1 Tringular norms and triangular conorms

2 Résumé

Les normes triangulaires sont des outils indispensables pour l'interprétation des conjonctions et disjonctions dans la logique floue. Par la suite, pour l'intersection des ensembles flous. Ce sont cependant des objets mathématiques intéressants pour eux-mêmes. Les normes triangulaires, telles que nous les utilisons aujourd'hui, jouent également un rôle important dans la prise de décision.

Dans cet aperçu on étudie quelques aspects algébriques, analytiques et logiques des normes triangulaires.

3 Tringular norms

3.1 Basic definitions and properties

Definition 1 *A triangular norm (t-norm for short) is a binary operation T on the unit interval $[0,1]$, i.e., it is a function $T : [0,1]^2 \to [0,1]$ such that for all $x, y, z \in [0,1]$: the following four axioms are satisfied:*

(T1) $T(x,y) = T(y,x)$. *(commutativity)*

(T2) $T(x, T(y,z)) = T(T(x,y), z)$. *(associativity)*

(T3) $T(x,y) \leq T(x,z)$ *whenever* $y \leq z$ *(monotonicity)*

(T4) $T(x,1) = x$. *(boundary condition)*

Example 2

The following are the four basic t-norms T_M, T_P, T_L, and T_D given by, respectively:

$T_M(x,y) = \min(x,y)$	*(Minimum)*
$T_P(x,y) = x \cdot y$	*(Product)*
$T_L(x,y) = max(x+y-1, 0)$	*(Łukasiewicz t-norm)*
$T_D(x,y) = \begin{cases} 0 & if\ (x,y) \in [0,1[^2 \\ \min(x,y) & otherwise \end{cases}$	*(Drastic product)*

Example 3

$T(x,y) = \dfrac{xy}{(2-x-y+xy)}$	*Einstein*
$T(x,y) = \dfrac{xy}{(x+y-xy)}$	*Hamacher*
$T(x,y) = \dfrac{xy}{\max(x,y,\alpha)}$	*Dubois and Parade (*1986*)* $\alpha \in [0,1]$

Proposition 4

Any t-conorm T *satisfies* $T(0,x) = T(x,0) = 0$*, for all* $x \in [0,1]$*.*

Proof.

We know that $T(x,0) \in [0,1]$, so $T(x,0) \geq 0$, and we use the axiom (S3)(monotonicity), we obtient $T(x,0) \leq T(1,0) = 0$. ■

Proposition 5

Let A *be a set with* $]0,1[\subseteq A \subseteq [0,1]$*, and assume that* $F : A^2 \to A$ *is a binary operation on* A *such that for all* $x,y,z \in A$ *the properties (T1) - (T3) and*

$F(x,y) \leq \min(x,y)$ *(*)*

are satisfied. Then the function $T : [0,1]^2 \to [0,1]$ *defined by*

$$T(x,y) = \begin{cases} F(x,y) & \text{if } (x,y) \in (A\backslash\{1\})^2, \\ \min(x,y) & \text{otherwise.} \end{cases}$$

is a t-norm.

Proof.

The commutativity (T1) and the boundary condition (T4) are satisfied by definition. Concerning the associativity (T2), observe that for $x,y,z \in A\backslash\{0,1\}$ we have $T(T(x,y),z) = T(x,T(y,z))$ as aÂ· consequence of the associativity of F, If $0 \in \{x,y,z\}$ then we get $T(x,T(y,z)) = 0 = T(T(x,y),z)$, and if $1 \in \{x,y,z\}$then T(T(x,y), z) = T(x,T(y ,z)) follows from (T4). Concerning the monotonicity (T3), suppose $y \leq z$. In the cases $x,y,z \in A\backslash\{1\}$ or $x \in \{0,1\}$ or $y = 0$, the inequality $T(x,y) \leq T(x,z)$ is inherited from the monotonicity of F and min. The only non-trivial case is when $x,y \in A\backslash\{1\}$ and $z = 1$, in which case $T(x,y) \leq T(x,z)$ follows from (*). ■

Definition 6 *A function* $f : [0,1]^2 \to [0,1]$ *which satisfies, for all* $x,y,z \in [0,1]$*, the properties (T1)- (T3) and* $f(x,y) \leq \min(x,y)$ *is called a t-subnorm.*

Example 7

1- $f(x,y) = 0.$

2- $f(x,y) = \dfrac{x.y}{3}.$

3- $f(x,y) = x.y.$

Remark 8

Clearly, each t-norm is a t-subnorm, but not vice versa: for example, the function $f : [0,1]^2 \to [0,1]$ *given by* $f(x,y) = 0$*, is a t-subnorm but not a t-norm because (T4) not satisfies* ($f(x,1) = 0 \neq x$).

