$$\vec{\vartheta} = \begin{vmatrix} \frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z} \\ \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} \\ \frac{1}{r}\left(\frac{\partial (r u_\theta)}{\partial r} - \frac{\partial u_r}{\partial \theta}\right) \end{vmatrix} \qquad (I.31)$$

For a plane flow, this vector has only one non-zero component since
ω =0 and u and v do not depend on z :

$$\overrightarrow{rot}\vec{V} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{e}_z \qquad (I.32)$$

I.9.3 Current function - Incompressible flow
I.9.3.1Definition :
If the flow of an incompressible fluid is conservative, then the continuity equation is:
$\vec{\nabla}.\vec{V} = 0$ (eq (I.28))

If we put $\vec{V} = \vec{\nabla} \wedge \vec{A}$,? $\vec{A}$ *alors* $\vec{\nabla}.(\vec{\nabla} \wedge \vec{A}) = 0$

$\vec{A}$ Is called vector potential
In Cartesian coordinates :

$$\vec{V} = \vec{\nabla} \wedge \vec{A} = \begin{vmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{vmatrix} \wedge \begin{vmatrix} A_x \\ A_y \\ A_z \end{vmatrix} = \begin{vmatrix} \frac{\partial A_z}{\partial y} - \cancel{\frac{\partial A_y}{\partial z}} = u \\ \cancel{\frac{\partial A_x}{\partial z}} - \frac{\partial A_z}{\partial x} = v \\ \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = \cancel{w} \end{vmatrix} \qquad (I.33)$$

If we consider a flow in the plane ⊥ to Oz, and therefore invariant by translation along z, then: $w = 0$ *et* $\frac{\partial}{\partial z} = 0$ from which :

$u = \frac{\partial A_z}{\partial y}$ *et* $v = -\frac{\partial A_z}{\partial x}$ then: $A_z(x,y) = \psi(x,y)$, the function **ψ** is called the **current function** . Therefore :

$$\begin{cases} u = \dfrac{\partial \psi}{\partial y} \\ v = -\dfrac{\partial \psi}{\partial x} \end{cases} \qquad \text{(I.34)}$$

Is the velocity field in Cartesian coordinates.

In cylindrical coordinates, this velocity field is written as :

$$\begin{cases} u_r = \dfrac{1}{r}\dfrac{\partial \psi}{\partial \theta} \\ u_\theta = -\dfrac{\partial \psi}{\partial r} \end{cases}$$ Or **ψ(r, θ)** (I.35)

I.9.3.2 Properties of the current function

As we posed $\vec{\nabla}.\vec{V} = \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0$ et $u = \dfrac{\partial \psi}{\partial y}, v = -\dfrac{\partial \psi}{\partial x}$

Then :

$$\frac{\partial^2 \psi}{\partial x \partial y} = \frac{\partial^2 \psi}{\partial y \partial x}$$

This relationship constitutes **Schwartz's** theorem**.** And so dψ is an exact total differential:

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy$$

In the plane(x, y), the set of points for which the value of ψ is constant **ψ(x,y) = cte** corresponds to the curve **y(x)** along which **dψ=0**

On this curve, check that :

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = -v dx + u dy = 0 \qquad \text{(I.36)}$$

Or : $-v dx + u dy = 0 \Rightarrow \dfrac{dy}{dx} = \dfrac{v}{u}$

ψ(x,y) = cte then **y(x)** is such that :

$$\underbrace{\frac{dy}{dx}}_{\text{pente dela courbe } y=f(x)} = \underbrace{\frac{v}{u}}_{\text{pente du vecteur vitesse } \vec{V}}$$

(I.37)

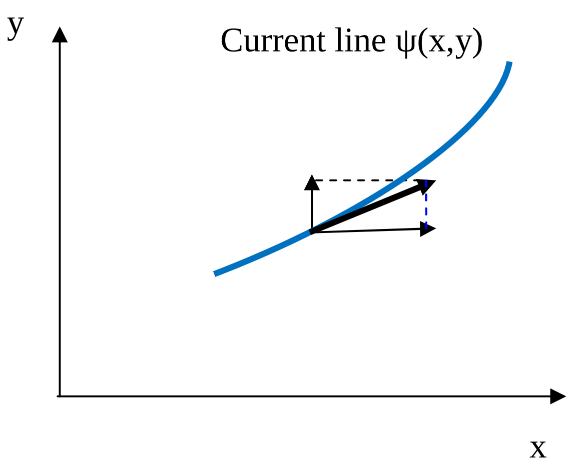


Figure I.9 Qualitative representation of the current line in the (x, y) plane

Let's calculate the flow between two infinitely adjacent current lines:
Let ψ(x,y) be the current function L and ψ+dψ the adjacent current function M. The velocity vector V is perpendicular to the line AB and has components u and v in the x and y directions.

We know that $q_v = \int \vec{V}.\vec{n}ds$

Flow through AB= flow through AO+flow through OB

Vds= udy-vdx

$$Vds = \frac{\partial \psi}{\partial y}dy + \frac{\partial \psi}{\partial x}dx = d\psi$$

And so $\mathbf{dq_v}$ =dψ therefore, between any two current lines of constants ψ_A and ψ_B :

$$q_v = \int_A^B dq_v = \int_A^B d\psi = \psi_B - \psi_A$$

(I.38)

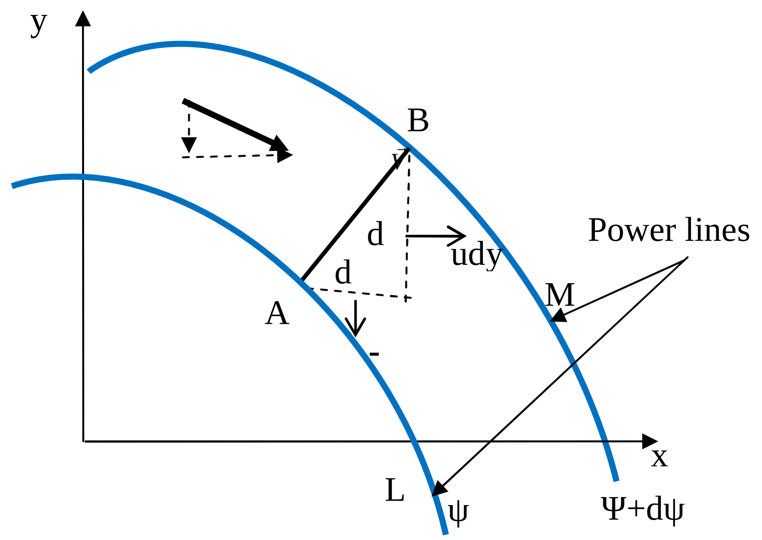


Figure I.10 Flow between two points and its relationship to current lines

I.9.4 Irrotational flows - velocity potential

I.9.4.1 Definition:

Flow is said to be **irrotational** when the fluid particles do not undergo **pure rotations**: Ω=0, i.e. $\overrightarrow{rot}\vec{V} = 0$

$$\omega = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} = 0 \Rightarrow \begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix} = \vec{0}$$

$$\Rightarrow \vec{\omega} = \tfrac{1}{2}\vec{\nabla} \wedge \vec{V} = \vec{0}$$

In other words, the rotation rate **ω** is zero in an irrotational flow.

