Chapter 03

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d Functions of a Real Chapter 03
Real-Valued Functions of a Real
Variable Variable **Chapter 03**
 Real-Valued Functions of a Real
 Variable
 3.1 Generalities

Let $D \subset \mathbb{R}$. A function f of a real variable is a rule which associates with each $x \in D$ one

and only one $y \in \mathbb{R}$.

Notation: $f:$ **and only one y** $\in \mathbb{R}$ **.**
 And only one y $\in \mathbb{R}$ **.**
 And only one y $\in \mathbb{R}$ **.**
 A *D* is called the domain of the function.
 Example
 $\in D$ is called the domain of the function.
 Example
 $\in D$ *j* ∞

3.1 Generalities

Variable

1 **Generalities**
 $D \subset \mathbb{R}$. A function f of a real variable is a rule which associates wit

only one $y \in \mathbb{R}$.

Notation: $f: D \to \mathbb{R}$.
 D is called the domain of the function.
 Example
 $1) f :]0, \$ **1 Generalities**
 $D \subset \mathbb{R}$. A function f of a real variable is a rule which associates wit

i only one $y \in \mathbb{R}$.

Notation: $f: D \to \mathbb{R}$.

D is called the domain of the function.
 Example
 $1) f :]0, \infty[\to \mathbb{R}$

Example

 $x \longmapsto \ln(x)$

$$
x \longmapsto \frac{1}{1-x}
$$

 $D \subseteq \mathbb{R}$. A function f of a feat variable is a function associates which each for
 x footation: $f: D \to \mathbb{R}$.

Sotation: $f: D \to \mathbb{R}$.
 ∞
 ∞ $f:]0, \infty[\to \mathbb{R}$
 $x \mapsto \ln(x)$
 ∞ $f:]1, \infty[\to \mathbb{R}$
 ∞ Solution: $f: D \to \mathbb{R}$.
 D is called the domain of the function.

Example
 $1) f: [0, \infty[\to \mathbb{R}$
 $x \mapsto \ln(x)$
 $2) g: [1, \infty[\to \mathbb{R}$
 $x \mapsto \frac{1}{1-x}$

Graph of function

Graph of function

Graph of function
 G write
 Example
 $f:]0, \infty[\rightarrow \mathbb{R}$
 $x \longmapsto \ln(x)$
 $2) g: [1, \infty[\rightarrow \mathbb{R}$
 $x \longmapsto \frac{1}{1-x}$

Graph of function

Graph of function
 f is a set of ordered pairs of real numbers $(x, f(x))$, where $x \in D(f)$.

write
 $graph f = \{(x, f(x)) \mid x \in D(f)\$

 $3.1. \text{ Genera}$
Monotone functions
Function $f(x)$ defined on the set D is called increasing, if
 $\forall x_1, x_2 \in D, x_1 < x_2 \Longrightarrow f(x_1) < f(x_2).$ $3.1. \textbf{Generalities}$
 Monotone functions
 $\text{Function } f(x) \text{ defined on the set } D \text{ is called} \text{increasing, if} \qquad \qquad \forall x_1, x_2 \in D, x_1 < x_2 \Longrightarrow f(x_1) < f(x_2).$
 non-increasing, if increasing, if 3.1. Generalities

Monotone functions

Function $f(x)$ defined on the set D is called

increasing, if
 $\forall x_1, x_2 \in D, x_1 < x_2 \implies f(x_1) < f(x_2)$.
 non-increasing, if
 $\forall x_1, x_2 \in D, x_1 < x_2 \implies f(x_1) \ge f(x_2)$.

decreasing, if $\begin{aligned} \text{3.1. } & \textbf{Generalities} \\ \text{Nonotone functions} \\ \text{Function } f(x) \text{ defined on the set } D \text{ is called} \\ \text{increasing, if} \\ & \forall x_1, x_2 \in D, x_1 < x_2 \Longrightarrow f(x_1) < f(x_2). \\ \text{docreasing, if} \\ & \forall x_1, x_2 \in D, x_1 < x_2 \Longrightarrow f(x_1) \geq f(x_2). \\ \text{docreasing, if} \\ & \forall x_1, x_2 \in D, x_1 < x_2 \Longrightarrow f(x_1) > f(x_2). \\ \text{non-decreasing, if} \\ & \forall x_1, x_2 \in D, x_1 < x_$ 3.1. Generalities

Function $f(x)$ defined on the set D is called

increasing, if
 $\forall x_1, x_2 \in D, x_1 < x_2 \Longrightarrow f(x_1) < f(x_2)$.

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decreasing, if
 $\forall x_1, x_2 \in D, x_1 < x_2 \Longrightarrow f(x$

$$
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$$

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\forall x_1, x_2 \in D, x_1 < x_2 \Longrightarrow f(x_1) > f(x_2)
$$

$$
\forall x_1, x_2 \in D, x_1 < x_2 \Longrightarrow f(x_1) \le f(x_2).
$$

3.1. Generalities

Monotone functions

Function $f(x)$ defined on the set D is called

increasing, if
 $\forall x_1, x_2 \in D, x_1 \prec x_2 \Longrightarrow f(x_1) \prec f(x_2).$

non-increasing, if
 $\forall x_1, x_2 \in D, x_1 \prec x_2 \Longrightarrow f(x_1) \ge f(x_2).$

decreasing, if
 $\forall x_$ $\forall x_1, x_2 \in D, x_1 < x_2 \Longrightarrow f(x_1) < f(x_2).$
 non-increasing, if
 $\forall x_1, x_2 \in D, x_1 < x_2 \Longrightarrow f(x_1) \ge f(x_2).$
 decreasing, if
 $\forall x_1, x_2 \in D, x_1 < x_2 \Longrightarrow f(x_1) > f(x_2).$
 non-decreasing, if
 $\forall x_1, x_2 \in D, x_1 < x_2 \Longrightarrow f(x_1) \le f(x_2).$

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 $\forall x_1, x_2 \in D, x_1 < x_2 \Longrightarrow f(x_1) > f(x_2).$

The above functions are said to be monotone on D,

incresing and decresing functions are said to be strictly m $\forall x_1, x_2 \in D, x_1 < x_2 \Longrightarrow f(x_1) \ge f(x_2).$
 non-decreasing, if
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The above functions are said to be monotone on D,
 incresing and decresing functions are said to be **strictly monotone**.
 number $-x$. $\forall x_1, x_2 \in D, x_1 < x_2 \Longrightarrow f(x_1) > f(x_2).$
 oon-decreasing, if
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The above functions are said to be monotone on D,
 oncressing and **decreasing** functions are said to be **strictly monoton** $x_2 \Longrightarrow f(x_1) \le f(x_2).$

one on *D*,

id to be **strictly monotone**.

excreasing) functions is an increasing (decreas-
 D, which contains with any number *x* also
 $x \in D \Longrightarrow f(-x) = f(x)$
 $x \in D \Longrightarrow f(-x) = -f(x)$

with respect to the *y* $\forall x_1, x_2 \in D, x_1 < x_2 \Longrightarrow f(x_1) \le f(x_2).$

The above functions are said to be monotone on D,
 necresing and **decresing** functions are said to be **strictly monotone.**
 Proposition: A sum of two increasing (decreasing) fun

$$
\forall x \in D \Longrightarrow f(-x) = f(x)
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\forall x \in D \Longrightarrow f(-x) = -f(x)
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one on *D*,
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ecreasing) functions is an increasing (decreas-
D, which contains with any number *x* also
 $x \in D \implies f(-x) = f(x)$
 $x \in D \implies f(-x) = -f(x)$
with respect to the *y*-axis, while graph of
to t incresing and decresing functions are said to be strictly monotone.

Proposition: A sum of two increasing (decreasing) functions is an increasing (decreas-

function f(x) be defined on the set D, which contains with any n **Proposition:** A sum of two increasing (decreasing) functions is an increasing (decreasing) function.

Even and odd functions

Let function $f(x)$ be defined on the set D , which contains with any number x also

number For and odd functions

Even and odd functions

Let function $f(x)$ be defined on the set D, which contains with any nun

ober $-x$.

Function $f(x)$ is said to be **even** on D, if
 $\forall x \in D \Longrightarrow f(-x) = f(x)$

Function $f(x)$ is said t Even and odd functions

Let function $f(x)$ be defined on the set D, which contains with any number x also

beher -x.

Function $f(x)$ is said to be even on D, if
 $\forall x \in D \Longrightarrow f(-x) = f(x)$

Function $f(x)$ is said to be odd on D, Function $f(x)$ be defined on the set D, which contains with any number x also
beher-x.
 \bullet Function $f(x)$ is said to be **even** on D, if
 $\forall x \in D \Longrightarrow f(-x) = f(x)$
 \bullet Function $f(x)$ is said to be **odd** on D, if
 $\forall x \in D \Longrightarrow f(-x) =$ $f(x)$ is said to be **even** on *D*, if
 $\forall x \in D \implies f(-x) = f(x)$

tion $f(x)$ is said to be **odd** on *D*, if
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of an **even** function is **symmetric with respect to** the *y*-axis, while graph of

metion is **sy** 2. for all x is said to be **even** on *D*, if
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of an **even** function is **symmetric with respect to the y-axis**, while graph of

metion i

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 $\begin{tabular}{l} \bf 3.2.~Limits of function \\ \hline \end{tabular}$ above, bounded 3.2. Limits of 3.2. Limits of 3.2. Limits of 3.2. Limits of Bounded function $f(x)$ defined on the set $D(f)$ is called bounded (bounded above, bow), iff there exists a real number K such that, for all x from $D(f)$ holds: 3.2. Limits of function

Bounded function

Function

Function

Function
 $f(x)$ defined on the set $D(f)$ is called bounded (bounded above, bounded

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 Bounded function

Function $f(x)$ defined on the set $D(f)$ is called bounded (bounded above, bounded

below), iff there exists a real number K such that, for all x from $D(f)$ holds:
 $|f$ 3.2. Limits of function

Bounded function

Function $f(x)$ defined on the set $D(f)$ is called bounded (bounded above, bounded

w), iff there exists a real number K such that, for all x from $D(f)$ holds:
 $|f(x)| \le K$

It mean **Bounded function**
 Runcion $f(x)$ defined on the set $D(f)$ is called bounded (bounded above, bounded below), iff there exists a real number K such that, for all x from $D(f)$ holds:
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 It means, that a fun 3.2. Limits of function

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Bounded function

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It means, **Bounded function**
 Example 10 Function $f(x)$ defined on the set $D(f)$ is called bounded (bounded

ww), iff there exists a real number K such that, for all x from $D(f)$ ho.
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It means, that a function is Bounded function

Function $f(x)$ defined on the set $D(f)$ is called bounded (bounded above, bound

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 $|f(x)| \le K$

It means, that a function is **bou**

$$
|f(x)| \le K
$$

 $|f(x)| \leq K$

as, that a function is **bounded** (bounded above, bounded below), if
 ounded (bounded above, bounded below) set of real numbers.

in that is **not bounded** is called **unbounded** function

ations on functions
 $|f(x)| \leq K$ It means, that a function is **bounded** (bounded above, bounded below), if its range
 $R(f)$ is a **bounded** (bounded above, bounded below) set of real numbers.
Function that is not **bounded** is called unbounded fu

Let
$$
f, g: D \to \mathbb{R}
$$

1. $(f \pm g)(x) = f(x) + g(x)$.
2. $(f, g)(x) = f(x) g(x)$

3.
$$
(\frac{f}{g})(x) = \frac{f(x)}{g(x)}, g(x) \neq 0.
$$

Punction that is not bounded is called unbounded function
 $\text{Operations on functions}$

Let $f, g: D \to \mathbb{R}$

1. $(f \pm g)(x) = f(x) + g(x)$.

