CHAPTER 1

LOGIC AND REASONING

1.1 Logic

1.1.1 Assertion (proposition)

Definition 1.1. *Proposition (Assertion) is a declarative statement declaring some fact. It is either true or false but not both.*

Example 1.2.

- *The proposition* $2+2=5$ *is false*
- *The proposition if* $x>4$ *then* $x>2$ *is true. .*
- *The proposition M'sila is in France is false.*
- *The expression* $x>0$ *is not a proposition, because contains a free variable* x*.*

Remark 1.3. *Following kinds of statements are not propositions-*

- ***Command**, example: Close the door.*
- ***Question**, example : Do you speak French?*
- ***Exclamation**, example: What a beautiful picture!*
- ***Inconsistent**, example : I always tell lie.*
- ***Predicate or Proposition Function**, example:* $P(x): x+3=5$

1.1.2 Logical Connectives

Logical connectives are the operators used to combine one or more propositions. In propositional logic. there are 5 basic connectives

① **Negation**

Definition 1.4. *The negation of a proposition P is the proposition, denoted $\overline{P}$, defined by :*

• *$\overline{P}$ is true whenever P is false ;*

• *$\overline{P}$ is false whenever P is true.*

Truth Table

P	$\overline{P}$
V	F
F	V

② **Conjunction:**

Let P and Q be two propositions. The conjunction of P and Q is the proposition, denoted $P \wedge Q$ (P and Q), defined as follow :

•$P \wedge Q$ is true when both P and Q are true.:

•$P \wedge Q$ is false when at least one of the two propositions is false.

Truth Table :

P	Q	$P \wedge Q$
V	V	V
V	F	F
F	V	F
F	F	F

③ **Th disjunction:**

Let P and Q be two propositions. The disjunction of P and Q is the proposition, denoted $P \vee Q$ (P or Q), defined as follow :

•$P \vee Q$ is true when at least one of the two propositions is true.

•$P \vee Q$ is false when both P and Q are false.

Truth Table :

P	Q	$P \vee Q$
V	V	V
V	F	V
F	V	V
F	F	F

④ **Implication**

Definition 1.5. *Let P and Q be two propositions.*

1. *The assertion $P \implies Q$ means that if P is true then Q must be true too. the mathematical definition of " $P \implies Q$" is the assertion $(\overline{P} \vee Q)$".*

Truth Table :

P	$\overline{P}$	Q	$P \implies Q$
V	F	V	V
V	F	F	F
F	V	V	V
F	V	F	V

2. ***The contrapositive** of the implication "$P \implies Q$" is the implication $\overline{Q} \implies \overline{P}$*

3. ***The converse** of the implication "$P \implies Q$" is the implication $Q \implies P$.*

4. ***The negation** of the implication "$Q \implies P$": is "$P \wedge \overline{Q}$"*

⑤ **Equivalent**

Let P and Q be two propositions. The equivalent $P \iff Q$ is the assertion $P \implies Q$ and $Q \implies P$. It is the statement that is true when the implication $P \implies Q$ and its converse $Q \implies P$ are both true simultaneously. We say that P is equivalent to Q, or in other words, P is true if and only if Q is true.

- $P \iff Q$ is true when either both P and Q are true or both P and Q are false.
- $P \iff Q$ is false in all other cases.

Truth Table :

P	Q	$P \iff Q$
V	V	V
V	F	F
F	V	F
F	F	V

Proposition 1.6. *Let P and Q be two propositions then:*

• $\overline{P \wedge Q} \iff \overline{P} \vee \overline{Q}$

• $\overline{P \vee Q} \iff \overline{P} \wedge \overline{Q}$

• $\overline{P \implies Q} \iff P \wedge \overline{Q}$

• $(P \implies Q) \iff (\overline{Q} \implies \overline{P})$

1.1.3 Quantifiers

If $P(x)$ is a predicate, then

1. **Existential Quantifier** $\exists x : P(x)$ (there is, there exist(s),) means, "There exists at least one element x such that $P(x)$ holds. Example: $\exists x \in \mathbb{R}, x \leq 0$.

2. **Universal Quantifiers** $\forall x : P(x)$ (for every , for all, for each) means, "For all x, the predicate $P(x)$ holds. Example: $\forall x \in \mathbb{R}, x^2 \geq 0$.

