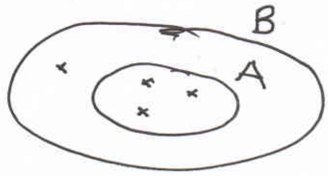
Fac. Tech.

Ener. Ren

Math 1. T. HERAIZ

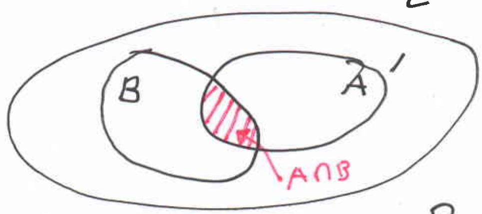
Série 02 (La Théorie des ensembles et les applications)

Rappel, A, B deux parties de E

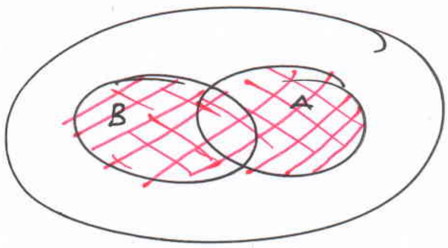


$A \subset B \iff x \in A \Rightarrow x \in B$

$A \cap B = \{x \in E, \quad x \in A \underset{\wedge}{\text{et}} x \in B\}$

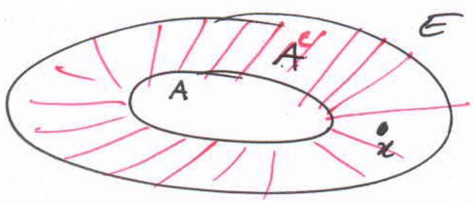


$A \cup B = \{x \in E, \quad x \in A \underset{\vee}{\text{ou}} x \in B\}$



- $A^c_E = A^c = \{x \in E \quad , x \notin A\} = E \setminus A =$

- $A \cup A^c_E = ?$
- $A \cap A^c_E = ?$



$$\boxed{(A^c_E)^c_E = A}$$

$(A^c)^c = A$

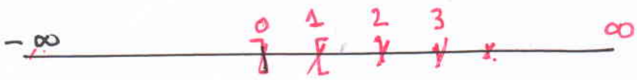
Par exemple : $\mathbb{N} \subset \mathbb{R}$

$\mathbb{N}^c_{\mathbb{R}} = \{x \in \mathbb{R} \ , x \notin \mathbb{N}\} = \mathbb{R} \setminus \mathbb{N}$

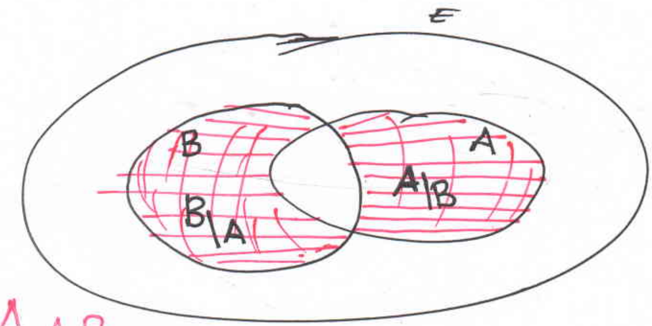
$=]-\infty, 0[\cup]0, 1[\cup]1, 2[\cup \ldots]n, n+1[\cup]n+1, \ldots$

$=]-\infty, 0[\cup \left(\bigcup_{i=0}^{\infty}]i, i+1[\right)$

$$\boxed{\textbf{Remarque} : A \setminus B = A \cap B^c_E}$$



$A \triangle B = A_{\setminus B} \cup B_{\setminus A} = \dots\dots$



$A \triangle B = A \cup B \setminus A \cap B$.

Par exemple $A = \{1, 2, 3, 4, 5\}$, $B = \{4, 5, 6, 7, 8, 9\}$

$A \cap B = \{4, 5\}$

$A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

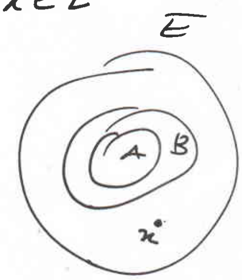
$A_{\setminus B} = \{1, 2, 3\}$, $B_{\setminus A} = \{6, 7, 8, 9\}$

$A \triangle B = \{1, 2, 3, 6, 7, 8, 9\}$

Exercice 01 , $A, B \subset E$ Montrer que.