2. $(f,g)(x) = f(x).g(x)$.

3. $\frac{f}{g}(x) = \frac{f(x)}{g(x)}, g(x) \neq 0$.
 3.2 Limits of function

Définition 3.2.1

Let $f:$ **Condition Solution:**

Let $f, g : D \to \mathbb{R}$

1. $(f \pm g)(x) = f(x) \cdot g(x)$.

2. $(fg)(x) = f(x) \cdot g(x) \neq 0$.
 2. Limits of function

finition 3.2.1

Let $f: D \to \mathbb{R}$ be a function. Let $x_0 \in D$. Then, $L \in \mathbb{R}$ is called the limite Let $f, g : D \to \mathbb{R}$

1. $(f \pm g)(x) = f(x) \cdot g(x)$.

2. $(fg)(x) = f(x) \cdot g(x)$.

3. $\frac{f}{g}(x) = \frac{f(x)}{g(x)}$, $g(x) \neq 0$.

3. 2 Limits of function

Définition 3.2.1

Let $f: D \to \mathbb{R}$ be a function. Let $x_0 \in D$. Then, $L \in \mathbb{R}$ is cal x) \neq 0.
 action

tion. Let $x_0 \in D$. Then, $L \in \mathbb{R}$ is called the limite of f as x

0, we can find $\delta > 0$ such that
 x , $|x - x_0| < \delta \Longrightarrow |f(x) - L| < \varepsilon$.
 $\lim_{x \to x_0} f(x) = L$ 3. $(\frac{f}{g})(x) = \frac{f(x)}{g(x)}$, $g(x) \neq 0$.

2 Limits of function

finition 3.2.1

Let $f: D \to \mathbb{R}$ be a function. Let $x_0 \in D$. Then, $L \in \mathbb{R}$ is called the l

roaches x_0 , if for any $\varepsilon > 0$, we can find $\delta > 0$ such *i D*. Then, $L \in \mathbb{R}$ is called the limite of f as x
 $\delta > 0$ such that
 $\Rightarrow |f(x) - L| < \varepsilon$.
 $f(x) = L$

$$
\forall x, |x - x_0| < \delta \Longrightarrow |f(x) - L| < \varepsilon.
$$

$$
\lim_{x \to x_0} f(x) = L
$$

3.2.1 One-sided Limits
 $3.2.1$ One-sided Limits

Right-Hand Limit

The right-hand limit of f at x_0 is L , denoted by 3.2.1 One-sided Limits

Right-Hand Limit

Right-Hand limit of f at x_0 is L , denoted by
 $\lim_{x \to a} f(x) = L$, $or(\lim_{x \to a} f(x) = L)$

\n- 3.2.1 One-sided Limits
\n- Right-Hand Limit
\n- The right-hand limit of
$$
f
$$
 at x_0 is L , denoted by
\n- $$
\lim_{x \to x_0^+} f(x) = L
$$
, or
$$
(\lim_{x \to x_0} f(x) = L)
$$
\n- Left-Hand Limit
\n- The left-hand limit of f at x_0 is L , denoted by
\n- $$
\lim_{x \to x_0^-} f(x) = L
$$
, or
$$
\lim_{x \to x_0^+} f(x) = L
$$
\n

\n- 3.2.1 One-sided Limits
\n- Right-Hand Limit
\n- The right-hand limit of
$$
f
$$
 at x_0 is L , denoted by
\n- $$
\lim_{x \to x_0^+} f(x) = L, or(\lim_{x \to x_0} f(x) = L)
$$
\n
\n- Left-Hand Limit
\n- The left-hand limit of f at x_0 is L , denoted by
\n- $$
\lim_{x \to x_0^-} f(x) = L, or \lim_{x \to x_0} f(x) = L
$$
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\n

i, denoted by
 $f(x) = L, or(\lim_{x \to x_0} f(x) = L)$

denoted by
 $f(x) = L, or \lim_{x \to x_0} f(x) = L$

limits coincide, we say the common value as the limit $f(x) = L$
 $f(x) = L$
we say the common value as the limit **EXECUTE:**

If The right-hand limit of f at x_0 is L , denoted by
 $\lim_{x \to x_0^+} f(x) = L$, or $(\lim_{x \to x_0} f(x) = L)$
 EXECUTE:
 IF The right-hand limit of f at x_0 **is** L **, denoted by
** $\lim_{x \to x_0^-} f(x) = L$ **, or \lim_{x \to x_0} f(x** The right-hand limit of f at x_0 is L , denoted by
 $\lim_{x\to z_0^+} f(x) = L$, or $(\lim_{x\to z_0} f(x) = L)$

Left-Hand Limit

The left-hand limit of f at x_0 is L , denoted by
 $\lim_{x\to z_0^-} f(x) = L$, or $\lim_{x\to z_0} f(x) = L$

If Th $\lim_{x \to x_0^+} f(x) = L$, or $(\lim_{x \to x_0} f(x) = L)$

s L, denoted by
 $\lim_{x \to x_0^-} f(x) = L$, or $\lim_{x \to x_0} f(x) = L$

aand limits coincide, we say the common value as the limit
 $f(x) = \lim_{x \to x_0^-} f(x) = \lim_{x \to x_0} f(x) = L$

aique if it exists. $or(\lim_{x \to x_0} f(x) = L)$
by
 $or \lim_{x \to x_0} f(x) = L$
 $x^{\frac{c}{c}}$
onicide, we say the common value as the limit
 $f(x) = \lim_{x \to x_0} f(x) = L$
sts. = L)

= L

ay the common value as the limit $f(x) = L$ **t**-Hand Limit

left-hand limit of f at x_0 is L , denoted by
 $\lim_{x \to x_0} f(x) = L$, or $\lim_{x \to x_0} f(x) = L$

If The right-hand and left-hand limits coincide, we say the common value as the limit
 $f(x)$ at x_0 and denote If The right-hand and left-hand limits coincide, we say the common value as the limit
 $f(x)$ at x_0 and denote it by
 $\lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x) = \lim_{x \to x_0} f(x) = L$
 Proposition

The limit of a function is unique if and left-hand limits coincide, we say the common value as the limit
ote it by $\lim_{x\to x_0^+} f(x) = \lim_{x\to x_0^-} f(x) = \lim_{x\to x_0} f(x) = L$
action is unique if it exists.
 $f(x) = \frac{1}{x}$
 $f(x) \text{ doesn't exists, because } \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{1}{x} = +\$ value as the limit
 $\frac{1}{x} = +\infty,$ and $\frac{1}{x} = -\infty.$

$$
\lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x) = \lim_{x \to x_0} f(x) = L
$$

Proposition

Examples

1) $f : \mathbb{R}/\{0\} \to \mathbb{R}$

The limit, $\lim_{x\to 0} f(x)$ doesn't exists, because $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{1}{x} = +\infty$, and

$$
= \lim_{x \to x_0} f(x) = L
$$

$$
\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{1}{x} = +\infty, \text{ and}
$$

$$
\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{1}{x} = -\infty.
$$

 $\lim_{x\to x_0^+} f(x) = \lim_{x\to x_0^-} f(x) = \lim_{x\to x_0} f(x) = L$
 Proposition

The limit of a function is unique if it exists.
 Examples
 $1) \ f: \mathbb{R}/\{0\} \to \mathbb{R}$
 $\lim_{x\to 0^-} f(x) = \frac{1}{x}$

The limit, $\lim_{x\to 0} f(x)$ doesn't exists, **osition**

mit of a function is unique if it exists.
 R/{0} \rightarrow **R**
 $x \rightarrow f(x) = \frac{1}{x}$

mit, $\lim_{x\to 0} f(x)$ doesn't exists, because $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{1}{x} = +\infty$, a
 $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{1}{x} = -\infty$.
 -Compute $\lim_{x\to 0} g(x)$.

3.2. Limits of function

-Compute $\lim_{x\to 0+} g(x)$.
 $\lim_{x\to 0+} g(x) = \lim_{x\to 0+} x = 0$, and $\lim_{x\to 0-} g(x) = \lim_{x\to 0-} (-x) = 0$

Since $\lim_{x\to 0+} g(x) = \lim_{x\to 0-} g(x) = 0$. Then, g has a limit at 0.
 Properties
 $f, g: D \to \mathbb{R}$, i

5.2. Limits of function
\nCompute
$$
\lim_{x\to 0} g(x)
$$
.
\n
$$
\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+} x = 0, \text{ and } \lim_{x\to 0^-} g(x) = \lim_{x\to 0^-} (-x) = 0
$$
\nSince $\lim_{x\to 0^+} g(x) = \lim_{x\to 0^-} g(x) = 0$. Then, g has a limit at 0.
\n**Properties**
\n
$$
f, g: D \to \mathbb{R}, \text{ if } \lim_{x\to x_0} f(x) = L_1 \text{ and } \lim_{x\to x_0} g(x) = L_2, \text{ then}
$$
\n
$$
\bullet \lim_{x\to x_0} [f(x) + \beta g(x)] = \alpha \lim_{x\to x_0} f(x) + \beta \lim_{x\to x_0} g(x) = \alpha L_1 + \beta L_2, (\forall \alpha, \beta \in \mathbb{R}).
$$
\n
$$
\bullet \lim_{x\to x_0} [f(x)g(x)] = (\lim_{x\to x_0} f(x))(\lim_{x\to x_0} g(x)) = L_1 L_2.
$$
\n
$$
\bullet \lim_{x\to x_0} (\frac{f(x)}{g(x)}) = \frac{\lim_{x\to x_0} f(x)}{\lim_{x\to x_0} g(x)} = \frac{L_1}{L_2}, \text{ if } L_2 \neq 0.
$$
\n**Theorem(Squaree Theorem)**
\nLet $f, g, h: \mathbb{R} \to \mathbb{R}$ be functions. Suppose that
\n
$$
\bullet g(x) \le f(x) \le h(x) \text{ for all } x \neq x_0
$$
\n
$$
\bullet \lim_{x\to x_0} g(x) = \lim_{x\to x_0} h(x) = L
$$
\n**Example**
\nShow $\lim_{x\to 0} x^2 \sin(\frac{1}{x}) = 0$
\nWe have
\n
$$
-1 \le \sin(\frac{1}{x}) \le 1 \implies -x^2 \le x^2 \sin(\frac{1}{x}) \le x^2, \text{ for all } x \neq 0.
$$
\nTherefore, by **Suppose Theorem** we conclude that $\lim_{x\to 0} x^2 \sin(\frac{1}{x}) = 0$

$$
\bullet \lim_{x \to x_0} \left(\frac{f(x)}{g(x)} \right) = \frac{\lim_{x \to x_0} f(x)}{\lim_{x \to x_0} g(x)} = \frac{L_1}{L_2}, \text{ if } L_2 \neq 0.
$$

lim^x ^x⁰ ; for all ^x = 0:

Then, $\lim_{x\to x_0} f(x) = L$.