From a mathematical point of view, the relationship $\vec{\nabla} \wedge (\vec{\nabla}\varphi) = \vec{0}, \ \forall\varphi$

We can then define a scalar φ such that : $\vec{V} = \vec{\nabla}\varphi$ **φ** is called the velocity potential. In the Cartesian reference frame and considering a plane flow, we can therefore write :

$$\vec{V} = \vec{\nabla}\varphi \Rightarrow \quad u = \frac{\partial\varphi}{\partial x}, \quad v = \frac{\partial\varphi}{\partial y} \quad \textbf{et} \quad w = \frac{\partial\varphi}{\partial z} \qquad \text{(I.39)}$$

If we assume that the fluid is incompressible, we must verify :

$$\vec{\nabla}.\vec{V} = 0 \quad \Rightarrow \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

This leads to the relationship :

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} = 0 \Rightarrow \Delta\varphi = 0 \qquad \textit{Laplace equation}$$

We therefore conclude that the velocity potential must satisfy ***Laplace*'s equation**.

Note:
If the flow is irrotational, the current function must also satisfy ***Laplace*'s equation :**

$$\vec{V} = \begin{cases} \partial\varphi/\partial y \\ -\partial\varphi/\partial x \\ 0 \end{cases} \quad \text{and} \quad \vec{\nabla} \wedge \vec{V} = \vec{0} \Rightarrow \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ 0 \end{pmatrix} \wedge \begin{pmatrix} \partial\varphi/\partial y \\ -\partial\varphi/\partial x \\ 0 \end{pmatrix} = \vec{0}$$

$$\Delta\varphi = 0 \Leftrightarrow -\frac{\partial^2\varphi}{\partial x^2} - \frac{\partial^2\varphi}{\partial y^2} = 0$$

I.9.4.2 Properties of the velocity potential

When a flow is plane, the equation $\varphi(x, y) = C^{te}$ defines, in the plane of the flow, a curve called "**equipotential**".

Along this curve, since $\varphi(x, y) = C^{te}$, we must verify : $d\varphi = 0$

The differential can be written as : $d\varphi = \frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy$

And as along an equipotential $d\varphi = 0$, then :

$$d\varphi = \frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy = 0 \Rightarrow u\,dx + v\,dy = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{u}{v} \qquad \text{(I.40)}$$

So $\frac{dy}{dx} = -\frac{u}{v}$ relationship to be verified at any point on the equipotential.

At any point M(x,y) in the flow plane, the ***streamline and equipotential are orthogonal***.

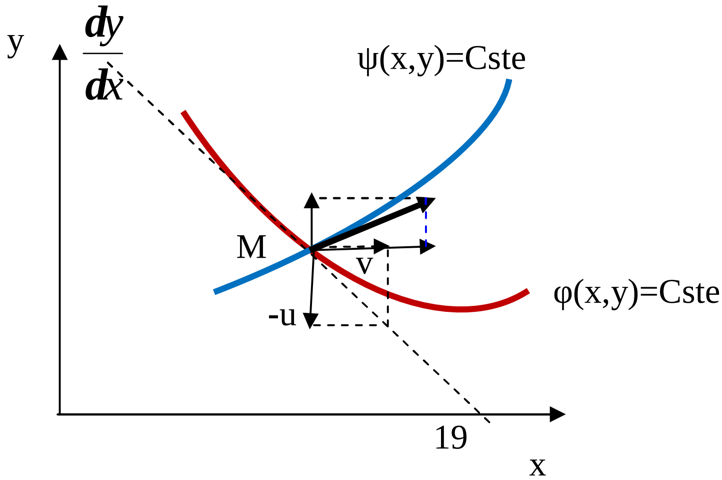


Figure I.11 Qualitative representation of the current line and equipotential in the (x, y) plane

I.9.4.3 Cauchy-Riemann equations

We can conclude from what we have seen above that :

- The velocity potential(φ) exists only for an irrotational flow.
- The current function(ψ) is applied for rotational and irrotational flow (stationary and incompressible).
- In the case of irrotational flow, the current function and velocity potential both satisfy **Laplace's equation.**

  Therefore, for an irrotational and incompressible flow, the following relationship can be verified:

$$\begin{aligned} u &= \frac{\partial \varphi}{\partial x} = \frac{\partial \psi}{\partial y} \\ v &= -\frac{\partial \varphi}{\partial y} = -\frac{\partial \psi}{\partial x} \end{aligned} \qquad (I.41)$$

  These equations are called Cauchy-Riemann equations.

I.9.4.4 Calculating the length of an arc element along a current line

We want to calculate the arc on the current line (ψ(x,y)=cste).

We have : $ds_{\varphi = C^{te}} = \sqrt{dx^2 + dy^2}$

Gold : $$\begin{aligned} d\varphi &= \frac{\partial \varphi}{\partial x} dx + \frac{\partial \varphi}{\partial y} dy \\ &= u\,dx + v\,dy \end{aligned}$$

In addition, along the current line we have **ψ(x,y)=cste ,** i.e.: $\frac{dy}{dx} = \frac{v}{u}$ therefore:

$dy = \frac{v}{u} dx$ by replacing we then obtain

$$d\varphi = u\,dx + \frac{v^2}{u} dx = \frac{u^2 + v^2}{v} dy \qquad (I.42)$$

Hence : $\mathbf{d}y = \frac{v}{u^2 + v^2}\mathbf{d}\varphi$

$\mathbf{d}x = \frac{u}{u^2 + v^2}\mathbf{d}\varphi$

Then :

$$\mathbf{d}s_{\varphi = c^{te}} = \sqrt{\mathbf{d}x^2 + \mathbf{d}y^2} = \sqrt{\frac{u^2 + v^2}{\left(u^2 + v^2\right)^2}\mathbf{d}\varphi^2} = \frac{\mathbf{d}\varphi}{\sqrt{u^2 + v^2}} \qquad (I.43)$$

So : $\mathbf{d}s_{\varphi = c^{te}} = \frac{\mathbf{d}\varphi}{V}$

The distance between two equipotentials is inversely proportional to the flow velocity.

- One of the properties of the current function is that the difference in the current function between two points represents the fluid flow through any line joining the points.
- If two points lie in the same streamline, in this case there is no flow between these two points and therefore $\psi - \psi_{12} = 0$ we then have **ψ(x,y)=cste**
- Similarly, **φ=cste** , represents the case where the velocity potential is the same at each point, and is said to represent an **equipotential line**.