① $A \subset B \iff B^c \subset A^c$

a) $\Rightarrow$) On Considère que $A \subset B$, Soit $x \in E$



$x \in B^c \Rightarrow x \notin B \Rightarrow x \notin A$ (Car $A \subset B$)

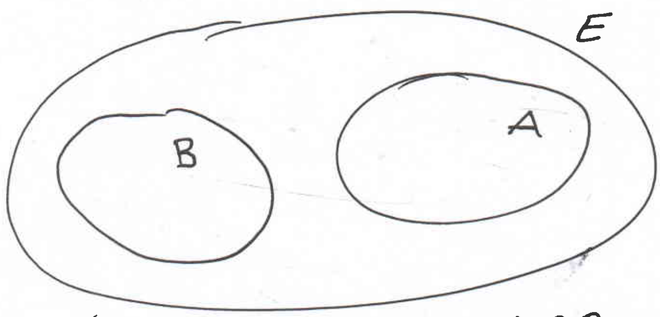
Alors $x \in A^c$. ainsi $B^c \subset A^c$

b) $\Leftarrow$) . On Considère que $B^c \subset A^c$, Soit $x \in E$

$x \in A \Rightarrow x \notin A^c \Rightarrow x \notin B^c$ (Car $B^c \subset A^c$)

Alors $x \in B$. Ainsi . $A \subset B$

2) $A \cap B = \emptyset \iff A \subset \complement_E B$



← A et B sont disjoint (séparés).

a) $\Longrightarrow$) On considère $A \cap B = \emptyset$, soit $x \in E$

$\forall x \in E$.

$x \in A \Longrightarrow x \notin B$ (car $A \cap B = \emptyset$)

$\Longrightarrow x \in \complement B$ Alors $A \subset \complement B$

b $\Longleftarrow$) On Considère $A \subset \complement_E B$, $\forall x \in E$.

$x \in A \Longrightarrow x \in \complement_E B$ (Car $A \subset \complement_E B$)

$\Longrightarrow x \notin B$. Alors $A \cap B = \emptyset$

On dit dans ce cas que A et B sont disjoint (séparés)

3) $\complement_E (A \cap B) = \complement A \cup \complement B$

$\complement_E (A \cap B) = \{x \in E, \; x \notin A \cap B\}$, $x \in A \cap B \Rightarrow x \in A$ et $x \in B$.

$= \{x \in E \; x \notin A$ ou $x \notin B\}$

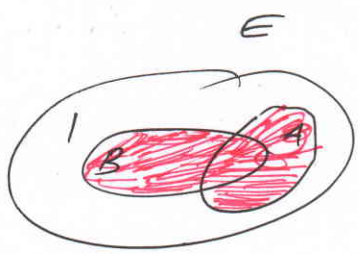
$= \complement_E A \cup \complement_E B$. $\boxed{\text{On peut montrer que: } \complement(A \cup B) = \complement A \cap \complement B \text{ aussi}}$

4) $A \cup (\complement_E A \cap B) = A \cup B$

$A \cup (\complement A \cap B) = (A \cup \complement A) \cap (A \cup B)$

$= E \cap (A \cup B)$

$= A \cup B$ (Car $A \cup B \subset E$).



Exercice 02

a) b) c) (Devoir)

d) Montrer que $A^c \Delta B^c = A \Delta B$

Rappel $M, F \subset E$

$$M \Delta F = M_{\setminus F} \cup F_{\setminus M}$$

1/. $A^c \Delta B^c = A^c_{\setminus B^c} \cup B^c_{\setminus A^c}$

$= \{x \in A^c \text{ et } x \notin B^c\} \cup \{x \in B^c \text{ et } x \notin A^c\}$

$= \{x \in B \text{ et } x \notin A\} \cup \{x \in A \text{ et } x \notin B\}$

$= B_{\setminus A} \cup A_{\setminus B} = A_{B} \cup B_{\setminus A}$

$$A^c \Delta B^c = A \Delta B$$

2/ 2ème Méthode $(M_{\setminus F} = M \cap F^c)$

$A^c \Delta B^c = A^c_{\setminus B^c} \cup B^c_{\setminus A^c}$

$= A^c \cap (B^c)^c \cup B^c \cap (A^c)^c$

$= (A^c \cap B) \cup (B^c \cap A) = (B \cap A^c) \cup (A \cap B^c)$

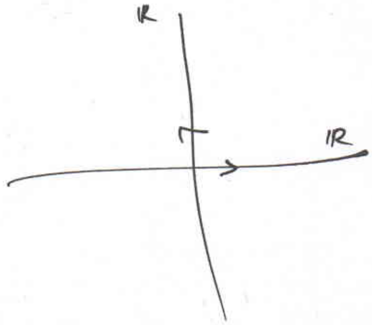
$= B_{\setminus A} \cup A_{\setminus B} = A \Delta B.$