Example

Show $\lim_{x\to 0} x^2 \sin(\frac{1}{x}) = 0$

• $\lim_{x \to x_0} [f(x)g(x)] = (\lim_{x \to x_0} f(x))(\lim_{x \to x_0} g(x)) = L_1L_2.$

• $\lim_{x \to x_0} \left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x \to x_0} f(x)}{\lim_{x \to x_0} g(x)} = \frac{L_1}{L_2}$, if $L_2 \neq 0$.
 Theorem(Squeeze Theorem)

Let $f, g, h : \mathbb{R} \to \mathbb{R}$ be functions. Suppos $1 \leq \sin(\frac{1}{x}) \leq 1 \Longrightarrow -x^2 \leq x^2 \sin(\frac{1}{x}) \leq x^2$, for all $x \neq 0$. $\begin{split} \mathcal{L}_1 &= \frac{\lim_{x\to x_0} f(x)}{\lim_{x\to x_0} g(x)} = \frac{L_1}{L_2}, \text{ if } L_2 \neq 0. \end{split}$
 $\begin{split} \textbf{R} &= \textbf{Theorem} \textbf{)} \\\\ \textbf{R} &= \textbf{function}. \text{ Suppose that} \\\\ \bullet g(x) &\leq f(x) \leq h(x) \text{ for all } x \neq x_0 \\\\ \bullet \lim_{x\to x_0} g(x) & = \lim_{x\to x_0} h(x) = L \\\\ \textbf{M}(\frac{1}{x}) &= 0 \\\\ \textbf{M}(\frac{1}{x$ $\lim_{x\to 0}(-x^2) = \lim_{x\to 0}x^2 = 0$ $\overline{g(x)}$ = $\frac{1}{\lim_{x \to x_0} g(x)} = \overline{L_2}$, if $L_2 \neq 0$.

Squeeze Theorem)
 $\mathbb{R} \to \mathbb{R}$ be functions. Suppose that
 $\bullet g(x) \leq f(x) \leq h(x)$ for all $x \neq x_0$
 $\bullet \lim_{x \to x_0} f(x) = L$.
 $\int_0^2 x^2 \sin(\frac{1}{x}) = 0$
 $\sin(\frac{1}{x}) \leq$ $\frac{1}{n_{x \to x_0} g(x)} = \frac{1}{L_2}$, if $L_2 \neq 0$.
 acorem)

metions. Suppose that
 $\bullet g(x) \le f(x) \le h(x)$ for all $x \ne x_0$
 $\bullet \lim_{x \to x_0} g(x) = \lim_{x \to x_0} h(x) = L$

L.
 0
 $\Rightarrow -x^2 \le x^2 \sin(\frac{1}{x}) \le x^2$, for all $x \ne 0$.
 $x^2 = 0$
 Th Theorem(Squeeze Theorem)

Let $f, g, h : \mathbb{R} \to \mathbb{R}$ be functions. Suppose that
 $\bullet g(x) \le f(x) \le h(x)$ for all $x \ne x_0$
 $\bullet \lim_{x \to x_0} f(x) = L$.

Then, $\lim_{x \to x_0} f(x) = L$.

Example

Show $\lim_{x \to 0} x^2 \sin(\frac{1}{x}) = 0$

We have
 $-1 \le$ $\begin{aligned} &\textbf{Theorem(Squareze Theorem)}\\ &\textbf{Let } f,g,h:\mathbb{R}\rightarrow\mathbb{R} \text{ be functions.}\text{ Suppose that}\\ &\bullet g(x)\leq f(x)\leq h(x) \text{ for all } x\neq x_0\\ &\bullet \lim_{x\rightarrow x_0}g(x)=\lim_{x\rightarrow x_0}h(x)=L\\ &\textbf{Example}\\ &\textbf{We have}\lim_{x\rightarrow 0}x^2\sin(\frac{1}{x})=0\\ &\textbf{We have}\lim_{x\rightarrow 0}(-x^2)=\lim_{x\rightarrow 0}x^2\equiv 0\\ &\textbf{Hence, by }\textbf{Squeezc Theorem},\text{ we conclude that }\lim_{$ hen, $\lim_{x\to x_0} f(x) = L$.

sample

w $\lim_{x\to 0} x^2 \sin(\frac{1}{x}) = 0$

have
 $-1 \le \sin(\frac{1}{x}) \le 1 \implies -x^2 \le x^2 \sin(\frac{1}{x}) \le x^2$, for all $x \ne 0$.
 $x \ne 0, -x^0 = \lim_{x\to 0} x^2 = 0$

orcfore, by **Squeeze Theorem**, we conclude that $\lim_{x\to 0$ Then, $\lim_{x\to x_0} f(x) = L$.

Example

Show $\lim_{x\to 0} x^2 \sin(\frac{1}{x}) = 0$

We have
 $-1 \le \sin(\frac{1}{x}) \le 1 \implies -x^2 \le x^2 \sin(\frac{1}{x}) \le x^2$, for all $x \ne 0$.
 $\lim_{x\to 0} (-x^2) = \lim_{x\to 0} x^2 = 0$

Therefore, by **Squeeze Theorem**, we conclude Show $\lim_{x\to 0} x^2 \sin(\frac{1}{x}) = 0$

We have
 $-1 \le \sin(\frac{1}{x}) \le 1 \implies -x^2 \le x^2 \sin(\frac{1}{x}) \le x^2$, for all $x \ne 0$.
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Therefore, by **Squeeze Theorem**, we conclude that $\lim_{x\to 0} x^2 \sin(\frac{1}{x}) = 0$.
 Show lim_{x-0} x² sin($\frac{1}{x}$) = 0

We have
 $-1 \le \sin(\frac{1}{x}) \le 1 \implies -x^2 \le x^2 \sin(\frac{1}{x}) \le x^2$, for all $x \ne 0$.
 $\lim_{x\to 0}(-x^2) = \lim_{x\to 0}x^2 = 0$

Therefore, by **Squeeze Theorem**, we conclude that $\lim_{x\to 0}x^2 \sin(\frac{1}{x}) =$ for all $x \neq 0$.
that $\lim_{x \to 0} x^2 \sin(\frac{1}{x}) = 0$.
Then it becomes,

 $^{2}\sin(\frac{1}{x}) = 0.$

 $0 \quad \infty$ $0 \quad \infty$

Inditerminate From $\frac{0}{0}$ and $\frac{\infty}{\infty}$ (Hospital).

Inditerminate From 0. ∞ ; transformation to $\frac{0}{0}$. Then it becomes,

3.2. Limits of function
 $\frac{g(x)}{x-x_0}$. $\lim_{x\to x_0} f(x)g(x) = \lim_{x\to x_0} \frac{f(x)}{\frac{1}{x}}.$ 3.2. Limits of func:
 $f(x)g(x) = \lim_{x \to x_0} \frac{f(x)}{\frac{1}{g(x)}}$.

formation to $\frac{\infty}{\infty}$. Then it becomes, $\lim_{x \to x_0} f(x)g(x) = \lim_{x \to x_0} \frac{g(x)}{\frac{1}{f(x)}}$.

minate From 1^{∞} ; $(\lim_{x \to x_0} f(x) = 1, \lim_{x \to x_0} g(x) = \infty)$. Transformati $\mathbf{1}$, the contract of th $g(x)$
g(x)
it becomes $\lim_{x \to a} f(x)g(x) = \lim_{x \to a} g(x)$ $\overline{1_{-}}$ to the contract of the contract of $\overline{1_{-}}$ to the contract of $\overline{1_{-}}$ $f(x)$ 3.2. Limits of furtherminate $\text{Hom}_{x \to x_0} \frac{f(x)}{g(x)}$.

Indication to $\frac{\infty}{\infty}$. Then it becomes, $\lim_{x \to x_0} f(x)g(x) = \lim_{x \to x_0} \frac{g(x)}{f(x)}$.

Inditerminate From 1^{∞} ; $(\lim_{x \to x_0} f(x) = 1, \lim_{x \to x_0} g(x) = \infty)$. Transforma

3.2. Limits of function
 $\lim_{x\to x_0} f(x)g(x) = \lim_{x\to x_0} \frac{f(x)}{\frac{f(x)}{g(x)}}$.

or transformation to $\frac{\infty}{\infty}$. Then it becomes, $\lim_{x\to x_0} f(x)g(x) = \lim_{x\to x_0} \frac{g(x)}{\frac{f(x)}{f(x)}}$.
 Inditerminate From 1^{∞} ; $(\lim_{x\to x_0} f(x) = 1$ 3.2. Limits of function
 $f(x)g(x) = \lim_{x \to x_0} \frac{g(x)}{\frac{1}{f(x)}}.$
 $f(x)g(x) = \infty$). Transformation to $\frac{0}{0}$.
 $\frac{\ln f(x)}{\frac{1}{g(x)}}$. 3.2. Limits of function
 $\lim_{x\to x_0} f(x)g(x) = \lim_{x\to x_0} \frac{f(x)}{\frac{1}{g(x)}}.$

or transformation to $\frac{\infty}{\infty}$. Then it becomes, $\lim_{x\to x_0} f(x)g(x) = \lim_{x\to x_0} \frac{g(x)}{f(x)}.$
 Inditerminate From 1^{∞} ; ($\lim_{x\to x_0} f(x) = 1$, $\lim_{$ 3.2. Limits of function

mes, $\lim_{x \to x_0} f(x)g(x) = \lim_{x \to x_0} \frac{g(x)}{\frac{f(x)}{f(x)}}$.
 $f(x) = 1, \lim_{x \to x_0} g(x) = \infty$). Transformation to $\frac{0}{0}$.
 $= \exp(\lim_{x \to x_0} \frac{\ln f(x)}{\frac{1}{g(x)}})$.
 $(x) = 0^+, \lim_{x \to x_0} g(x) = 0$. Transformation to $\frac{0$ 3.2. Limits of function
 $g(x) = \lim_{x \to x_0} \frac{g(x)}{\frac{1}{f(x)}}$.
 $g(x) = \infty$). Transformation to $\frac{0}{0}$.
 $g(x) = 0$). Transformation to $\frac{0}{0}$.
 $\frac{1}{\sqrt{1-\frac{1}{1-\frac{1}{2}}}}$. $0₁$ $3.2.$ Limits of function
 $\lim_{x\to x_0} f(x)g(x) = \lim_{x\to x_0} \frac{f(x)}{g(x)}$ or transformation to $\frac{\infty}{\infty}$.