Given two curves **φ=cste** and **ψ=cste ,** these two curves intersect at every point.
At the point of intersection of these curves, the slopes are :

For the curve **φ=cste: slope** $= \frac{\partial y}{\partial x} = \frac{\frac{\partial \varphi}{\partial x}}{\frac{\partial \varphi}{\partial x}} = \frac{u}{v}$

For curve **ψ=cste: slope** $= \frac{\partial y}{\partial x} = \frac{\frac{\partial \psi}{\partial x}}{\frac{\partial \psi}{\partial x}} = \frac{-v}{u} = -\frac{v}{u}$

The product of the slopes of these curves is : $\frac{u}{v} \times -\frac{v}{u} = -1$

This shows that equipotential lines and current lines form an orthogonal network at all points of intersection.

I.10 Flow representation by complex functions

Many classical plane flows can be represented by complex functions. Let
$f(z) = \varphi(x, y) + i\psi(x, y)$ Where z=x+iy is the complex variable associated with the complex potential function f(z) (φ and ψ represent the potential and current functions

respectively). For this function f(z) to be analytic, its derivative must be defined everywhere, i.e.

$\lim_{\delta z\to 0}\left(\frac{\delta f}{\delta z}\right)$ tends towards the same value regardless of how δz tends towards 0.

If we ask: $\delta z \to 0 \Leftrightarrow \begin{cases}\delta x \to 0\\ \delta y = 0\end{cases}$ **ou** $\begin{cases}\delta x = 0\\ \delta y \to 0\end{cases}$

And δz can be made to tend towards 0 in the following two ways:

Therefore :

$$\lim_{\delta z\to 0}\left(\frac{\delta f}{\delta z}\right)=\lim_{\substack{\delta x\to 0\\ \delta y=0}}\left(\frac{\delta\varphi+i\,\delta\psi}{\delta x+i\,\delta y}\right)=\lim_{\substack{\delta x=0\\ \delta y\to 0}}\left(\frac{\delta\varphi+i\,\delta\psi}{\delta x+i\,\delta y}\right)=\frac{\mathbf{d}f}{\mathbf{d}z}$$

$$\lim_{\delta x\to 0}\left(\frac{\delta\varphi}{\delta x}+i\frac{\delta\psi}{\delta x}\right) \qquad , \quad \lim_{\delta y\to 0}\left(-i\frac{\delta\varphi}{\delta y}+\frac{\delta\psi}{\delta y}\right)$$

$$\frac{\partial\varphi}{\partial x}+i\frac{\partial\psi}{\partial x}=\frac{\partial f}{\partial x} \qquad\qquad -i\frac{\partial\varphi}{\partial y}+\frac{\partial\psi}{\partial y}=-i\frac{\partial f}{\partial y}$$

This gives :

$$\frac{\partial\varphi}{\partial x}+i\frac{\partial\psi}{\partial x}=-i\frac{\partial\varphi}{\partial y}+\frac{\partial\psi}{\partial y} \quad \text{hence : } \frac{\partial\varphi}{\partial x}=\frac{\partial\psi}{\partial y} \quad \textbf{et} \quad \frac{\partial\varphi}{\partial y}=-\frac{\partial\psi}{\partial x}$$

This system of equations constitutes the ***Cauchy-Riemann relations*** which verify the relations found above.

Finally, for $f(z)=\varphi(x,y)+i\,\psi(x,y)$ to be an analytic function, $\varphi(x,y)$ and $\psi(x,y)$ must verify these ***Cauchy*** relations**.**

For a plane flow, which can be described by means of a current function $\psi(x,y)$ and a velocity potential $\varphi(x,y)$, these *Cauchy* relations are well verified:

$$u=\frac{\partial\varphi}{\partial x}=\frac{\partial\psi}{\partial y} \quad \textbf{et} \quad v=\frac{\partial\varphi}{\partial y}=-\frac{\partial\psi}{\partial x}$$

Consequently, the flow can also be described by means of the complex analytical function :

$f(z)=\varphi(x,y)+i\,\psi(x,y)$ Where $z=x+iy$

This function is known *as the "**complex velocity potential**"*.

Properties :

We have seen that for a flow to be described by means of a current function ψ and a velocity potential φ, these two functions must verify Laplace's equation (Δψ=0 and Δφ=0).

Let there be two flows such that :

$$\begin{cases}\Delta\varphi_1 = 0 \quad et \quad \Delta\psi_1 = 0 \\ \Delta\varphi_2 = 0 \quad et \quad \Delta\psi_2 = 0\end{cases} \Leftrightarrow \begin{cases} f_1(z) = \varphi_1 + i\psi_1 \\ f_2(z) = \varphi_2 + i\psi_2\end{cases}$$

Since the Laplacian operator is linear, this implies that :

$$\begin{cases}\Delta(\lambda_1\varphi_1 + \lambda_2\varphi_2) = \lambda_1\Delta\varphi_1 + \lambda_2\Delta\varphi_2 \\ \Delta(\lambda_1\psi_1 + \lambda_2\psi_2) = \lambda_1\Delta\psi_1 + \lambda_2\Delta\psi_2\end{cases}$$

We pose : $\begin{cases}\varphi = \lambda_1\varphi_1 + \lambda_2\varphi_2 \\ \psi = \lambda_1\psi_1 + \lambda_2\psi_2\end{cases} \Leftrightarrow \begin{cases}\Delta\varphi = 0 \\ \Delta\psi = 0\end{cases}$

And so: $f(z) = \varphi + i\psi = \lambda_1 f_1(z) + \lambda_2 f_2(z)$, f(z) describes the flow resulting from the superposition of the two flows f_1 and f_2 . Consequently, several elementary flows can be superimposed to create more complex flows, simply by adding the corresponding complex potentials.

I.10.1 Uniform flow

Consider the plane flow modeled by the complex velocity potential :

$$f(z) = Uz$$

We then have : $\varphi(x, y) + i\,\psi(x, y) = U(x + i\,y) = Ux + i\,Uy$

By identification, we obtain :

$$\varphi(x, y) = Ux$$
$$\psi(x, y) = Uy$$

The current lines are such that $\psi(x, y) = Uy = \mathbf{C^{te}}$

$\Leftrightarrow \quad y = \mathbf{C^{te}}\ \forall x$ These are horizontal lines.

The equipotentials are such that : $\varphi(x, y) = Ux = \mathbf{C^{te}}$

$\Leftrightarrow \quad x = \mathbf{C^{te}}\ \forall y$ are vertical straight lines.