Corollary 9

If f is a t-subnorm then the function $T : [0,1]^2 \to [0,1]$ *defined by*

$$T(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in [0,1[^2, \\ \min(x,y) & \text{otherwise,} \end{cases}$$

is a triangular norm.

3.1.1 Comparison of t-norms

Definition 10

(i) If, for two t-norms T_1 *and* T_2*, the inequality* $T_1(x,y) \leq T_2(x,y)$ *holds for all* $(x,y) \in [0,1]^2$*, then we say that* T_1 *is weaker than* T_2 *or, equivalently, that* T_2 *is stronger than* T_1*, and we write in this case* $T_1 \leq T_2$.

(ii) We shall write $T_1 < T_2$ *whenever* $T_1 \leq T_2$ *and* $T_1 \neq T_2$*, i.e., if* $T_1 \leq T_2$ *and for some* $(x_0, y_0) \in [0,1]^2$ *we have* $T_1(x_0,y_0) < T_2(x_0,y_0)$

Lemma 11

(i) The minimum T_M *is the strongest t-norm* ($T_M \geq T$).

(ii) The drastic product T_D *is the weakest t-norm* ($T_D \leq T$).

Proof.

(i) For each t-norm T and for each $(x,y) \in [0,1]^2$ we have both $T(x,y) \leq T(x,1) = x$ and $T(x,y) \leq T(1,y) = y$, so $T(x,y) \leq \min(x,y) = T_M(x,y)$.

(ii) All t-norms coincide on the boundary of $[0,1]^2$ and for all $(x,y) \in]0,1[^2$ we trivially have $T(x,y) \geq 0 = T_D(x,y)$.

■

Example 12

- $T_0\,(x,y) = \begin{cases} 0 \quad \textit{if} \ \ (x,y) \in [0,1[^2, \\ \min(x,y) \quad \textit{otherwise.} \end{cases}$ *(Drastic product of weber).*

- $T_1(x,y) = \max(x+y-1,0)$ *(Łukasiewicz).*

- $T_{1.5}(x,y) = \dfrac{xy}{2-x-y+xy}$ *(Einstein).*

- $T_2(x,y) = xy$ *(Algebraic or probaliste).*

- $T_{2.5}(x,y) = \dfrac{xy}{x+y-xy}$ *(Hamacher).*

- $T_3(x,y) = \min(x,y)$ *(Zadeh).*

We have: $T_0 \leq T_1 \leq T_{1.5} \leq T_2 \leq T_{2.5} \leq T_3$.

Definition 13 *(Domination of t-norm)*
Let T_1 *and* T_2 *be two t-norms. Then we say that* T_1 *dominates* T_2 *(in symbols* $T_1 \gg T_2$*) if for all* $x,y,u,v \in [0,1]$

$$T_1(T_2(x,y),T_2(u,v)) \geq T_2(T_1(x,u),T_1(y,v)). \qquad \text{((Equ 1))}$$

Lemma 14

(i) For each t-norm T *we have* $T_M \gg T$ *and* $T \gg T_D$.

(ii) If for two t-norms T_1 *and* T_2 *we have* T_1 *dominates* T_2 *(* $T_1 \gg T_2$ *) then,* T_1 *,is stronger than* T_2 *(* $T_1 \geq T_2$ *).*

(iii) The relation $\gg$ *on the set of all t-norms is reflexive and antisymmetric.*

Proof.

(i) Trivially, (par separation des cas)

(ii) If for two t-norms T_1 and T_2 we have $T_1 \gg T_2$ then, putting $y = u = 1$ in (Equ 1), we immediately see that also $T_1 \geq T_2$ holds.

(iii) from the commutativity (T_1) and the associativity (T_2) we obtain for each t-norm T and all $x, y, u, v \in [0,1]$ the equality $T(T(x,y), T(u,v)) = T(T(x,u), T(y,v))$, ($T(T(x,y), T(u,v)) = T(x, T(y, T(u,v)) = T(x, T(T(y,u), v)) = T(x, T(T(u,y), v) = T(x, T(u, T(y,v)) = T(T(x,u), T(y,v)))$. i.e., $T \gg T$, and the assumptions $T_1 \gg T_2$ and $T_2 \gg T_1$ imply, as a consequence of (ii), $T_1 = T_2$.