Exercice 03

Rappel : Le Produit Cartésien

E; F deux ensembles

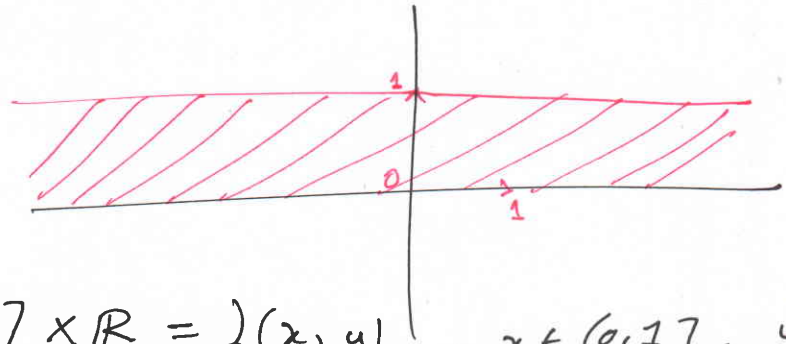
$$E \times F = \{ (x, y) , x \in E, y \in F \}$$



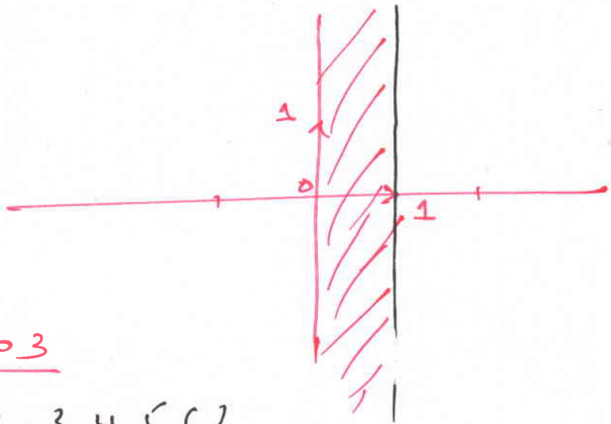
Par exemple

① $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{ (x, y) ; x \in \mathbb{R}, y \in \mathbb{R} \}$

② $\mathbb{R} \times [0, 1] = \{ (x, y), x \in \mathbb{R}, y \in [0, 1] \}$



③ $[0, 1] \times \mathbb{R} = \{ (x, y), x \in [0, 1], y \in \mathbb{R} \}$



Exercice 03

$E = \{1, 2, 3, 4, 5, 6\}$

$E^2 = E \times E = \{ (1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1), (2,2), (2,3),$
$(2,4), (2,5), (2,6), (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (4,1), (4,2), (4,3),$
$(4,4), (4,5), (4,6), (5,1), (5,2), (5,3), (5,4), (5,5), (5,6),$
$(6,1), (6,2), (6,3), (6,4), (6,5), (6,6) \}$

$F = \{ (x, y) \in \mathbb{R}^2$ tq y devise $x \} = \{ (1,1), (2,1), (2,2), (3,1), (3,3),$
$(4,1),$ [illegible] $, (4,4), (5,1), (5,5), (6,1), (6,2), (6,3), (6,6) \}$

II. Les applications

Exercice 01 $f : E_1 \longrightarrow E_2$; $A_1, A_2 \subset E_1$
$B_1, B_2 \subset E_2$.

1). $A_1 \subset A_2 \Longrightarrow f(A_1) \subset f(A_2)$

On Considère que $A_1 \subset A_2$, soit $y \in E_2$

~~So~~ $y \in f(A_1) \Longrightarrow \exists x \in A_1$, t.q $y = f(x)$

$\Longrightarrow x \in A_2$, $y = f(x)$ (Car $A_1 \subset A_2$)

$\Longrightarrow y \in f(A_2)$; Alors $f(A_1) \subset f(A_2)$.

2 , 3 (Devoir)

4 $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$

a) $\subset$) $(f^{-1}(B_1 \cap B_2) \subset f^{-1}(B_1) \cap f^{-1}(B_2))$, Soit $x \in E_1$

$x \in f^{-1}(B_1 \cap B_2) \Rightarrow \exists y \in B_1 \cap B_2$ tel que $y = f(x)$.