Determining the velocity field :

$$\vec{v} = \begin{cases} u = \dfrac{\partial\varphi}{\partial x} = \dfrac{\partial\psi}{\partial y} = U \\ v = \dfrac{\partial\varphi}{\partial y} = -\dfrac{\partial\psi}{\partial x} = 0 \end{cases}$$

The speed is uniform: $\vec{v} = U\vec{e}_x$

Current lines: $\psi(x, y) = Uy = \mathbf{C^{te}} \Leftrightarrow y = \mathbf{C^{te}}\ \forall x$ (horizontal lines)

Equipotential: $\varphi(x, y) = Ux = \mathbf{C^{te}} \Leftrightarrow x = \mathbf{C^{te}}\ \forall y$ (vertical lines)

Speed field : $\vec{V} = U\,\vec{e}_x$

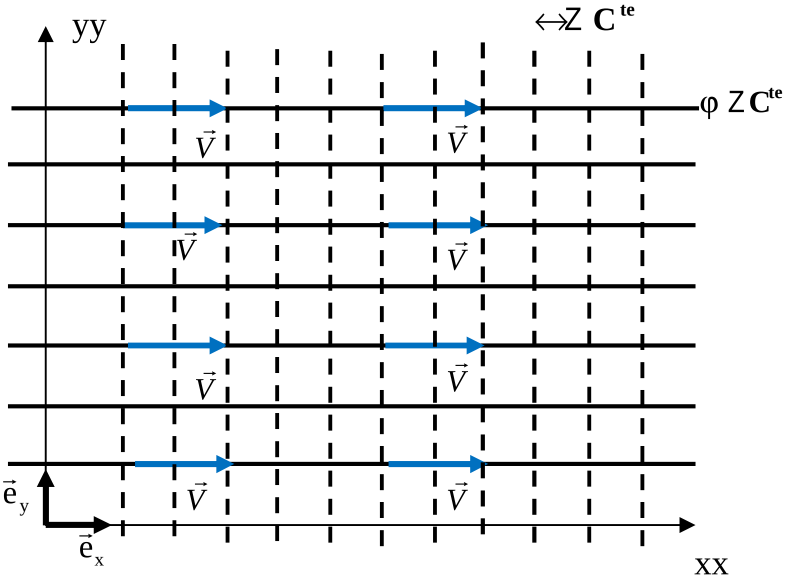


Figure I.12 Uniform flow $f(z) = Uz$

I.10.2 Plane flow around a source or well

Consider the plane flow modeled by the complex velocity potential :

$f(z) = C\,\ln z$ Where $z = x + iy = re^{i\theta}$ and C a real constant.

$\Rightarrow\ f(z) = C\,\ln\left(re^{i\theta}\right) = C\left[\ln r + i\theta\right]$

We can then deduce the current function and velocity potential :

$\varphi(r,\theta) = C\,\ln r$

$\psi(r,\theta) = C\theta$

The current lines are such that : $\psi(r,\theta) = C\theta = C^{te}$

$\Rightarrow\ \theta = C^{te}\ \forall r$ these are straight lines passing through the origin

The equipotentials are such that : $\varphi(r,\theta) = C\,\ln r = C^{te}$

$\Rightarrow\ r = C^{te}\ \forall\theta$ These are concentric circles centered on the origin.

Determining the velocity field :

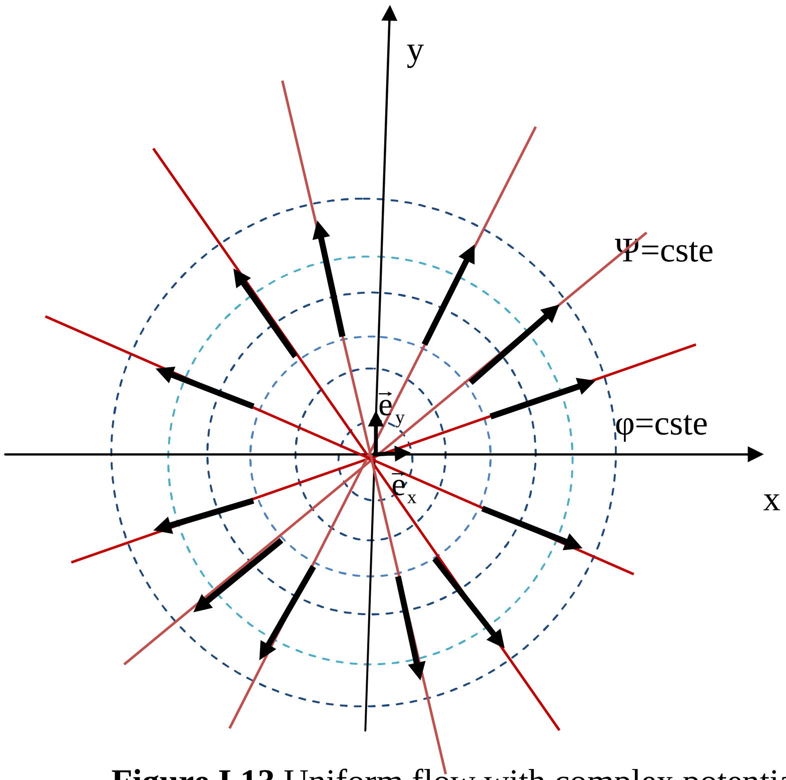


Figure I.13 Uniform flow with complex potential
$f(z) = C\ln z$

$$\vec{v} = \begin{cases} v_r = \dfrac{\partial \varphi}{\partial r} = \dfrac{1}{r}\dfrac{\partial \psi}{\partial \theta} \\ v_\theta = \dfrac{1}{r}\dfrac{\partial \varphi}{\partial \theta} = -\dfrac{\partial \psi}{\partial r} \end{cases}$$

Or : $\vec{v} = \begin{cases} v_r = C/r \\ v_\theta = 0 \end{cases} \Rightarrow \vec{V} = \dfrac{C}{r}\vec{e}_r$

Speed is therefore radial and inversely proportional to distance from the origin.

If C>0, then flow is directed outwards

Divergent flow ***source*** at origin.

If C<0, then the flow is directed towards the origin

Convergent flow ***well*** at origin.

Physical meaning of the constant C :

The volume flow of this radial flow (source or well) is calculated:

$q_v = \oiint_S \vec{V}.\vec{n}\,dS$ Where S is a closed surface surrounding the origin.