Then it becomes, $\lim_{x\to x_0} f(x)g(x) = \lim_{x\to x_0} \frac{g(x)}{f(x)}$.

Inditerminate From 1^{∞} ; $(\lim_{x\to x_0} f(x) = 1, \lim_{x\to x_0} g(x$ 3.2. Limits of function
 $\lim_{x_0} \frac{f(x)}{g(x)}$.

hen it becomes, $\lim_{x \to x_0} f(x)g(x) = \lim_{x \to x_0} \frac{g(x)}{f(x)}$.
 $\lim_{x \to x_0} f(x) = 1$, $\lim_{x \to x_0} g(x) = \infty$). Transformation to $\frac{0}{0}$.
 $(f(x))^{g(x)} = \exp(\lim_{x \to x_0} \frac{\ln f(x)}{f(x)})$.
 $(\lim_{x \to x$ $\frac{\ln f(x)}{\frac{1}{g(x)}}$. 3.2. Limits of function

comes, $\lim_{x\to x_0} f(x)g(x) = \lim_{x\to x_0} \frac{g(x)}{\frac{1}{f(x)}}$.
 $\sum_{x_0} f(x) = 1$, $\lim_{x\to x_0} g(x) = \infty$). Transformation to $\frac{0}{0}$.
 $f(x) = 0^+, \lim_{x\to x_0} g(x) = 0$). Transformation to $\frac{0}{0}$.
 $f(x) = 0^+, \lim_{x\to$ 3.2. Limits of function
 $g(x) = \lim_{x \to x_0} \frac{g(x)}{\frac{1}{f(x)}}.$
 $g(x) = \infty$). Transformation to $\frac{0}{0}$.
 $g(x) = 0$). Transformation to $\frac{0}{0}$.
 $g(x) = 0$). Transformation to $\frac{0}{0}$.
 $\lim_{x_0} g(x) = 0$). Transformation to \frac

Inditerminate From $0^0(\lim_{x\to x_0} f(x) = 0^+, \lim_{x\to x_0} g(x) = 0)$. Transformation to $\frac{0}{0}$. 0° $\frac{g(x)}{1}$. $\ln f(x)$

3.2. Limits of function
 $\lim_{x\to x_0} f(x)g(x) = \lim_{x\to x_0} \frac{f(x)}{\frac{1}{q(x)}}.$

or transformation to $\frac{\infty}{\infty}$. Then it becomes, $\lim_{x\to x_0} f(x)g(x) = \lim_{x\to x_0} \frac{g(x)}{f(x)}.$
 Inditerminate From 1^∞ ; $(\lim_{x\to x_0} f(x) = 1, \lim_{x\to x_0$ 3.2. Limits of function
 $\int_{x_0}^{x_0} \frac{f(x)}{y(x)} dx$.

Alternative becomes, $\lim_{x \to x_0} f(x)g(x) = \lim_{x \to x_0} \frac{g(x)}{f(x)}$.
 $\int_{x_0}^{x_0} \frac{f(x)}{f(x)} dx = \int_{x_0}^{x_0} f(x) dx = \int_{x_0}^{x_0} f(x) dx$. Transformation to $\frac{0}{0}$.
 $\int_{x_0}^{x_0} f$ $\begin{array}{l} 3.2. \text{ Limits of fun} \\ \lim_{x\to x_0} f(x)g(x) = \lim_{x\to x_0} \frac{f(x)}{\frac{1}{g(x)}}. \end{array}$

or transformation to \sum_{∞}^{∞} .
 Inditerminate From 1^{∞} ; ($\lim_{x\to x_0} f(x) = 1, \lim_{x\to x_0} g(x) = \infty$). Transformation

Then it becomes, $\lim_{x\to x$ **Inditerminate From** ∞^0 , ($\lim_{x\to x_0} f(x) = \infty$, $\lim_{x\to x_0} g(x) = 0$). Transformation to $\frac{0}{0}$. 3.2. Limits of function

mes, $\lim_{x\to x_0} f(x)g(x) = \lim_{x\to x_0} \frac{g(x)}{\frac{1}{f(x)}}$.
 $f(x) = 1, \lim_{x\to x_0} g(x) = \infty$). Transformation to $\frac{0}{0}$.
 $\exp(\lim_{x\to x_0} \frac{g(x)}{\frac{1}{g(x)}})$.
 $x) = 0^+, \lim_{x\to x_0} g(x) = 0$. Transformation to $\frac{0}{0}$ 3.2. Limits of function

(x) = $\lim_{x \to x_0} \frac{g(x)}{\frac{1}{f(x)}}$.

(x) = ∞). Transformation to $\frac{0}{0}$.

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x) = 0). Transformation to $\frac{0}{0}$.

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g(x) = 0). Transformation to $\frac{0}{0}$.

). $0 \cdot$ 3.2. Limits of function
 $\lim_{x\to x_0} f(x)g(x) = \lim_{x\to x_0} \frac{f(x)}{\overline{s}(x)}$.

or transformation to $\frac{\infty}{\infty}$. Then it becomes, $\lim_{x\to x_0} f(x)g(x) = \lim_{x\to x_0} \frac{g(x)}{f(x)}$.
 Inditerminate From 1^∞ ; $(\lim_{x\to x_0} f(x) = 1, \lim_{x\to x_0}$ 3.2. Limits of function
 $\lim_{x\to \frac{\pi}{g(x)}}$

hen it becomes, $\lim_{x\to x_0} f(x)g(x) = \lim_{x\to x_0} \frac{g(x)}{f(x)}$.
 $\lim_{x\to x_0} f(x) = 1$, $\lim_{x\to x_0} g(x) = \infty$). Transformation to $\frac{0}{0}$.
 $(f(x))^{g(x)} = \exp(\lim_{x\to x_0} \frac{\ln f(x)}{\frac{1}{g(x)}})$.
 $(\lim_{x\to$ $\frac{g(x)}{1}$. $\lim_{x\to x_0} f(x)g(x) = \lim_{x\to x_0} \frac{f(x)}{g(x)}$.

or transformation to $\frac{\infty}{\infty}$. Then it becomes, $\lim_{x\to x_0} f(x)g(x) = \lim_{x\to x_0} \frac{g(x)}{f(x)}$.
 Inditerminate From 1^{∞} ; $(\lim_{x\to x_0} f(x) = 1, \lim_{x\to x_0} g(x) = \infty)$. Transformation
 $\begin{array}{l} \lim_{x\rightarrow x_{0}}f(x)g(x)=\lim_{x\rightarrow x_{0}}\frac{g(x)}{f(x)}.\\ \\ =1,\lim_{x\rightarrow x_{0}}g(x)=\infty). \text{ Transformation to }\frac{0}{0}.\\ \\ \lim_{x\rightarrow x_{0}}\frac{\ln f(x)}{g(x)}).\\ \\ 0^+,\lim_{x\rightarrow x_{0}}g(x)=0). \text{ Transformation to }\frac{0}{0}.\\ \\ \lim_{x\rightarrow x_{0}}\frac{g(x)}{\ln f(x)}).\\ \\ =\infty, \lim_{x\rightarrow x_{0}}g(x)=0). \text{ Transformation to }\frac{0}{0}.\\ \\ \lim_{x\rightarrow x_{0}}\frac{g(x)}{\ln f(x)}).\\ \\ f(x)=$ = $\lim_{x \to x_0} \frac{g(x)}{\frac{1}{f(x)}}$.

= ∞). Transformation to $\frac{0}{0}$.

0). Transformation to $\frac{0}{0}$.

= 0). Transformation to $\frac{0}{0}$.
 $g(x) = \infty$). Transformation or transformation to $\sum_{\overline{g}(0)}^{\infty} \overline{f_{\overline{g}(0)}}$
 Inditerminate From 1^{∞} ; $(\lim_{x \to x_0} f(x) = 1, \lim_{x \to x_0} g(x) = \infty)$. Transf

Then it becomes, $\lim_{x \to x_0} (f(x))^{g(x)} = \exp(\lim_{x \to x_0} \frac{\ln f(x)}{\frac{f(x)}{g(x)}})$.
 Inditerminate Fr hen it becomes, $\lim_{x\to x_0} f(x)g(x) = \lim_{x\to x_0} \frac{g(x)}{f(x)}$.
 $\int_{0}^{\infty} (\lim_{x\to x_0} f(x) = 1, \lim_{x\to x_0} g(x) = \infty)$. Transformation to $\frac{0}{0}$.
 $(f(x))^{g(x)} = \exp(\lim_{x\to x_0} \frac{\ln f(x)}{\frac{1}{g(x)}})$.
 $(\lim_{x\to x_0} f(x) = 0^+, \lim_{x\to x_0} g(x) = 0)$. Tran **Inditerminate From** 1^{∞} ; $(\lim_{x\to x_0} f(x) = 1, \lim_{x\to x_0} g(x) = \infty)$. Transformation to $\frac{0}{0}$.
Then it becomes, $\lim_{x\to x_0} (f(x))^{g(x)} = \exp(\lim_{x\to x_0} \frac{\ln f(x)}{x^{\frac{1}{10}}})$.
 Inditerminate From $0^0(\lim_{x\to x_0} f(x) = 0^+, \lim_{x\to x$ Then it becomes, $\lim_{x\to x_0} (f(x))^{g(x)} = \exp(\lim_{x\to x_0} \frac{\ln f(x)}{g(x)})$.

Inditerminate From $0^0(\lim_{x\to x_0} f(x) = 0^+, \lim_{x\to x_0} g(x) = 0)$. Tra:

Then it becomes, $\lim_{x\to x_0} (f(x))^{g(x)} = \exp(\lim_{x\to x_0} \frac{g(x)}{\ln f(x)})$.