Remark 1.7. • *The negation of* $\forall x : P(x)$ *is* $\exists x : \overline{P(x)}$.
• *The negation of* $\exists x : P(x)$ *is* $\forall x : \overline{P(x)}$. •

Example 1.8. • *There is an integer x for which* $5\breve{}x = 2$
. • *$2n$ is an even number for all natural numbers n.*
The existential quantifier is used in the first sentence, indicating that at least one integer x fulfills the equation $5\breve{}x = 2$. The second statement uses the universal quantifier, which states that $2n$ is an even integer for every natural number n.

Remark 1.9. *"We sometimes encounter the symbol $\exists!$, which means 'there exists a unique.' For example: $\exists! x \in \mathbb{R}, x^3 = 1$." This symbol is used to indicate that there is one and only one element satisfying a particular condition. In the given example, it means "there exists a unique real number x such that $x^3 = 1$." The unique real number satisfying this equation is 1, since $1^3 = 1$.*

1.2 Modes of reasoning

1.2.1 Direct reasoning

A direct proof is one of the most familiar forms of proof. We use it to prove statements of the form "if p then q" or "p implies q," which we can write as $p \Rightarrow q$. The method of the proof is to take an original statement p, which we assume to be true, and use it to show directly that another statement q is true. So, a direct proof has the following steps:

- Assume the statement p is true.
- Use what we know about p and other facts as necessary to deduce that another statement q is true, that is, show $p \Rightarrow q$ is true.

Example 1.10. *Directly prove that if n is an odd integer, then n^2 is also an odd integer.*

Solution: Let p be the statement that n is an odd integer, and q be the statement that n^2 is an odd integer. Assume that n is an odd integer, then by definition, $n = 2k+1$ for some integer k. We will now use this to show that n^2 is also an odd integer.

$$\begin{aligned} n^2 &= (2k+1)^2 \text{ since} n = 2k+1 \\ &= 4k^2+4k+1 \\ &= 2(2k^2+2k)+1 \end{aligned}$$

.

Hence we have shown that n^2 has the form of an odd integer since $2k^2+2k$ is an integer. Therefore, we have shown that $p \Rightarrow q$, and so we have completed our proof.

1.2.2 Proof by Cases:

If one wishes to verify a statement $P(x)$ for all x in a set E, they show the statement for all x in a subset A of E and then for x not belonging to A. This is the proof by cases.

Example 1.11. *Prove by cases that, for all$n \in \mathbb{N}$, n^2+3n+7 is odd.*

Solution If $n \in \mathbb{N}$, then either n is even or n is odd.
Case 1: If n is even, then $n = 2k$ for some $k \in \mathbb{N}$. Thus,

$$n^2+3n+7 = (2k)^2+3(2k)+7 = 2(2k^2+3k)+7 = 2p+7,$$

where $p = 2k^2 + 3k$. So since $2k^2 + 3k \in \mathbb{N}$, we have $n^2 + 3n + 7$ is odd.

Case 2: Similarly, if n is odd, then $n = 2k + 1$ for some $k \in \mathbb{Z}$. Thus,

$$n^2 + 3n + 7 = (2k+1)^2 + 3(2k+1) + 7 = 2(2k^2 + 5k) + 1 + 7 = 2p + 1,$$

where $p = 2k^2 + 5k$. So since $2k^2 + 5k \in \mathbb{N}$, we have $n^2 + 3n + 7$ is odd.

1.2.3 Proof by Contrapositive

The reasoning by Contrapositive is based on the following equivalence:

The statement "$P \implies Q$" is equivalent to "$\overline{Q} \implies \overline{P}$".

So, if we want to prove the statement "$P \implies Q$"", we actually show that if "$\overline{Q}$" is true, then "$\overline{P}$" is true.

Example 1.12. *Let $n \in \mathbb{N}$. Show that if n^2 is even, then n is even.*

Solution: we assume that n is not even. We want to show that in this case, n^2 is not even. Since n is not even, it is odd, and therefore, there exists $k \in \mathbb{N}$ such that $n = 2k + 1$. Then $n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2m + 1$, where $m = 2k^2 + 2k \in \mathbb{N}$. Thus, n^2 is odd. Conclusion: We have shown that if n is odd, then n^2 is odd. By Contrapositive, this is equivalent to: if n^2 is even, then n is even.

1.2.4 Proof by Contradiction

In this method, you assume the negation of the statement you want to prove and then show that this assumption leads to a contradiction. Since a contradiction cannot be true, it follows that the original statement must be true.