$\vec{V} = \frac{C}{r}\vec{e}_r \quad et \quad \vec{n} = \vec{e}_r$

This is a linear flow taking place in the direction ⊥ to the z axis, in the (xy) plane we can consider as the integration surface a cylinder of height Δz=1, and therefore :

$$\oiint_S \ldots dS = \oint_\ell \ldots \Delta z\, d\ell$$

Since the flow is on a plane, we integrate on a circle of any radius r centered on the origin.

$$q_v = \Delta z \oint_\ell \vec{V}.\vec{n}\, r\, d\theta = \Delta z\, r \int_0^{2\pi} \vec{V}.\vec{n}\, d\theta \quad \text{where} \quad \begin{cases} \vec{V} = C/r\, \vec{e}_r \\ \vec{n} = \vec{e}_r \end{cases}$$

$$\Rightarrow q_v = \Delta z\, r \int_0^{2\pi} \frac{C}{r} d\theta = \Delta z\, r \frac{C}{r} \int_0^{2\pi} d\theta = 2\pi C \Delta z$$

1

volumetric flow rate per unit height

$\Rightarrow C = \frac{q_v}{2\pi}$ and therefore: $f(z) = \frac{q_v}{2\pi} \ln z$ **$q_v > 0$**: source flow rate

$q_v < 0$: well flow rate

I.10.3 Vortex or free vortex

Consider the plane flow modeled by the complex velocity potential :

$f(z) = -iC \ln z$ where $z = x + iy = re^{i\theta}$ and C a real constant.

$$\Rightarrow f(z) = -iC \ln\left(re^{i\theta}\right) = -iC\left[\ln r + i\theta\right] = C\theta - iC \ln r$$

We can then deduce the current function and the velocity potential :

$$\begin{cases} \varphi(r,\theta) = C\theta \\ \psi(r,\theta) = -C \ln r \end{cases}$$

The current lines are such that : $\psi(r,\theta) = -C \ln r = C^{te}$

$\Rightarrow \quad r = C^{te} \ \forall\theta$ These are concentric circles centered on the origin.

The equipotentials are such that : $\varphi(r,\theta) = C\theta = C^{te}$

$\Rightarrow \quad \theta = C^{te} \ \forall r$ These are straight lines passing through the origin.

Determining the velocity field :

$$\vec{V} = \begin{cases} v_r = \dfrac{\partial \varphi}{\partial r} = \dfrac{1}{r}\dfrac{\partial \psi}{\partial \theta} \\ v_\theta = \dfrac{1}{r}\dfrac{\partial \varphi}{\partial \theta} = -\dfrac{\partial \psi}{\partial r} \end{cases}$$

Or : $\vec{V} = \begin{cases} v_r = 0 \\ v_\theta = \dfrac{C}{r} \end{cases} \Rightarrow \vec{V} = \dfrac{C}{r}\vec{e}_\theta$

Velocity is therefore ortho-radial and inversely proportional to distance from the origin.

If **C>0**, then the flow is around the origin in the ***trigonometric direction***.

If **C<0**, then the flow is ***clockwise*** around the origin.

Physical meaning of the constant C :

Let's calculate the velocity "circulation" around the origin:

$\Gamma = \oint_{\ell} \vec{V}.\mathbf{d}\vec{\ell}$ Where runs an arbitrary current line, *i.e.* a circle of radius *r*.

With: $\vec{V} = \dfrac{C}{r}\vec{e}_\theta$ and $\boldsymbol{d}\vec{\ell} = r\,\boldsymbol{d}\theta\,\vec{e}_\theta \Rightarrow \quad \Gamma = \int_0^{2\pi} \dfrac{C}{r} r\,\mathbf{d}\theta = 2\pi C$

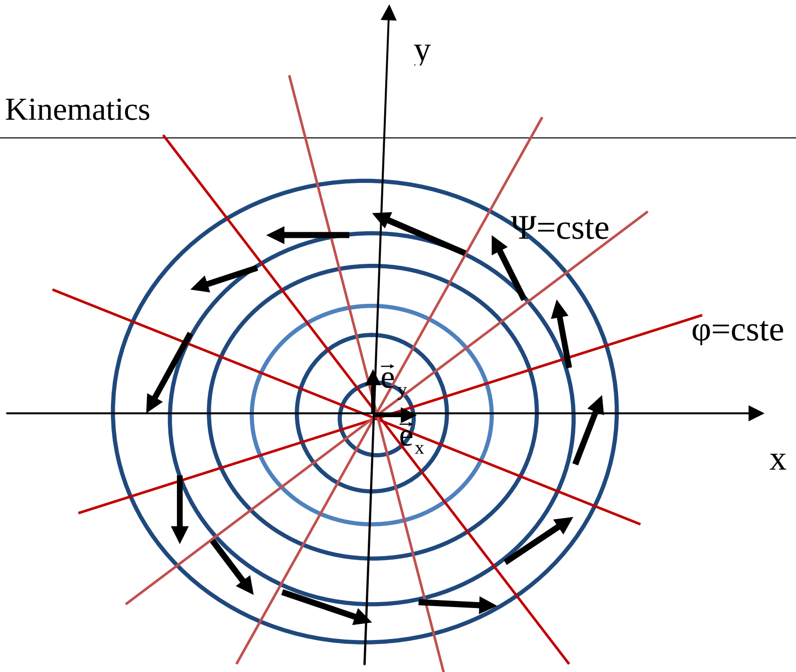


Figure I.14 Uniform flow with complex potential
$f(z) = -iC\ln z$

So $C = \dfrac{\Gamma}{2\pi}$ and therefore $f(z) = -i\dfrac{\Gamma}{2\pi}\ln z$ where Γ is the vortex **circulation** (free vortex).

If $\Gamma > \mathbf{0}$, the vortex rotates in the **trigonometric direction**.

If $\Gamma < \mathbf{0}$, the vortex rotates **clockwise**.

I.10.4 Corners and stopping points

A "**stopping point" is** a point where the speed is zero.

Consider the plane flow modeled by the complex velocity potential :

$f(z) = \mathbf{C} z^{m+1}$ Where $m \geq -\dfrac{1}{2}$

In cylindrical coordinates: $z = r e^{i\theta}$ and therefore $f(z) = \mathbf{C} r^{m+1} e^{i(m+1)\theta}$

Then we have : $\begin{cases} \psi(r,\theta) = \mathbf{C} r^{m+1}\cos[(m+1)\theta] \\ \varphi(r,\theta) = \mathbf{C} r^{m+1}\sin[(m+1)\theta] \end{cases}$

The velocity field is obtained by : $\vec{V} = \begin{cases} v_r = \dfrac{\partial\psi}{\partial r} = \dfrac{1}{r}\dfrac{\partial\varphi}{\partial\theta} \\ v_\theta = \dfrac{1}{r}\dfrac{\partial\psi}{\partial\theta} = -\dfrac{\partial\varphi}{\partial r} \end{cases}$

We find :

$$\vec{V} = \begin{cases} v_r = \mathbf{C}(m+1) r^m \cos[(m+1)\theta] \\ v_\theta = -\mathbf{C}(m+1) r^m \sin[(m+1)\theta] \end{cases}$$

Note that $v_r = v_\theta = 0$ for $r = 0$ the **origin is the stopping point**.