■

Remark 15

The converse is false: $T_1 \geq T_2$ *does not imply* $T_1 \gg T_2$.
consider the t-norm T_P *and the t norm* T *given by:*

$$T(x,y) = \begin{cases} \dfrac{xy}{2} & \text{if } (x,y) \in [0,1[^2, \\ \min(x,y) & \text{otherwise,} \end{cases}$$

we have $T_P \geq T$ *but* $T_P \gg T$ *is false.*
let $(x,y) \in [0,1]^2$ *if* $(x,y) \in [0,1[^2$ *hence* $T_P = xy > \dfrac{xy}{2} = T(x,y)$
if $\max(x,y) = 1$ $T_P(x,y) = \min(x,y) = T(x,y)$.
So $\forall (x,y) \in [0,1]^2$ *we have* $T_P(x,y) \geq T(x,y)$ *i,e,.* $T_P \geq T$
but $T_P(T(x,y), T(u,v)) \ngeq T(T_P(x,u), T_P(y,v))$,
because if $(x,y) \in [0,1[^2$ *and* $(u,v) \in [0,1[^2$ *we get* $T_P(Tx,y), T(u,v)) = \dfrac{xyuv}{4}$ *and* $T(T_P(x,u), T_P(y,v)) = \dfrac{xyuv}{2}$.

Proposition 16

(i) The only t-norm T satisfying $T(x,x) = x$ *for all* $x \in [0,1]$ *is the minimum* T_M.

(ii) The only t-norm T satisfying $T(x,x) = 0$ *for all* $x \in [0,1[$ *is the drastic product* T_D.

Proof.

(i) If for a t-norm T we have $T(x,x) = x$ for each $x \in [0,1]$, then for all $(x,y) \in [0,1]^2$ with $y \leq x$ the monotonicity (T3) implies $y = T(y,y) \leq T(x,y) \leq T_M(x,y) = y$,
which, together with (T1), means $T = T_M$.

(ii) Assume $T(x,x)=0$ for each $x\in[0,1[$. Then for all $(x,y)\in[0,1[^2$ with $y\leq x$ we have $0\leq T(x,y)\leq T(x,x)=0$, hence, together with (T1) and (T4), yielding $T=T_D$.

■

4 Triangular conorms

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4.1 Basic definitions and properties

Definition 17 *A triangular conorm (t-conorm for short) is a binary operation S on the unit interval $[0,1]$, i.e., it is a function $S:[0,1]^2\to[0,1]$ such that for all $x,y,z\in[0,1]$: the following four axioms are satisfied:*

(S1) $S(x,y)=S(y,x)$. *(commutativity)*

(S2) $S(x,S(y,z))=S(S(x,y),z)$. *(associativity)*

(S3) $S(x,y)\leq S(x,z)$ *whenever* $y\leq z$ *(monotonicity)*

(S4) $S(x,0)=x$. *(boundary condition)*

Example 18

The following are the four basic t-norms S_M, S_P, S_L, and S_D given by, respectively:

$S_M(x,y)=\max(x,y)$	*(maximum)*
$S_P(x,y)=x+y-x\cdot y$	*(probabilistic sum)*
$S_L(x,y)=min(x+y,1)$	*(Lukasiewicz t-conorm, bounded sum)*
$S_D(x,y)=\begin{cases}1 & if\ (x,y)\in]0,1]^2\\ \max(x,y) & otherwise\end{cases}$	*(drastic sum)*

Example 19

$T(x,y)=\dfrac{x+y}{(1+xy)}$	*Einstein*
$T(x,y)=\dfrac{x+y-2xy}{(1-xy)}$	*Hamacher*
$T(x,y)=\dfrac{x+y+xy-\min(x,y,1-\alpha)}{\max(1-\alpha,1-y,\alpha)}$	*Dubois and Parade (*1986*)* $\alpha\in[0,1]$

Proposition 20

Any t-conorm S satisfies $S(1,x) = S(x,1) = 1$, for all $x \in [0,1]$.

Proof.

We know that $S(x,1) \in [0,1]$, so $S(x,1) \leq 1$, and we use the axiom (S3)(monotonicity), we obtient $S(x,1) \geq S(0,1) = 1$. ■

Proposition 21

A function $S : [0,1]^2 \to [0,1]$ is a t-conorm if and only if there exists a t-norm T such that for all $(x,y) \in [0,1]^2$

$$S(x,y) = 1 - T(1-x, 1-y). \qquad (*)$$

Proof.