Inditerminate From ∞^0 ies, $\lim_{x\to x_0} (f(x))^{g(x)} = \exp(\lim_{x\to x_0} \frac{\ln f(x)}{\frac{1}{g(x)}}).$

ie From $0^0(\lim_{x\to x_0} f(x) = 0^+, \lim_{x\to x_0} g(x) = 0).$ Transformation to $\frac{6}{6}$.

ees, $\lim_{x\to x_0} (f(x))^{g(x)} = \exp(\lim_{x\to x_0} \frac{g(x)}{\ln f(x)})$.

ie From ∞^0 , $(\lim_{x\to x_0} f(x) = \in$ (a) $\cos \left(\frac{\sin x}{\sin x} \right)$ (b) $\cos \left(\frac{\sin x}{\sin x} \right)$.
 $\cos \left(\frac{\sin x}{\sin x} \right)$.
 $\sin \left(\frac{\sin x}{\sin x} \right)$.
 $\cos \left(\frac$ Inditerminate From ∞^0 , $(\lim_{x\to x_0} f(x) = \infty, \lim_{x\to x_0} g(x) = 0)$. Transformation to $\frac{0}{0}$.

Then it becomes, $\lim_{x\to x_0} (f(x))^{g(x)} = \exp(\lim_{x\to x_0} \frac{g(x)}{\overline{x_1(x)}})$.

Inditerminate From $\infty - \infty$, $(\lim_{x\to x_0} f(x) = \infty, \lim_{x\to x$

Inditerminate From $\infty - \infty$, $(\lim_{x \to x_0} f(x) = \infty, \lim_{x \to x_0} g(x) = \infty)$. Transformation to $\frac{0}{0}$. 0.

 $\Big(\frac{\left(\frac{1}{g(x)} - \frac{1}{f(x)}\right)}{\frac{1}{f(x)g(x)}}\Big).$), where \mathcal{L}

 $f(x)$ of the Indeterminate Form $\frac{0}{0}$ or $\frac{\infty}{\infty}$,

$$
\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}
$$

Examples

0 Compute $\lim_{x\to\pi}\frac{x^2-\pi^2}{\sin x}$. π^2 $\sin x$: 2000 - 2000 - 2000 - 2000 - 2000 - 2000 - 2000 - 2000 - 2000 - 2000 - 2000 - 2000 - 2000 - 2000 - 2000 - 20
- 2000 - 2000 - 2000 - 2000 - 2000 - 2000 - 2000 - 2000 - 2000 - 2000 - 2000 - 2000 - 2000 - 2000 - 2000 - 20

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Solution: we use L'Hopital's Rule)

Solution: we use L'Hopital's Rule.

Solution: we use L'Hopital's Rule.

Solution: we use L'Hopital's Rule. Since the numerator and denomirator both approach
 $\lim_{x\to\$ zero. Form $\frac{0}{0}$ or $\frac{\infty}{\infty}$,
 $= \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$

orm $\frac{0}{0}$
 \therefore the numerator and denomirator both approach
 $\frac{H}{x} \lim_{x \to \pi} \frac{2x}{\cos x} = -2\pi$.

orm $\frac{\infty}{\infty}$
 $\frac{\frac{3}{3}}{\frac{5}{5}} \frac{H}{\frac{1}{5}} \lim_{x \to 0^+} \$

$$
\lim_{x \to \pi} \frac{x^2 - \pi^2}{\sin x} = \frac{0}{0} \stackrel{H}{=} \lim_{x \to \pi} \frac{2x}{\cos x} = -2\pi.
$$

For a lim_{z--ro} $\frac{f(x)}{g(x)}$ of the Indeterminate Form $\frac{0}{0}$ or $\frac{\infty}{\infty}$,
 $\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)}$
 Examples

1) L'Hopital's Rule and Indeterminate Form $\frac{0}{0}$

Compute $\lim_{x \to \pi} \frac{x^2 -$ Compute $\lim_{x\to 0^+} \frac{\frac{1}{x^2}}{\ln x}$. x^2 $\ln x$. : **Solution:** $\lim_{x\to 0^+} \frac{\frac{1}{x^2}}{\ln x} = \frac{\infty}{\infty} \stackrel{H}{=} \lim_{x\to 0^+} \frac{\frac{-2}{x^3}}{\frac{1}{x}} \stackrel{H}{=} \lim_{x\to 0^+} \frac{\frac{-2}{x^3}}{\frac{1}{x}}$ $\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f(x)}{g'(x)}$
terminate Form $\frac{0}{0}$
s Rule. Since the numerator and denomirator both approach
 $\frac{x^2 - \pi^2}{\sin x} = \frac{0}{0} \stackrel{H}{=} \lim_{x \to \pi} \frac{2x}{\cos x} = -2\pi.$
terminate Form $\frac{\infty}{\infty}$
 $\frac{H}{=}$ $\begin{aligned}\n&\lim_{x\to x_0} \frac{f(x)}{g'(x)}\\
&\lim_{x\to \pi} \frac{0}{\cos x} \\
&\lim_{x\to \pi} \frac{2x}{\cos x} = -2\pi.\\
&\lim_{x\to 0} \frac{\infty}{\infty}\\
&\lim_{x\to 0^+} \frac{\frac{-2}{\pi^2}}{\frac{1}{x}}\\
&= \lim_{x\to 0^+} \frac{-\frac{2}{\pi^2}}{\frac{1}{x}} = -\infty.\n\end{aligned}$ 2 x^3 terminate Form $\frac{0}{0}$

Rule. Since the numerator and denomirator both approach
 $\frac{2-\pi^2}{\sin x} = \frac{0}{0} \frac{H}{x \to \pi} \frac{\sin x}{\cos x} = -2\pi.$

terminate Form $\frac{\infty}{\infty}$
 $\frac{H}{\sin x \to 0^+} \frac{\frac{-2}{x^2}}{\frac{1}{x}} \frac{H}{\sin x \to 0^+} \frac{\frac{-2}{x$ $\frac{-6}{\frac{x^4}{2}} = \lim_{x\to 0^+} \frac{-6}{x^2} = -\infty.$ be the numerator and denomirator both approach
 $\frac{H}{=} \lim_{x \to \pi} \frac{2x}{\cos x} = -2\pi.$

Som $\frac{\infty}{\infty}$
 $\frac{\pi^2}{\frac{1}{x}} = \lim_{x \to 0^+} \frac{\frac{-2}{x}}{\frac{x^2}{x}} = \lim_{x \to 0^+} \frac{-6}{x^2} = -\infty.$

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Compute $\lim_{x\to 0^+} x \ln x$.

3.2. Limits of function
3) Inditerminate From $0.\infty$, (transformation to $\frac{\infty}{\infty}$)
Compute $\lim_{x\to 0^+} x \ln x$.
Solution: $\lim_{x\to 0^+} x \ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} = \frac{\infty}{\infty}$ 3) Inditerminate From 0. ∞ , (transformation to $\frac{\infty}{\infty}$)

Compute $\lim_{x\to 0^+} x \ln x$.

Solution: $\lim_{x\to 0^+} x \ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} = \frac{\infty}{\infty}$
 $\frac{H}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{x^2} = \lim_{x\to 0^+} (-x) = 0$.

4) Indit $rac{\ln x}{\frac{1}{x}} = \frac{\infty}{\infty}$

$$
\stackrel{H}{=} \lim_{x \to 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x \to 0^+} (-x) = 0.
$$

3.2. Limits of function
 $0.\infty$, (transformation to $\frac{\infty}{\infty}$)
 $x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} = \frac{\infty}{\infty}$
 $\frac{11}{\frac{1}{x}} \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{1}{x^2}} = \lim_{x\to 0^+} (-x) = 0.$
 1^∞ , (transformation to $\frac{0}{0}$)
 $\left(\frac{1}{$ $0⁷$) and the contract of \mathcal{L} Compute $\lim_{x\rightarrow 1^+} x^{\frac{1}{x-1}}$

3.2. Limits of function

3) Inditerminate From 0. ∞ , (transformation to $\frac{\infty}{\infty}$)

Compute $\lim_{x\to 0^+} x \ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{x}{x}} = \frac{\infty}{\infty}$
 $\frac{H}{2} \lim_{x\to 0^+} \frac{\frac{1}{x}}{x^2} = \lim_{x\to 0^+} (-x) = 0.$

4) Inditermin 3.2. Limits of function

3) Inditerminate From $0.\infty$, (transformation to $\frac{\infty}{\infty}$)

Solution: $\lim_{x\to 0^+} x \ln x = \lim_{x\to 0^+} \frac{\ln x}{x} = \frac{\infty}{\infty}$
 $= \lim_{x\to 0^+} \frac{1}{x^2} = \lim_{x\to 0^+} (-x) = 0$.

4) Inditerminate From 1^∞ $\frac{\ln x}{x-1} = \frac{0}{0}.$ $\overline{0}$ 0. 3.2. Limits of function

3) Inditerminate From 0. ∞ , (transformation to $\frac{\infty}{\infty}$)

Compute $\lim_{x\to 0^+} x \ln x$.

Solution: $\lim_{x\to 0^+} x \ln x = \lim_{x\to 0^+} \frac{\frac{h_x}{2}}{\frac{h_x}{2}} = \frac{\infty}{\infty}$
 $\frac{H}{2} \lim_{x\to 0^+} \frac{\frac{1}{\pi}}{\frac{$ $\lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} = \frac{\infty}{\infty}$
 $\lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{1}{x^2}} = \lim_{x\to 0^+} (-x) = 0.$

, (transformation to $\frac{0}{0}$)

, $\lim_{x\to 1^-} \lim_{x\to 1} \lim_{x\to 1^+} \ln(x^{\frac{1}{x-1}}) = \lim_{x\to 1^+} \frac{\ln x}{x-1} = \frac{0}{0}.$

The $L = \lim_{x\to 1^+$ $\begin{aligned} &\lim_{\alpha \to 0^+} (-x) = 0. \end{aligned}$
 $\begin{aligned} \text{and} &\lim_{x \to 1^+} \ln(x^{\frac{1}{x-1}}) = \lim_{x \to 1^+} \frac{\ln x}{x-1} = \frac{0}{0}. \end{aligned}$
 $\frac{H}{x-1^+}\lim_{1 \to 1^+} \frac{\frac{1}{x}}{1} = 1.$
 $\text{and} &\lim_{x \to 1^+} \frac{\frac{1}{x}}{1} = 1.$ Solution: $\lim_{x\to 0^+} x \ln x$.