The current line passing through the stop point must therefore verify :

$$\varphi(r,\theta) = C^{te} = \varphi_A \qquad \text{Where} \qquad \varphi_A = \varphi(r_A,\theta_A) = C\, r_A^{m+1} \sin[(m+1)\theta_A] = 0$$

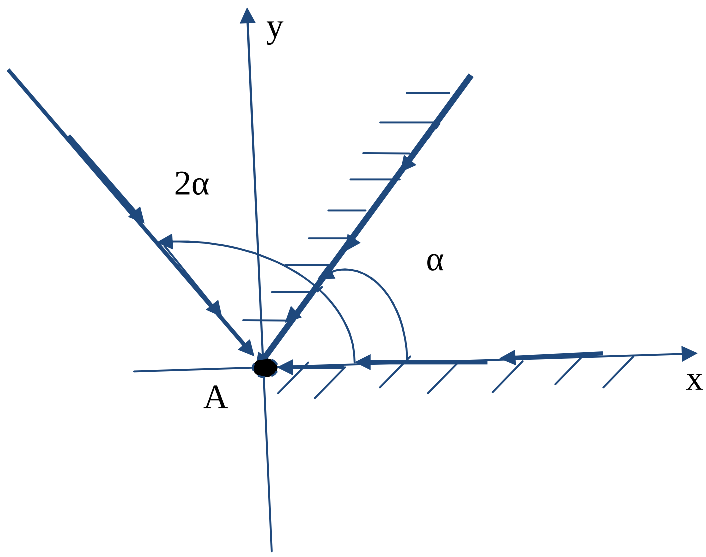
The equation for this current line is then written :

$$C\, r^{m+1} \sin[(m+1)\theta] = 0 \Rightarrow \begin{cases} r = 0 \ \forall\theta & \leftarrow \text{Stop } \textit{point} \\ \sin[(m+1)\theta] = 0 \ \forall r \end{cases}$$

$$\theta = \frac{n\pi}{(m+1)} \ \forall r \Rightarrow (m+1)\theta = n\pi \ \forall r$$

if $n=0$: $\theta = 0 \ \forall r$ half-right Ax

Since current lines can be likened to impassable barriers, those passing through the stopping point form "corners": these are the ***stopping corners***.



Let's now analyze the fluid flow between these stop wedges for a few specific values of *m*.

$$f(z) = C\, z^{m+1} \text{ where } m \geq -\frac{1}{2}$$

- **Case where m=1**

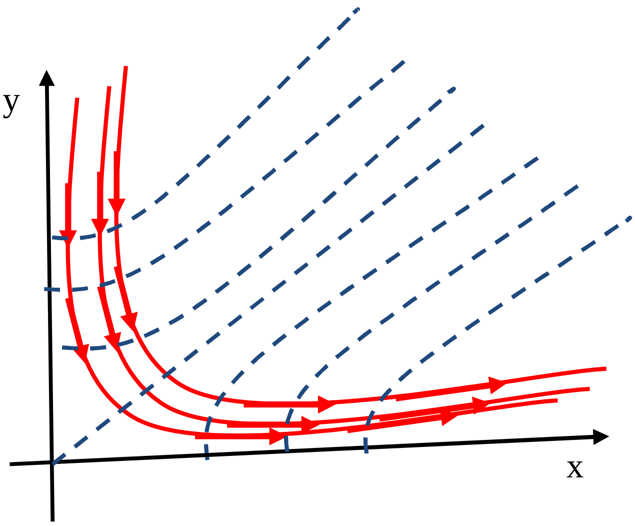
$$\psi(r,\theta) = C\,r^2 \sin(2\theta) = C^{te} \qquad \text{And } \alpha = \frac{\pi}{m+1} = \frac{\pi}{2} \Rightarrow \textit{right-angle corner}$$

$$\Rightarrow \quad \psi(r,\theta) = 2C\,r^2 \sin\theta \cos\theta = 2C \underbrace{r\sin\theta}_{y}\,\underbrace{r\cos\theta}_{x} = C^{te}$$

$$\psi(r,\theta) = C^{te} \quad \Leftrightarrow \quad 2C\,xy = C^{te}$$

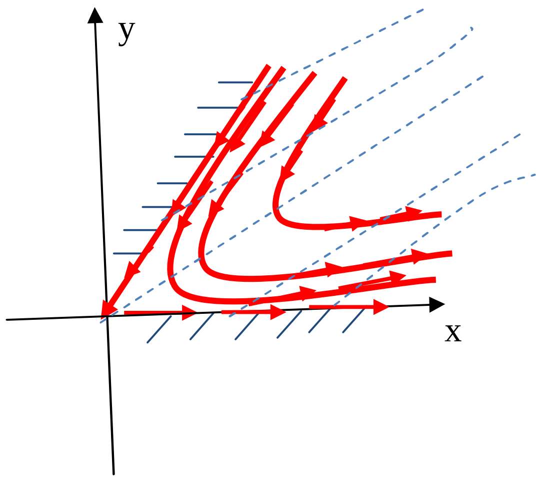
$y = \dfrac{C^{te}}{x}$ ***inside this corner, the current lines are hyperbolas***

As equipotentials are $\perp$ at all points, they are also hyperbolas.

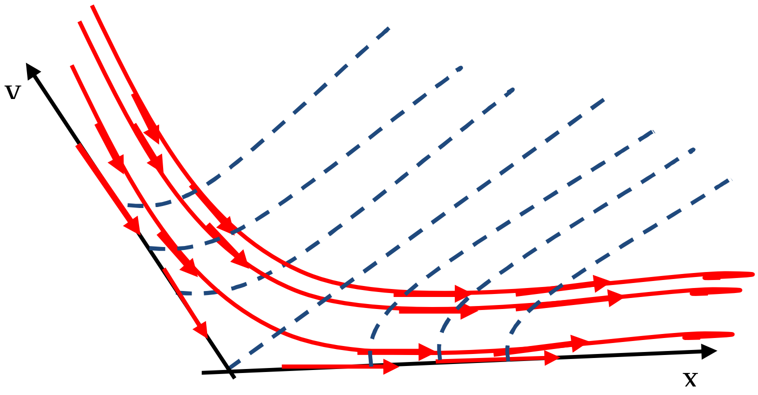


- **Case where** m>1

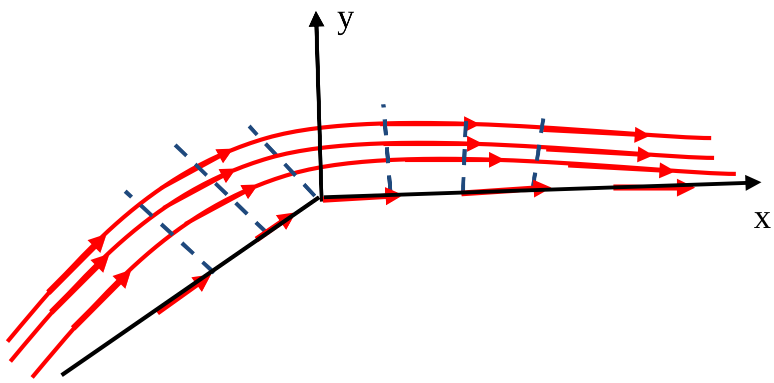
$$\alpha = \frac{\pi}{m+1} < \frac{\pi}{2}$$