$\Longrightarrow y \in B_1$, $y = f(x)$ et $y \in B_2$, $y = f(x)$

$\Longrightarrow x \in f^{-1}(B_1) \cap f^{-1}(B_2)$

Rappel

$f(A) = \{ y \in E_2.\ \exists x \in A\ \ y = f(x) \}$

$f^{-1}(B) = \{ x \in E_1, \exists y \in B, y = f(x) \}$.

Exercice 02 ; $f:]2,+\infty[\longrightarrow]2,+\infty[$

$$x \longmapsto f(x) = \frac{2x-1}{x-2}$$

$$g :]2,+\infty[\longrightarrow]2,\infty[$$

$$x \longmapsto g(x) = x^2 - 2$$

① $g(]3,4]) = \{ g(x) \in]2,+\infty[\ , \ x \in]3,4] \}$

$x \in]3.4] \Longrightarrow \quad 3 < x \leq 4$

$$9 < x^2 \leq 16$$

$$7 < x^2 - 2 \leq 14$$

$$\boxed{\text{Alors } g(]3,4]) =]7, 14]}$$

② $g^{-1}(]4,5]) = \{ x \in]2,+\infty[\ , \ g(x) \in]4,5] \}$

$g(x) \in]4,5] \Longrightarrow \quad 4 < x^2 - 2 \leq 5$

$$6 < x^2 \leq 7$$

$$\sqrt{6} < |x| \leq \sqrt{7}$$

alors $\begin{cases} \sqrt{6} < x \leq \sqrt{7} & \text{si } x \geq 0 \\ -\sqrt{7} \leq x < -\sqrt{6} & \text{si } x < 0 \end{cases}$ Annulé

Donc $g^{-1}(]4,5]) =]\sqrt{6}, \sqrt{7}]$

2) Montrer que f et g sont bijectives.

-o-

1) Pour f

a) L'injectivité ; Soient $x, x' \in]2, \infty[$

$$f(x) = f(x') \Rightarrow \frac{2x-1}{x-2} = \frac{2x'-1}{x'-2}$$

$$\Rightarrow \cancel{2xx'} - 4x - x' + \cancel{2} = \cancel{2x'x} - 4x' - x + \cancel{2}$$

$$\Rightarrow -4x - x' = -4x' - x$$

$$\Rightarrow -3x = -3x' \Rightarrow \boxed{x = x'}$$

Alors f est injective . . . . . . (1)

(b) Surjectivité ; Soit $y \in]2, +\infty[$

$$y = f(x) \Rightarrow y = \frac{2x-1}{x-2}$$

$$\Rightarrow y(x-2) = 2x - 1$$

$$\Rightarrow yx - 2y = 2x - 1$$

$$\Rightarrow yx - 2x = 2y - 1$$

$$\Rightarrow x(y-2) = 2y - 1$$

$$\Rightarrow \boxed{x = \frac{2y-1}{y-2}},$$

On a $\forall y \in]2, +\infty[, \exists x = \frac{2y-1}{y-2} \in]2, +\infty[$ tel que $y = f(x)$

Alors f est surjective . . . . (2)

f est injective et surjective, donc elle est bijective

Pour g

a) L'injectivité Soient $x, x' \in]2, +\infty[$.

$$g(x) = g(x') \Rightarrow x^2 - 2 = x'^2 - 2$$

$$\Rightarrow x^2 = x'^2 \Rightarrow |x| = |x'|$$

$$\Rightarrow x = x' \quad (\text{Car } x, x' > 0)$$

Alors g est injective.

b) Surjectivité Soit $y \in]2, +\infty[$.

$$y = g(x) \Rightarrow y = x^2 - 2$$

$$\Rightarrow x^2 = y + 2$$

$$\Rightarrow \boxed{x = \sqrt{y+2}} \text{ ou } \underline{x = -\sqrt{y+2}} < 0$$

Annulé

$\forall y \in]2, +\infty[$, $\exists\, x = \sqrt{y+2} \in]2, +\infty[$ tel que

$$y = g(x)$$

Alors g est surjective

g est injective et surjective, alors elle est bijective.