Solution: $\lim_{x\to 0^+} x \ln x$.

Solution: $\lim_{x\to 0^+} x \ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{x}{2}} = \frac{\infty}{\infty}$
 $\frac{H}{\sin \theta} \lim_{x\to 0^+} \frac{1}{\frac{x}{2}} = \lim_{x\to 0^+} (-x) = 0.$

4) Inditerminate From 1², (transformatio **5044404.** Independent $\lim_{x\to 0^+} x \ln x = \lim_{x\to 0^+} \frac{1}{\frac{x^2}{x^2}} = \lim_{x\to 0^+} (-x) = 0.$

4) Inditerminate From 1^∞ , (transformation to $\frac{0}{0}$)

Compute $\lim_{x\to 1^+} x^{\frac{1}{x-1}}$
 Solution: $L = \lim_{x\to 1^+} (x^{\frac{1}{x-1}}$ transformation to $\frac{0}{0}$)
 $\Rightarrow \ln L = \lim_{x \to 1^+} \ln(x^{\frac{1}{x-1}}) = \lim_{x \to 1^+} \frac{\ln x}{x-1} = \frac{0}{0}$.

Iopital's Rule:
 $L = \lim_{x \to 1^+} \frac{\ln x}{x-1} = \lim_{x \to 1^+} \frac{1}{x} = 1$.
 $\Rightarrow \lim_{x \to 1^+} x^{\frac{1}{x-1}} = e$.
 ∞ , (transformation to $\Rightarrow \ln L = \lim_{x \to 1^+} \ln(x^{\frac{1}{x-1}}) = \lim_{x \to 1^+} \frac{\ln x}{x-1} = \frac{0}{0}.$

pital's Rule:
 $= \lim_{x \to 1^+} \frac{\ln x}{x-1} = \lim_{x \to 1^+} \frac{\frac{1}{x}}{1} = 1.$
 $+ x^{\frac{1}{x-1}} = e.$
 \therefore (transformation to $\frac{0}{0}$)
 $= \frac{x - \sin x}{x \sin x} = \frac{0}{0}$
 $= \lim$

$$
\ln L = \lim_{x \to 1^+} \frac{\ln x}{x-1} \stackrel{H}{=} \lim_{x \to 1^+} \frac{\frac{1}{x}}{1} = 1.
$$

 $0⁷$) and $\overline{}$ ($\overline{}$) and $\overline{}$ ($\overline{\phantom{a$ Compute $\lim_{x\to 0^+} (\frac{1}{\sin x} - \frac{1}{x})$ 1 x^{\prime}) and the contract of \mathcal{L} Therefore, we can apply to replaus since:
 $\ln L = \lim_{x \to 1^+} \frac{\ln x}{x-1} \frac{u}{x} \lim_{x \to 1^+} \frac{1}{x} = 1.$

Thus, $\ln L = 1 \Longrightarrow L = \lim_{x \to 1^+} x^{\frac{1}{n} + 1} = e.$

5) Inditerminate From $\infty = \infty$, (transformation to $\frac{9}{9}$)

Compute $\ln L = 1 \Longrightarrow L = \lim_{x \to 1^+} x \to 1 = \lim_{x \to 1^+} 1 = 1.$
 $\ln \ln L = 1 \Longrightarrow L = \lim_{x \to 1^+} x^{\frac{1}{k-1}} = e.$

Inditerminate From $\infty - \infty$, (transformation to $\frac{0}{0}$)
 $\text{mputel } \lim_{x \to 0^+} \left(\frac{1}{\sin x} - \frac{1}{x} \right) = \frac{x - \sin x}{x \sin x} = \frac{0}{0}$
 $\frac{$ Thus, $\ln L = 1 \Longrightarrow L = \lim_{x \to 1^+} x^{\frac{1}{x-1}} = c$.

5) Inditerminate From $\infty - \infty$, (transformation to $\frac{0}{0}$)

Compute $\lim_{x \to 0^+} (\frac{1}{\sin x} - \frac{1}{x})$

Solution: $\lim_{x \to 0^+} (\frac{1}{\sin x} - \frac{1}{x}) = \frac{x - \sin x}{x \sin x} = \frac{0}{0}$
 $\frac{H}{2 \$

Solution: $\lim_{x \to 0^+} (\frac{1}{\sin x} - \frac{1}{x}) = \frac{x - \sin x}{x \sin x} = \frac{0}{0}$ 1 $\frac{1}{x} = \frac{x - \sin x}{x \sin x} = \frac{0}{0}$ 0 0

$$
\begin{aligned}\n\dot{z} &= \lim_{x \to 1^{+}} \frac{\ln x}{x - 1} \stackrel{H}{=} \lim_{x \to 1^{+}} \frac{\frac{1}{x}}{1} = 1. \\
\text{and } x \to 1^{+}} x \frac{\frac{1}{x - 1}}{1} &= e. \\
\infty, \text{ (transformation to } \frac{0}{0}) \\
&= \frac{x - \sin x}{x \sin x} = \frac{0}{0} \\
\frac{H}{x} \lim_{x \to 0^{+}} \frac{1 - \cos x}{\sin x + x \cos x} = \frac{0}{0} \\
\frac{H}{x} \frac{\sin x}{2 \cos x - x \sin x} &= \frac{0}{2} = 0. \\
\text{which are not indeterminate are:}\n\end{aligned}
$$

 $\frac{0}{\infty}$; $\frac{1}{\infty}$ have limite as 0.

3.3 Continuity and IVT
3.3.1 Continuity of Function
Définition 3.3.1 *continuous at a point*

3.3. Continuity and IVT
3.3.1 Continuity of Function
Définition 3.3.1 continuous at a point
Let $f: D \to \mathbb{R}$ be a function, f is continuous at a point a if 3.3 **Continuity and IVT**

3.3.1 **Continuity of Function**

Définition 3.3.1 *continuous at a point*

Let $f: D \to \mathbb{R}$ be a function, f is continuous at a point *a* if
 $\lim_{x \to a} f(x) = f(a).$

$$
\lim_{x \to a} f(x) = f(a).
$$

3.3. Continuity and IVT

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3.1 Continuity of Function

finition 3.3.1 *continuous at a point*

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On the other 3.3. Continuity and **IVT**

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On the othe

Figure 3.3. (a) A continuous function, (b) A function with a discontinuity at $x = 1$.
 Continuous S. (a) A continuous function, (b) A function with a discontinuity at $x = 1$.
 Continuous Continuous function, (b) A fun Figure 3.3. (a) A continuous function, (b) A function with a discontinuity at $x = 1$.
 Graphically, you can think of continuity as being able to draw your function without having to lift your pencil off the paper. If yo Figure 3.3. (a) A continuous function, (b) A function with a discontinuity at $x = 1$.
 Graphically, you can think of continuity as being able to draw your function without thaving to lift your pencil off the paper. If y Figure 3.3. (a) A continuous function, (b) A function with a discontinuity at $x = 1$.
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drawing the function, Figure 3.3. (a) A continuous function, (b) A function with a discontinuity at $x = 1$.
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drawing the function,

3.3. Continuity and IVT
nuous on the entire real 3.3. Continuity and IVT
Furthermore, a function is everywhere continuous if it is continuous on the entire real
aber line $]-\infty, +\infty[$.

$$
\lim_{x \to a^{+}} f(x) = f(a)
$$

$$
\lim_{x \to a^{-}} f(x) = f(a)
$$

Example 01
 $f(x)$
 $\lim_{x\to a^+} f(x) = f(a)$ (right continuous at a point *a*)

Example 01
 $f(x) = \begin{cases} x-2, & x \le 0 \\ 2, & x > 0 \\ 2, & x > 0 \end{cases}$

Away from $x = 0$, we see that *f* is continuous. Therefore, we look at $x = 0$
 $\lim_{x\to 0^$ $\lim_{x \to a^+} f(x) = f(a) \text{ (right continuous at a point } a$
 $\lim_{x \to a^+} f(x) = f(a) \text{ (right continuous at a point } a$
 Example 01 $f(x) = \begin{cases} x - 2, & x \leq 0 \\ 2, & x > 0 \\ 2, & x > 0 \end{cases}$
 $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 2 = 2$
 $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} x - 2 = -2$ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $x-2$, $x \le 0$ $\begin{array}{ccc} 2, & x > 0 \end{array}$ $\lim_{x \to a^+} f(x) = f(a) \text{ (right continuous at a point } a)$
 $\lim_{x \to a^+} f(x) = f(a) \text{ (right continuous at a point } a)$
 $x - 2, \quad x \le 0$
 $2, \quad x > 0$
 $= 0, \text{ we see that } f \text{ is continuous. Therefore, we look at } x = 0.$
 $= \lim_{x \to 0^+} 2 = 2$ $\lim_{x\to a^+} f(x) = f(a) \text{ (right continuous at a point } a)$
 $x = 2, \quad x \le 0$
 $x = 2, \quad x > 0$
 $x > 0$
 $\lim_{x\to 0^+} 2 = 2$
 $\lim_{x\to 0^-} x = 2 = -2$ $\lim_{x \to a^+} f(x) = f(a) \text{ (right continuous at a point } a \text{)}$
 $\text{Example 01} \label{eq:2.1} f(x) = \left\{ \begin{array}{ll} x - 2, & x \leq 0 \\ 2, & x > 0 \\ \text{away from } x = 0 \text{, we see that } f \text{ is continuous. Therefore, we look at } x = 0. \end{array} \right.$
 $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 2 = 2$
 $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} x - 2 = -2$
 $\lim_{x \to 0} \text$ $\lim_{x \to a^+} f(x) = f(a) \text{ (right continuous at a point } a \text{) }$
 $\text{Example 01} \\ f(x) = \left\{ \begin{array}{ll} x - 2, & x \leq 0 \\ 2, & x > 0 \\ -1, & x = 0, \text{ we see that } f \text{ is continuous. Therefore, we look at } x = 0. \\ \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x - 2 = -2 \\ \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} x - 2 = -2 \\ \lim_{x \to 0^-} \text{does not exist. We conclude that } f \text{ is discontinuous at } x = 0. \text$ $\lim_{x \to a^+} f(x) = f(a) \text{ (right continuous at a point } a \text{) }$
 Example 01 $\begin{aligned} &f(x) = \left\{ \begin{array}{ll} x - 2, & x \leq 0 \\ 2, & x > 0 \\ x - 2, & x \leq 0 \end{array} \right. \\ &\text{Any from } x = 0, \text{ we see that } f \text{ is continuous. Therefore, we look at } x = 0. \\ &\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x - 2 = -2 \\ &\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} x - 2 = -2 \\ &\lim$ $\lim_{x \to a^-} f(x) = f(a) \text{ (right continuous at a point } a \text{) }$
 Example 01
 $\text{Example 01} \hspace{2.2cm} x - 2, \quad x \leq 0$
 $\text{Now, from } x = 0, \text{ we see that } f \text{ is continuous. Therefore, we look at } x = 0.$
 $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 2 - 2 = -2$
 $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x - 2 = -2$
 $\lim_{x \to 0^+} \text{does not exist. We conclude that } f \text{ is discontin$ $\lim_{x\to a^+} f(x) = f(a) \text{ (right continuous at a point } a \text{) }$
 Example 01
 $f(x) = \begin{cases} x-2, & x \le 0 \\ 2, & x > 0 \end{cases}$