- **Case where 0<m<1** $\dfrac{\pi}{2} < \alpha = \dfrac{\pi}{m+1} < \pi$



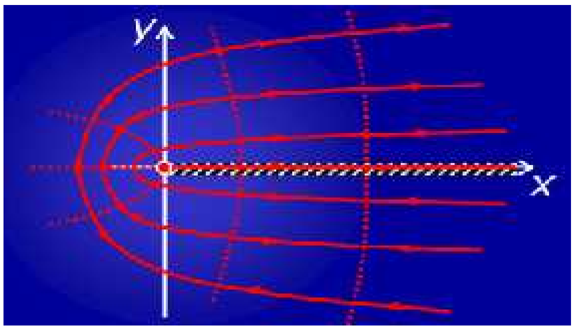
➢ **Case where** J $\frac{1}{2}$ **<m<0,** ® Y ~ Z $\frac{®}{m\,H\,1}$ Y 2®



mZ J$\frac{1}{2}$

~ Z 2®

➢ Where



I.10.5 Doublet and dipole

We know that for a flow to be described by a current function and a velocity potential, both functions must satisfy *Laplace*'s equation :

$\Delta\psi = 0$ And $\Delta\varphi = 0 \Rightarrow f(z) = \varphi + i\psi$

Let's consider 2 flows such as :

1) $\Delta\psi_1 = 0$ and $\Delta\varphi_1 = 0 \Rightarrow f_1(z) = \varphi_1 + i\psi_1$

2) $\Delta\psi_2 = 0$ and $\Delta\varphi_2 = 0 \Rightarrow f_2(z) = \varphi_2 + i\psi_2$

Since *Laplace*'s equation is linear :

$\Delta(\lambda_1\varphi_1 + \lambda_2\varphi_2) = \lambda_1\Delta\varphi_1 + \lambda_2\Delta\varphi_2 = 0$

$\Delta(\lambda_1\psi_1 + \lambda_2\psi_2) = \lambda_1\Delta\psi_1 + \lambda_2\Delta\psi_2 = 0$

So if we put $\varphi = \lambda_1\varphi_1 + \lambda_2\varphi_2$ and $\psi = \lambda_1\psi_1 + \lambda_2\psi_2$ then :

$\Delta\psi = 0$ And $\Delta\varphi = 0 \Rightarrow f(z) = \varphi + i\psi = \lambda_1 f_1(z) + \lambda_2 f_2(z)$

Consequently, *f(z)* describes the flow resulting from the superposition of the two flows f_1 and f_2

Several elementary flows can therefore be superimposed to create more advanced flows, simply by adding the corresponding complex potentials.

I.10.6 Association of a source and a well :

Let's consider a source with flow rate *+q*, located at *x=a*, onto which we superimpose a sink with flow rate *-q*, located at *x=-a*.

The resulting complex potential is written as :

$$f(z) = +\frac{q}{2\pi}\ln(z-a) - \frac{q}{2\pi}\ln(z+a) \quad \text{Let's say : } \begin{cases} z_1 = z - a = r_1\mathbf{e}^{i\theta_1} \\ z_2 = z + a = r_2\mathbf{e}^{i\theta_2} \end{cases}$$

Hence :

$$f(z) = \frac{q}{2\pi}\left[\ln z_1 - \ln z_2\right] = \frac{q}{2\pi}\left[\ln r_1 + i\theta_1 - \ln r_2 - i\theta_2\right]$$

$$\Rightarrow \quad f(z) = \frac{q}{2\pi}\left\{\ln\frac{r_1}{r_2} + i(\theta_1 - \theta_2)\right\} \Rightarrow \begin{cases} \varphi = \frac{q}{2\pi}\ln\frac{r_1}{r_2} \\ \psi = \frac{q}{2\pi}(\theta_1 - \theta_2) \end{cases}$$

$$\begin{cases} \varphi = \frac{q}{2\pi}\ln\frac{r_1}{r_2} \\ \psi = \frac{q}{2\pi}(\theta_1 - \theta_2) \end{cases}$$

Therefore, the current lines are such that :

$$\psi = \frac{q}{2\pi}(\theta_1 - \theta_2) = \mathbf{C^{te}}$$

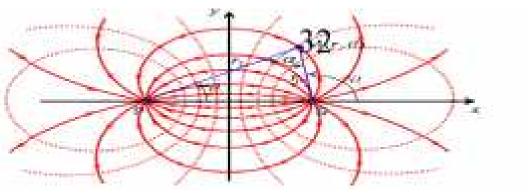


Figure I.15 Current lines for a source and a sink

Let's extend the distance between the well and the source to 0.

$$f(z) = +\frac{q}{2\pi}\ln(z-a) - \frac{q}{2\pi}\ln(z+a) = \frac{q}{2\pi}\ln\left[\frac{z-a}{z+a}\right] = \frac{q}{2\pi}\ln\left[\frac{z(1-a/z)}{z(1+a/z)}\right]$$

$$f(z) = \frac{q}{2\pi}\ln\left[\frac{1-a/z}{1+a/z}\right] \quad \text{where} \quad \frac{1}{1+a/z} \xrightarrow[a \to 0]{} 1-a/z$$

So $f(z) \approx \frac{q}{2\pi}\ln\left[(1-a/z)^2\right] = \frac{q}{2\pi}2\ln(1-a/z) \approx \frac{q}{2\pi}2\left[-\frac{a}{z}\right] \approx \frac{q}{2\pi}2\left[-\frac{a}{z}\right]$

Let *2aq* = *p* be the dipole moment : $f(z) = -\frac{1}{2\pi}\frac{p}{z}$

$$f(z) = -\frac{1}{2\pi}\frac{p}{z} = -\frac{1}{2\pi}\frac{p}{re^{i\theta}} = -\frac{1}{2\pi}\frac{p}{r}e^{-i\theta} = -\frac{1}{2\pi}\frac{p}{r}(\cos\theta - i\sin\theta) = \varphi + i\psi$$

Hence $\begin{cases} \varphi = -\frac{1}{2\pi}\frac{p}{r}\cos\theta \\ \psi = \frac{1}{2\pi}\frac{p}{r}\sin\theta \end{cases}$ → $\varphi = C^{te} \Rightarrow \frac{1}{2\pi}\frac{p}{r}\sin\theta = C^{te}$

Current line equation

$$\Rightarrow \frac{1}{r}\sin\theta = C^{te} \Rightarrow r\sin\theta = C^{te} r^2 \Rightarrow y = C^{te}(x^2+y^2) \Rightarrow y = C^{te}(x^2+y^2)$$

$$\Rightarrow x^2+y^2 - C^{te}y = 0 \Rightarrow x^2+y^2 - Ky = 0 \Rightarrow x^2 + (y-K/2)^2 = (K/2)^2$$

$\psi = C^{te} \Rightarrow$ equation of a circle with center (0,K/2) and radius K/2

Current lines are circles all centered on the y axis, and all passing through the origin.