If T is a t-norm then obviously the operation S defined by (*) satisfies (S1)- (S3) and (S4)

(S_1) $S(x,y) = 1 - T(1-x,1-y) = 1 - (1-y,1-x) = S(y,x)$,

(S_2) $S(x, S(y,z)) = 1 - T(1-x, 1-S(y,z)) = 1 - T(1-x, 1-(1-T(1-y, 1-z))) = 1 - T(1-x, T(1-y,1-z))$,
$S(S(x,y),z) == 1 - T(1-s(x,y), 1-z) = 1 - T(1-(1-T(1-x,1-y)), 1-z) = 1 - T(T(1-x,1-y),1-z) = 1 - T(1-x, T(1-y,1-z))$,

(S_3) $S(x,y) = 1 - T(1-x,1-y) \leq 1 - T(1-x,1-z) = S(x,z)$ whenever $y \leq z$,

(S_4) $S(x,0) = 1 - T(1-x,1) = 1 - (1-x) = x$,

and is, therefore, a t-conorm. On the other hand, if S is a t-conorm, then define the function $T : [0,1]^2 \to [0,1]$ by

$$T(x,y) = 1 - S(1-x, 1-y), \qquad (**)$$

Again, it is trivial to T is a t-norm and that (*) holds. ■

Remark 22

(i) The t-conorm given by (*) is called the dual t-conorm of T and, analogously, the t-norm given by (**) is said to be the dual t-norm of S.

(ii) *The proof of* ***Proposition 21*** *makes it clear that also each t-norm is the dual operation of some t-conorm. Note that* (T_M, S_M), (T_P, S_P), (T_L, S_L), *and*(T_D, S_D) *are pairs of t-norms and t-conorms which are mutually dual to each other.*

Definition 23 *Let T be a t-norm and S be a t-conorm. Then we say that T is distributive over S if for all $x, y, z \in [0,1]$*

$$T\left(x, S\left(y, z\right)\right) = S\left(T\left(x, y\right), T\left(x, z\right)\right).$$

and that S is distributive over T if for all $x, y, z \in [0,1]$

$$S\left(x, T\left(y, z\right)\right) = T\left(S\left(x, y\right), S\left(x, z\right)\right).$$

Remark 24

If T is distributive over S and S is distributive over T, then (T,S) is called a distributive pair (of t-norms and t-conorms).

Proposition 25

Let T be a t-norm and S a t-conorm. Then we have:

(i) *S is distributive over T if and only if $T = T_M$.*

(ii) *T is distributive over S if and only if $S = S_M$.*

(iii) *(T, S) is a distributive pair if and only if $T = T_M$ and $S = S_M$.*

Proof.

Obviously, each t-conorm is distributive over T_M because of the monotonicity (S3) of the t-conorm.

($\subseteq$) we have

$$S(x, T_M(y, z)) \leq S(x, y) \qquad \text{(a)}$$

$$S(x, T_M(y, z)) \leq S(x, z) \qquad \text{(b)}$$

(a)and(b) given that $S(x, T(y, z)) \leq T_M(S(x, y), S(x, z))$.

($\supseteq$) ....

Conversely, if S is distributive over T then for all $x \in [0,1]$ we have $x = S(x, T(0,0)) = T(S(x,0), S(x,0)) = T(x,x)$, and from Proposition (..) we obtain $T = T_M$. An analogaus argument proves (ii), and (iii) is just the combination of (i) and (ii). ■

Remark 26

(i) The duality changes the order: if, for some t-norms T_1 and T_2 we have $T_1 \leq T_2$, and if S_1 and S_2 are the dual t-conorms of T_1 and T_2, respectively, then we get $S_1 \geq S_2$. Consequently, for each t-conorm S we have

$$S_M \leq S \leq S_D \tag{1}$$

i.e., the maximum S_M is the weakest and the drastic sum S_D is the strongest t-conorm.

(ii) For the t-conorms in ***example 18*** *we get this ordering:*

$$S_M < S_P < S_L < S_D. \tag{2}$$

The continuity of t-conorm S is equivalente to the continuity of the t-norm duale T.

Definition 27

A T-conorm $S : [0,1]^2 \to [0,1]$ is continue if for all the sequences convergentes $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \in [0,1]^{\mathbb{N}}$ we have :

$$S\left(\lim_{n\to\infty} x_n, \lim_{n\to\infty} y_n\right) = \lim_{n\to\infty} S(x_n, y_n).$$

Example 28

• *the t-conorms S_M, S_P, S_L are continues, and the drastic sum S_D is not continue.*

4.2 Elementary algebraic properties

Definition 29

(i) An element $a \in [0,1]$ is called an idempotent element of S if $S(a,a) = a$. The numbers 0 and 1 (which are idempotent elements for each t-conorm S) are called trivial idempotent elements of S, each idempotent element in $]0,1[$ will be called a non-trivial idempotent element of S.

(ii) *An element* $a \in]0,1[$ *is called a nilpotent element of* S *if there exists some* $n \in N$ *such that* $a_S^{(n)} = 0$.

(iii) *An element* $a \in]0,1[$ *is called a zero divisor of* S *if there exists some* $b \in]0,1[$ *such that* $S(a,b) = 0$.

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