Away from $x = 0$, we see that f is continuous. Therefore, we look at $x = 0$.
 $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 2 = 2$
 $\lim_{x\to 0^+} f(x) = \lim$ **Example 01**
 $f(x) = \begin{cases} x-2, & x \le 0 \\ 2, & x > 0 \end{cases}$

Away from $x = 0$, we see that f is continuous. Therefore, we look at $x = 0$.
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 $f(x) = \begin{cases} \n& x - 2, \quad x \le 0 \\ \n& 2, \quad x > 0 \n\end{cases}$

Away from $x = 0$, we see that f is continuous. Therefore, we look at $x = 0$.
 $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x - 2 = 2$
 $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x - 2 = -2$
 \lim_{x

Remark

 $\begin{minipage}[c]{0.9\linewidth} \textbf{Remark} \end{minipage} \begin{minipage}[c]{0.9\linewidth} \textbf{Remark} \end{minipage} \begin{minipage}[c]{0.9\linewidth} \textbf{The following functions are all continuous at all points of there domains:} \begin{align*} \textbf{(i) Polynomials}; \textbf{(ii) Rational Functions}; \textbf{(iii) Root Functions}; \textbf{(vi) Trigonometric Functions:} \begin{align*} \textbf{(v) Inverse Trigonometric Functions}; \textbf{(vi) Exponential Functions}; \textbf{(vii) Logarithmic Functions:} \end{align*} \end{minipage} \begin{minipage}[c]{0.9\linewidth} \textbf{Matrix} \end{minipage} \begin{$ 3.3. Continuity and IVT

The following functions are all continuous at all points of there domains:

(i) Polynomials;(ii) Rational Functions;(iii) Root Functions;(iv) Trigonometric Functions

(v) Inverse Trigonometric Func (v) Inverse Trigonometric Functions; (vi) Trigonometric Functions; (iii) Polynomials; (ii) Rational Functions; (iii) Root Functions; (iv) Trigonometric Functions; (vi) Inverse Trigonometric Functions; (vi) Exponential Fun tions 3.3. Contin
 Remark

The following functions are all continuous at all points of there domains:

(i) Polynomials;(ii) Rational Functions;(iii) Root Functions;(iv) Trigonomet

(v) Inverse Trigonometric Functions; (vi) Ex 3.3. Continuity and IVT
 Remark

The following functions are all continuous at all points of there domains:

(i) Polynomials;(ii) Rational Functions;(iii) Root Functions;(iv) Trigonometric Functions

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 Remark

The following functions are all continuous at all points of there domains:

(i) Polynomials;(ii) Rational Functions;(iii) Root Functions;(iv) Trigonome

(v) Inverse Trigonometric Functions; (vi) Exp tions are all continuous at all points of there domains:

Rational Functions; (iii) Root Functions; (ivi) Trigonometric Functions

metric Functions; (vii) Exponential Functions; (vii) Logarithmic Func-

1es of x for which

1)
$$
f(x) = \frac{2}{x^2+1}
$$
.

Contract Contract Contract

2)
$$
f(x) = \begin{cases} \frac{x^2 - 1}{x - 1}, & x \neq 1 \\ 2, & x = 1 \end{cases}.
$$

Soltion

2.3. Continuity and IVT

The following functions are all continuous at all points of there domains:

(i) Polynomials;(ii) Rational Functions;(iii) Root Functions;(iv) Trigonometric Functions

(v) Inverse Trigonometric Fun ; functions are all continuous at all points of there domains:

ls;(ii) Rational Functions;(iii) Root Functions;(iv) Trigonometric Functions

igonometric Functions; (vi) Exponential Functions; (vii) Logarithmic Func-

e v $\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{(x-1)(x+1)}{(x-1)} = \lim_{x\to 1} x+1 = 2$. Since $\lim_{x\to 1}$ rse Trigonometric Functions; (vi) Exponential Functions; (vii) Logarithm

se **01**

ine the values of *x* for which each function is continuous.
 $=\frac{2}{x^2+1}$.
 \therefore Since $x^2+1=0$ has no real solutions, we see that *f* c Functions; (vi) Exponential Functions; (vii) Logarithmic Func-
 x for which each function is continuous.

0 has no real solutions, we see that f is continuous for all $x \in \mathbb{R}$.
 $\neq 1$
 $\frac{-1)(x+1)}{(x-1)} = \lim_{x\to 1}$ (v) Inverse Trigonometric Functions; (vi) Exponential Functions; (vi) Logarithmic Func-

tions

Exercise 01

Determine the values of x for which each function is continuous.

1) $f(x) = \frac{2}{x+1}$.

Soltion: Since $x^2 + 1 = 0$ tons
 Exercise 01

Determine the values of x for which each function is continuous.

1) $f(x) = \frac{2}{x^2+1}$.
 Soltion: Since $x^2 + 1 = 0$ has no real solutions, we see that f is continuous for

2) $f(x) = \begin{cases} \frac{x^2-1}{x-1}, &$ Exercise 01

Determine the values of x for which each function is continuous.

1) $f(x) = \frac{2}{x^2+1}$.

Soltion: Since $x^2 + 1 = 0$ has no real solutions, we see that f is continuous for all $x \in \mathbb{R}$.

2) $f(x) = \begin{cases} \frac{x^2-$ **Soltion**: Since $x^2 + 1 = 0$ has no real solutions, we see that f is continuous for

2) $f(x) = \begin{cases} \frac{x^2-1}{x-1}, & x \neq 1 \\ 2, & x = 1 \end{cases}$.
 Soltion
 $\lim_{x \to 1} \frac{x^2-1}{x-1} = \lim_{x \to 1} \frac{(x-1)(x+1)}{(x-1)} = \lim_{x \to 1} x + 1 = 2$.

Sinc 2) $f(x) = \begin{cases} \frac{x^2-1}{x-1}, & x \neq 1 \\ 2, & x = 1 \end{cases}$
 Soltion
 $\lim_{x \to 1} \frac{x^2-1}{x-1} = \lim_{x \to 1} \frac{(x-1)(x+1)}{(x-1)} = \lim_{x \to 1} x + 1 = 2$. Since $\lim_{x \to 1} f(x)$

tinuous at $x = 1$. Additionally, we see that f is continuous everywhe $\begin{cases}\n\frac{x^2-1}{x-1}, & x \neq 1 \\
2, & x = 1\n\end{cases}$
 $\frac{1}{x} = \lim_{x \to 1} \frac{(x-1)(x+1)}{(x-1)} = \lim_{x \to 1} x + 1 = 2.$ Since $\lim_{x \to 1} f(x) = 2 = f(1), f$ is $x = 1$. Additionally, we see that f is continuous everywhere else. (because y function).
 2) $f(x) = \begin{cases} \frac{x-1}{x-1}, & x \neq 1 \\ 2, & x = 1 \end{cases}$

Soltion
 $\lim_{x \to 1} \frac{x^2-1}{x^2-1} = \lim_{x \to 1} \frac{(x-1)(x+1)}{(x-1)} = \lim_{x \to 1} x + 1 = 2$. Since $\lim_{x \to 1} f(x) = 2 = f(1), f$ is the final star $x = 1$. Additionally, we see that f is contin timuous at $x = 1$. Additionally, we see that f is continuous everywhere else. (because
selementary function).
Hence, f is continuous on \mathbb{R} .
Exercise 02
Consider the function
 $g(x) = \begin{cases} 2x - 3, & \text{if } x \le 1 \\ 0, & \text$

Consider the function \overline{f}

$$
g(x) = \begin{cases} 2x - 3, & \text{if } x < 1 \\ 0, & \text{if } x \ge 1 \end{cases}
$$

Solution

$$
\lim_{x \to 1^+} g(x) = \lim_{x \to 1^+} (0) = 0,
$$

and

Hence, f is continuous on R.

Exercise 02

 $g(x) = \begin{cases} 2x - 3, & \text{if } x < 1 \\ 2x - 3, & \text{if } x \ge 1 \\ 0, & \text{if } x \ge 1 \end{cases}$

Show that it is continuous at the point $x = 1$. Is g a continuous function?

Solution
 $\lim_{x \to 1^+} g(x$ **Exercise 02**

Consider the function
 $g(x) =\begin{cases} 2x-3, & \text{if } x < 1 \\ 0, & \text{if } x \ge 1 \end{cases}$

Show that it is continuous at the point $x = 1$. Is g a continuous function
 Solution
 $\lim_{x\to 1^+} g(x) = \lim_{x\to 1^-} (0) = 0,$

and
 \lim

Exercise 03

\nFind the values of
$$
\alpha
$$
 that make the function $f(x)$ continuous for all real numbers.