Flow generated by a dipole $f(z) = -\frac{1}{2\pi}\frac{p}{z}$

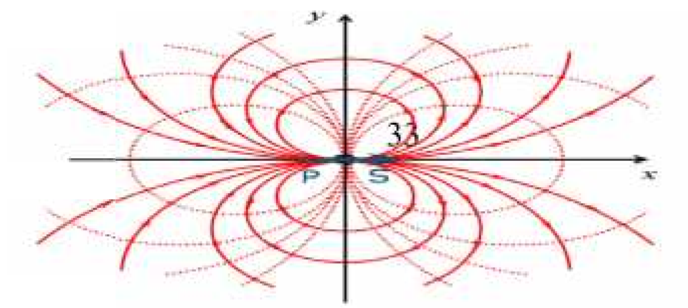


Figure I.16 Current lines for a dipole

I.10.7 Uniform flow around a circular cylinder with circulation

Let's consider a uniform flow around a circle in the presence of a circulation Γ centered at the origin. The complex potential function is written: $f(z) = V_0\left(z + \frac{a^2}{z}\right) - i\frac{\Gamma}{2\pi}\ln z$

In view of the logarithmic singularity, the complex plane will be equipped with the half-axis cutoff

$$x \geq 0$$

The complex velocity of this flow is expressed as:

$V(z) = V_0\left(1 - \frac{a^2}{z^2}\right) - i\frac{\Gamma}{2\pi z}$ It cancels out at points with affixes z_A such that :

$$z_A^2 - i\frac{\Gamma}{2\pi V_0}z_A - a^2 = 0$$

This provides two stopping points:

$$z_A = z_{A'} = i\frac{\Gamma}{2\pi V_0} \pm \frac{1}{2}\sqrt{4a^2 - \frac{\Gamma^2}{4\pi^2 V_0^2}}$$

There are several cases depending on the descriminant:

- 1^{er} $0 < \Gamma < 4\pi a V_0$ Cases

The discriminant is then positive and the affixes of the two points have the same modulus:

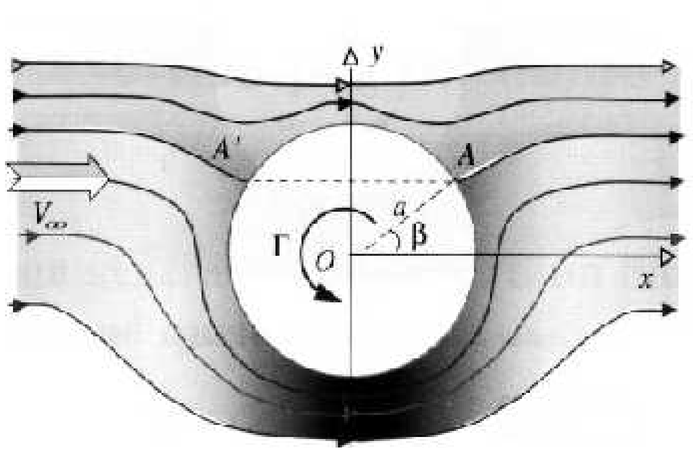


Figure I.17 Flow around a cylinder with circulation (low circulation)

$$\sqrt{x_A^2 + y_{A'}^2} = \sqrt{\frac{16\pi^2 a^2 V_0^2 - \Gamma^2}{16\pi^2 V_0^2} + \frac{\Gamma^2}{16\pi^2 V_0^2}} = a$$

The two stopping points are therefore on the circle of radius a, in symmetrical positions with respect to axis Oy. They are marked by the polar angles β and $\pi-\beta$ respectively with:

$$\sin\beta = \frac{\Gamma}{4\pi a V_0}$$

The general flow configuration is shown in figure (I.17).

Without traffic, there are two stopping points at the intersection of the circle and the real axis. We can therefore see that the influence of traffic is equivalent to shifting the two stopping points symmetrically with respect to Oy by an ordinate proportional to the value of the traffic.

- 2ème $\Gamma = 4\pi a V_0$ Cases

For this critical traffic value, the two stop points merge with the intersection of the circle and the Oy axis($\beta=\beta'=\pi/2$). This gives the configuration shown in figure (I.18).

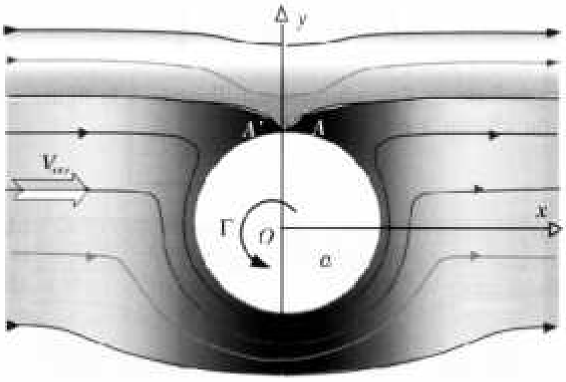


Figure I.18 Flow around a cylinder with circulation (critical circulation)

- 3ème $\Gamma > 4\pi a V_0$ Cases

The discriminant of the equation of the affixes of the stopping points is then negative, so the roots take the form:

$$z_A = z_{A'} = i\left[\Gamma \pm \sqrt{\Gamma^2 - 4a^2\pi^2 V_0^2}\right]/4\pi V_0$$

These are pure imaginary, which means that the two stopping points are on the oy axis. The product of the roots is worth in modulus a^2 . This leads to the configuration shown in figure(I.19).

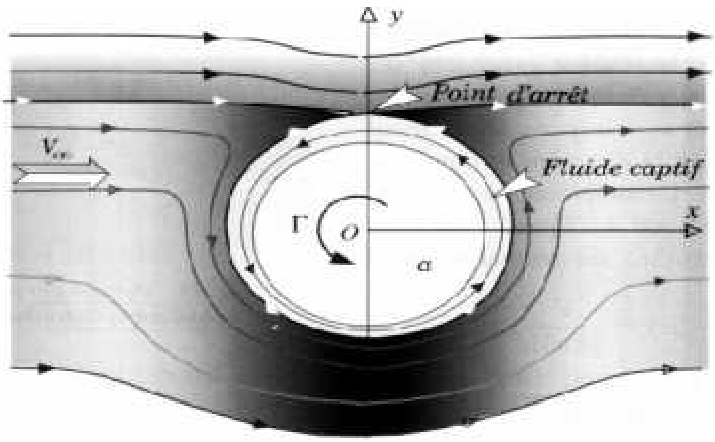


Figure I.19 Flow around a cylinder with circulation (strong circulation)