$f^{-1} :]2, +\infty[\longrightarrow]2, +\infty[$ $x \longrightarrow f^{-1}(x) = \dfrac{2x-1}{x-2}$	$g^{-1} :]2, +\infty[\longrightarrow]2, +\infty[$ $x \longmapsto g^{-1}(x) = \sqrt{x+2}$

③ f et g sont bijectives,

Alors $f \circ g$ et $g \circ f$ sont bijectives

$$f \circ g :]2, +\infty[\longrightarrow]2, \infty[$$

$$x \longrightarrow f \circ g(x)$$

$$f \circ g(x) = f(g(x)) = f(x^2 - 2)$$

$$= \frac{2(x^2-2) - 1}{(x^2-2) - 2} = \frac{2x^2 - 4 - 1}{x^2 - 4}$$

$$\boxed{f \circ g(x) = \frac{2x^2 - 5}{x^2 - 4}}$$

$$g \circ f :]2, +\infty[\longrightarrow]2, +\infty[.$$

$$x \longmapsto g \circ f(x)$$

$$g \circ f(x) = g(f(x)) = g\left(\frac{2x - 1}{x - 2}\right)$$

$$\boxed{g \circ f(x) = \left(\frac{2x - 1}{x - 2}\right)^2 - 2}$$

④ $f^{-1}(x) = \frac{2x-1}{x-2}$, $g^{-1}(x) = \sqrt{x+2}$, $f^{-1}, g^{-1} :]2,+\infty[\longrightarrow]2,+\infty[$.

Pour tout $x \in]2,+\infty[$.

$f \circ g(x) = \frac{2x^2-5}{x^2-4}$, Soit $y \in]2,+\infty[$

$f \circ g(x) = y \Rightarrow \frac{2x^2-5}{x^2-4} = y \Rightarrow 2x^2-5 = x^2y - 4y$

$\Longrightarrow 2x^2 - x^2y = 5-4y$

$\Rightarrow x^2(2-y) = (5-4y)$

$\Rightarrow x^2 = \frac{5-4y}{2-y} \Rightarrow x = \sqrt{\frac{5-4y}{2-y}}$

Alors $(f \circ g)^{-1} :]2,+\infty[\longrightarrow]2,+\infty[$

$$x \longmapsto \boxed{(f \circ g)^{-1}(x) = \sqrt{\frac{5-4x}{2-x}}}$$

On a. $g \circ f(x) = \left(\frac{2x-1}{x-2}\right)^2 - 2$, Soit $y \in]2,+\infty[$

$g \circ f(x) = y \Rightarrow \left(\frac{2x-1}{x-2}\right)^2 - 2 = y$

$\Longrightarrow \left(\frac{2x-1}{x-2}\right)^2 = y+2 \Rightarrow \frac{2x-1}{x-2} = \sqrt{y+2}$

$\Longrightarrow 2x - 1 = x\sqrt{y+2} - 2\sqrt{y+2}$.

Alors

$$2x - x\sqrt{y+2} = 1 - 2\sqrt{y+2}$$

$$x\left(2 - \sqrt{y+2}\right) = \left(1 - 2\sqrt{y+2}\right)$$

$$x = \frac{1 - 2\sqrt{y+2}}{2 - \sqrt{y+2}}$$

Donc : $g \circ f^{-1} :]2, +\infty[\longrightarrow]2, +\infty[$

$$\boxed{(g \circ f)^{-1}(x) = \frac{1 - 2\sqrt{x+2}}{2 - \sqrt{x+2}}}$$

- $f^{-1} \circ g^{-1}$, $\forall x \in]2, +\infty[$ $f^{-1}(x) = \frac{2x-1}{x-2}$, $g^{-1}(x) = \sqrt{x+2}$

$$f^{-1} \circ g^{-1}(x) = f^{-1}\left(g^{-1}(x)\right) = f^{-1}\left(\sqrt{x+2}\right)$$

$$\boxed{f^{-1} \circ g^{-1}(x) = \frac{2\sqrt{x+2} - 1}{\sqrt{x+2} - 2}} = \frac{1 - 2\sqrt{x+2}}{2 - \sqrt{x+2}}.$$

On remarque que

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1} ;$$

- $g^{-1} \circ f^{-1}$, $\forall x \in]2, +\infty[$.

$$g^{-1} \circ f^{-1}(x) = g^{-1}\left(f^{-1}(x)\right) = g^{-1}\left(\frac{2x-1}{x-2}\right) = \boxed{\sqrt{\frac{2x-1}{x-2} + 2}}$$

$$g^{-1} \circ f^{-1}(x) = \sqrt{\frac{4x-5}{x-2}} = (f \circ g)^{-1}(x)$$

On remarque que : $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$