\n
$$
f(x) = \begin{cases} 4x + 5, & \text{if } x \geq -2 \\ x^2 + \alpha, & \text{if } x < -2 \end{cases}
$$

\n**Solution**

\nFirst, we note that, for $x > -2$, $f(x) = 4x + 5$ is continuous. For $x < -2$, $f(x) = x^2 + \alpha$

Solution

3.3. Continuity and IVT

3

3

thes of α that make the function $f(x)$ continuous for all real numbers.
 $x + 5$, if $x \ge -2$
 $x + \alpha$, if $x < -2$

4

te that, for $x > -2$, $f(x) = 4x + 5$ is continuous. For $x < -2$, $f(x) = x^2 + \alpha$ **Exercise 03**

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 Solution

First, we note that, for $x > -2$, $f(x) = 4x + 5$ is cont 3.3. Continuity and IVT

ke the function $f(x)$ continuous for all real numbers.
 $\therefore -2$
 ≤ -2
 $\Rightarrow f(x) = 4x + 5$ is continuous. For $x < -2$, $f(x) = x^2 + \alpha$
 \Rightarrow of α . We now look at $x = 2$.
 $x + 5$) = -3.
 $x + 6$ + α $f(x) = 4x + 5$ is continuous. For $x < -2$, $f(x) = x^2 + \alpha$
We now look at $x = 2$.
= -3.
= 4 + α
herefore require 4 + $\alpha = -3 \implies \alpha$ has to be -7.
 $f(x) = f(-2) = -3$
all other choices of α , f would be discontinuous at First, we note that, for $x > -2$, $f(x) = 4x + 5$ is continuous. For $x < -2$, $f(x) = x^2 + \alpha$
will be continuous for all choices of α . We now look at $x = 2$.
 $\lim_{x \to -2^+} f(x) = \lim_{x \to -2^+} (4x + 5) = -3$.
 $\lim_{x \to -2^-} f(x) = \lim_{x \to -2} (x^2$

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\lim_{x \to -2^+} f(x) = \lim_{x \to -2^+} (4x + 5) = -3.
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\lim_{x \to -2^-} f(x) = \lim_{x \to -2^-} (x^2 + \alpha) = 4 + \alpha.
$$

3.3. Continuity and IVT

Exercise 03

Find the values of α that make the function $f(x)$ continuous for all real numbers.
 $f(x) = \begin{cases} 4x + 5, & \text{if } x \ge -2 \\ x^2 + \alpha, & \text{if } x < -2 \end{cases}$

Solution

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 Exercise 03

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 $\lim_{x \to -2^-} f(x) = \lim_{$ will be continuous for all choices of α . We now look at $x = 2$.
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 $\lim_{x \to -2^{-}} f(x) = \lim_{x \to -2} (x^2 + \alpha) = 4 + \alpha$

In order for the limit to exist, we therefore require $4 + \alpha = -3 \Longrightarrow \alpha$ has t $\begin{split} &\lim_{x\to -2^+}f(x)=\lim_{x\to -2^+}(4x+5)=-3.\\ &\lim_{x\to -2^-}f(x)=\lim_{x\to -2^-}(x^2+\alpha)=4+\alpha\\ &\text{In order for the limit to exist, we therefore require }4+\alpha=-3\Longrightarrow \alpha\text{ has}\\ &\text{With this choice of we get that}\\ &\lim_{x\to -2}f(x)=f(-2)=-3\\ &\text{and so would be continuous. For all other choices of }\alpha,\ f\text{ would be d}\\ &x=-2.\\ \textbf{Operations of Continuous Functions}\\ &\text{If f and g are continuous at a and λ is a constant, then the following funcointuous at a.}\\ &\qquad f$ $\lim_{x \to -2^{-}} (x^{2} + \alpha) = 4 + \alpha$

and to exist, we therefore require $4 + \alpha = -3 \implies \alpha$ has to be -7 .

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 $\lim_{x \to -2} f(x) = f(-2) = -3$

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 $f \pm g$; λf ; fg; $\frac{f}{g}(g \neq 0)$

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 ontinuity of Function Composition

is continuous at for Continuous Functions

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i. $fg: \frac{f}{g}(g \neq 0)$

of Function Composition

nois at a and f is continuous at $g(a)$, then the composition fu **Operations of Continuous Functions**

If f and g are continous at a and λ is a constant, then the following functions are also

continuous at a .
 $f \pm g$; λf ; $f g$; $\frac{f}{g}(g \neq 0)$
 Continuity of Function C

Example

 $g(x) = x^2$ is continuous on R since it is a polynomial, and $f(x) = \cos(x)$ is also continuous

Proposition

3.3. Continuity and IVT

If $f(x)$ is continuous at b and $\lim_{x\to a} g(x) = b$ then,
 $\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x))$

Example Evaluate the following limit.

3.3. Continuity and IVT
\n
$$
\lim_{x \to a} g(x) = b \text{ then},
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\lim_{x \to 0} e^{\sin x}
$$

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Example Evaluate the following limit.

Since we know that exponentials are continuous everywhere we can use the proposition
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Proposition

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 Example Evaluate the following limit.
 $\lim_{x\to 0} e^{i\ln x}$

Since we know that exponentials are continuous everywhere $\lim_{x\to a} e^{i\text{sin}x}$ $\lim_{x\to 0} e^{i\text{sin}x}$ Since we know that exponentials are continuous everywhere we can use the proposition

above.
 $\lim_{x\to 0} e^{i\text{sin}(x)} = e^{\lim_{x\to 0} \sin x} = e^0 = 1.$
 3.3.2 Theorem Intermediate Value Th $\lim_{x\to 0}e^{\sin x}$
Since we know that exponentials are continuous everywhere we can use the proposition
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3.3.2 Theorem Intermediate Value Theorem (IVT)
If f is cont

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Since we know that exponentials are continuous everywhere we can use the proposition
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S.2 Theorem Intermediate Value Theorem (IVT)
is continuous on the interv $\lim_{x\to 0} e^{\sin x}$ Since we know that exponentials are continuous everywhere we can use the proposition

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is continuous on the i Since we know that exponentials are continuous everywhere we can use the proposition

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 $\lim_{x\to 0} e^{\sin(x)} = e^{\lim_{x\to 0} \sin x} = e^0 = 1.$

3.2 **Theorem Intermediate Value Theorem (IVT)**

is continuous on the interval $[a, b]$ an above.
 $\lim_{x\to 0} e^{4n(x)} = e^{\lim_{x\to 0} \sin x} = e^0 = 1.$
 3.3.2 Theorem Intermediate Value Theorem (IVT)

If *f* is continuous on the interval $[a, b]$ and *N* is between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$,
 $f(b)$, then there is **3.3.2 Theorem Intermediate Value Theorem (IVT)**
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If f is continuous on the interval [a, b] and N is between $f(a)$ and $f(b)$, where $f(a) \neq$
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f is continuous **3.3.2 Theorem Intermediate Value Theorem (IVT)**

If f is continuous on the interval $[a, b]$ and N is between $f(a)$ and $f(b)$, where $f(a) \neq$
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f is continuous o **S.2 Theorem Intermediate Value Theorem (IVT)**

^F is continuous on the interval $[a, b]$ and N is between $f(a)$ and $f(b)$, where $f(a) \neq$
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 f is continuous on If f is continuous on the interval [a, b] and N is between $f(a)$ and $f(b)$, where $f(a) \neq$
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f is continuous on $|a, b|$ and $f(a) \le N \le f(b) \implies \exists c \in |a, b|$; $f(c) = N$),
then there is a number c in $]a, b]$ such that $f(c) = N$.
The Intermediate Value Theorem guarantees that if $f(x)$ is continuous and
 $f(a) \le N \le f(b)$.
The Intermediate Value Theorem guarantees that if $f(x)$ is continuous and
 f is continuous on $[a, b]$ and $f(a) \le N \le f(b) \implies \exists c \in [a, b] : f(c) = N$
The Intermediate Value Theorem guarantees that if $f(x)$ is continuous and
 $f(a) \le N \le f(b)$.The line $y = N$ intersects the function at some point $x = c$. Such a
mum The Intermediate Value Theorem guarantees that if $f(x)$ is continuous and
 $f(a) \le N \le f(b)$. The line $y = N$ intersects the function at some point $x = c$. Such

number is between a and b and has the property that $f(c) = N$. (See

Philadel Controllering and finally and the intermediate Value Theorem
 Philadel Controllering a $\mathbb{E}(\mathbf{x})$ is the intermediate value theorem is to verify the existence of

ont of an equation in a given interval. In **Example 01:** Use the IVT to show that the function; thus, $f(x)$ is clearly condition for the function fast is a proper metricular, the IVT theorem is used to see ther a given function has its zero $(f(x) = 0)$ within the gi **EM**

b verify the existence of

theorem is used to see

interval $]a, b[$. To verify

bositive).
 $3 + 2x - 1$ has a zero in
 $x^3 + 2x - 1$ has a zero in

hus, $f(x)$ is clearly con-**Application of Intermediate Value Theor**
The important application of the intermediate value theorem is to
a root of an equation in a given interval. In particular, the IVT
whether a given function has its zero ($f(x) = 0$ is important application of the intermediate value theorem is to verify the existence of
oot of an equation in a given interval. In particular, the IVT theorem is used to see
ther a given function has its zero $(f(x) = 0)$ w a root of an equation in a given interval. In particular, the IVT theorem is used to see
whether a given function has its zero $(f(x) = 0)$ within the given interval $[a, b]$. To verify

this, we follow the steps below:
 Step

tive and $f(b)$ is positive).

terval $]a, b[$.

function $f(x) = x^3 + 2x - 1$ has a zero in

omial function; thus, $f(x)$ is clearly con-
 $\lt 2$.

st exist a $c \in [0, 1]$ such that $f(c) = 0$.
 $x^3 + 3x^2 + x - 2$ has a root between nd $f(b)$ is positive).
 $a, b[$.

on $f(x) = x^3 + 2x - 1$ has a zero in

function; thus, $f(x)$ is clearly con-

st a $c \in [0, 1]$ such that $f(c) = 0$.
 $x^2 + x - 2$ has a root between 0 and
 $x^2 + x - 2$ has a root between 0 and
 x 1: **Example 01:** Use the IVT to show that the function $f(x) = x^3 + 2x - 1$ has a zero in interval [0, 1].
 Solution
 $f(x)$ is continuous everywhere as it is a polynomial function; thus, $f(x)$ is clearly con-

too son [0, 1].
 Example 01: Use the IVT to show that the function $f(x) = x^3 + 2x - 1$ has a zero in
the interval [0,1].
Solution
 $f(x)$ is continuous everywhere as it is a polynomial function; thus, $f(x)$ is clearly con-
tinuous on [0,1]

Solution

Example

is defined and continuous for all $x \neq 0$. As $\lim_{x\to 0} \frac{\sin x}{x} = 1$, it makes sense to define a \mathbf{R} for all $x \neq 0$. As $\lim_{x\to 0} \frac{\sin x}{x} = 1$, it makes sense to define a
 $\neq 0$
 $\{c\} \to \mathbb{R}$ be a function

ant $f(c)$ is not defined, we define a new function
 $F(x) = \begin{cases} f(x), & \text{for } x \neq c \\ L, & \text{for } x = c \end{cases}$

is called th is defined and continuous for all $x \neq 0$. As $\lim_{x \to 0} \frac{\sin x}{x} = 1$, it makes sense to define a

function
 $F(x) = \begin{cases} \frac{\sin x}{x}, & \text{for } x \neq 0 \\ 1, & \text{for } x = 0 \end{cases}$

finition 3.3.4 Let $f : D - \{c\} \to \mathbb{R}$ be a function

If \lim

$$
F(x) = \begin{cases} \frac{\sin x}{x}, & \text{for } x \neq 0 \\ 1, & \text{for } x = 0 \end{cases}
$$

$$
F(x) = \begin{cases} f(x), & \text{for } x \neq c \\ L, & \text{for } x = c \end{cases}
$$