Example 1.13. *Prove by Contradiction that $\sqrt{2}$ is not rational*

To prove that the square root of 2 is not rational by contradiction, we assume the opposite: suppose that $\sqrt{2}$ is rational. By definition, a rational number can be expressed as a fraction $\frac{a}{b}$, where a and b are integers, and $b \neq 0$. So, we can express $\sqrt{2}$ as a fraction: $\sqrt{2} = \frac{a}{b}$ where a and b are integers with $pcd(a, b) = 1$. Now, we square both sides of the equation: $(\sqrt{2})^2 = \left(\frac{a}{b}\right)^2 \implies 2 = \frac{a^2}{b^2} \implies a^2 = 2 \cdot b^2 =$. Since a^2 is equal to $2 \cdot b^2$, it means that a^2 is even, because $2 \cdot b^2$ is even. and by example **??** we deduce that a is even, that is $a = 2k$, $k \in \mathbb{N}$. Substituting this into the equation $a^2 = 2 \cdot b^2$, we get $b^2 = 2k^2$, so by Example (**??**) b is even. Therefore, both a and b are even, which contradicts our initial assumption that a and b have no common factors other than 1. So our initial assumption that $\sqrt{2}$ is rational must be false. Therefore, the square root of 2 is not rational.

1.2.5 Counterexample

In some cases, you can disprove a statement by providing a counterexample that shows the statement is false for at least one instance.

Example 1.14. *Consider real-valued functions defined on the interval $0 \leq x \leq 1$. Give a counterexample to disprove the following statement: "If the product of two functions is the zero function, then one of the functions is the zero function."*
(The zero function is the function which produces 0 for all inputs — i.e. the constant function $f = 0$.)
Here are two functions whose product is the zero function, neither of which is the zero function:

$$f(x) = \begin{cases} \frac{1}{2} - x & \text{if } 0 \leq x \leq \frac{1}{2} \\ 0 & \text{if } \frac{1}{2} < x \leq 1 \end{cases}$$

$$g(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq \frac{1}{2} \\ x - \frac{1}{2} & \text{if } \frac{1}{2} < x \leq 1 \end{cases}$$

Remark 1.15. *(a) A single example can't prove a universal statement (unless the universe consists of only one case!).*

(b) A single counterexample can disprove a universal statement.

1.2.6 Induction

: Let $P(n)$ be a given statement involving the natural number $n \geq n_0$ such that

i) The statement is true for $n = n_0$, i.e., $P(n_0)$ is true

ii) If the statement is true for $n = k$ (where k is a particular but arbitrary natural number), then the statement is also true for $n = k + 1$, i.e, truth of $P(k)$ implies the truth of $P(k + 1)$. Then $P(n)$ is true for all natural numbers n.

Example 1.16. *Let's prove by induction that*

$$\sum_{k=1}^{n} k^2 = 1^3 + 2^3 + \ldots + n^3.$$

Let $P(n)$ be the property: $(1 + 2 + \ldots + n)^2 = 1^3 + 2^3 + \ldots + n^3$. The property $P(1)$ is true because $1^2 = 1^3$. Now, let n be a nonzero natural number. We will show that the implication $P(n) \Rightarrow P(n + 1)$ is true.

Assume that the property is true for the rank n (this is the induction hypothesis) and let's prove that it is also true for the rank $n+1$, i.e., we want to show that

$$(1+2+\ldots+n+1)^2 = 1^3+2^3+\ldots+(n+1)^3.$$

Let's start from the left-hand side and arrive at the right-hand side (it's simpler this way). Let $S = 1+2+\ldots+n$. Then, we have:

$$(1+2+\ldots+n+1)^2 = (S+(n+1))^2 = S^2+2(n+1)S+(n+1)^2.$$

Now, S, which is the sum of the first n integers, is equal to $n(n+1)/2$. Also, according to the induction hypothesis, $S^2 = 1^3+2^3+\ldots+n^3$. Therefore, we get:

$$(1+2+\ldots+n+1)^2 = 1^3+2^3+\ldots+n^3+n(n+1)^2+(n+1)^2 = 1^3+2^3+\ldots+(n+1)^3.$$

Thus, we have shown that for any nonzero natural number n, the implication $P(n) \Rightarrow P(n+1)$ is true. The principle of mathematical induction then allows us to conclude that the equality
$(1+2+\ldots+n)^2 = 1^3+2^3+\ldots+n^3$ *holds true for all nonzero natural numbers n.*